Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 827-842, December 2015

ELA

THE ABSORPTION LAWS FOR THE GENERALIZED INVERSES IN RINGS*

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Abstract. In this paper, it is given equivalent conditions for the absorption laws in terms of the Moore-Penrose, group, core inverse, core inverse dual, {1}, {1,2}, {1,3}, and {1,4} inverses in rings. The results given here extend the results of [X. Liu, H. Jin, and D.S. Cvetković-Ilić. The absorption laws for the generalized inverses. *Appl. Math. Comp.*, 219:2053–2059, 2012.].

Key words. Rings, Absorption law, Generalized inverses.

AMS subject classifications. 16B99, 16W10, 15A09.

1. Introduction. It is known that for nonsingular matrices A and B of the same size, we have

$$A^{-1}(A+B)B^{-1} = A^{-1} + B^{-1}.$$

This equality is known as *the absorption law*.

Recently, the absorption law has been extended to generalized inverses. First, Chen et al. [3] investigated the absorption law for inner inverses. The authors found the maximal rank and the minimal rank of $A^- + B^- - A^-(A+B)B^-$ by using the singular value decomposition, and they obtained equivalent conditions for $A^- + B^- = A^-(A+B)B^-$ for some $A^- \in A\{1\}$, $B^- \in B\{1\}$. Making use of the rank of the generalized Schur complement, Chen et al. [4] studied the maximal and minimal ranks of $G + H - G(A+B)H$, where G and H are generalized inverses of A and B, respectively. In [8] and [9], the authors studied the mixed absorption laws for generalized inverses. The absorption law was extended to Hilbert spaces by Liu et al. in [10]. The authors gave necessary and sufficient conditions for the absorption laws and mixed absorption laws of the generalized inverses by decomposing the involved Hilbert spaces as a direct sum of two orthogonal closed subspaces.

*Received by the editors on August 17, 2015. Accepted for publication on November 29, 2015. Handling Editor: Oskar Maria Baksalary.

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The "absorption law" in a ring $\mathcal{R}$, says that if $a, b \in \mathcal{R}$ are both invertible, then $a^{-1} + b^{-1} = a^{-1}(a+b)b^{-1}$ and it is a natural question whether this has extension to generalized inverses.

In this paper, we study the absorption laws for the generalized inverses in rings. The outline of this paper is as follows. In Section 2, we define some preliminary notions used in the rest of the article. In particular, we first provide the definition of various types of generalized inverses. In Section 3, we study the absorption laws for the Moore-Penrose inverse, the group inverse, the core inverse and the dual core inverse. In Section 4, we consider the absorption laws for $\{1\}$, $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$-inverses. In Section 5, we study the mixed absorption laws.

The reader must not confuse the absorption law in rings with the absorption laws in Boolean algebras or lattices [5, 7], i.e.,

$$a \wedge (a \vee b) = a \quad \text{and} \quad a \vee (a \wedge b) = a,$$

where $\wedge$ and $\vee$ are two binary operations.

2. Preliminaries. Let $\mathcal{R}$ be a ring. Recall that if there is an element $\mathbb{1} \in \mathcal{R}$ such that

$$\mathbb{1}a = a\mathbb{1} = a \quad \text{for all a} \in \mathcal{R},$$

then $\mathbb{1}$ is called the *unity* and $\mathcal{R}$ is called a *unitary ring*. An element $a \in \mathcal{R}$ is said to be *von Neumann regular* if there exists $x \in \mathcal{R}$ such that $a = axa$. This x is said to be an *inner inverse* of a. In general x is not uniquely determined by a. We shall denote by $\mathcal{R}^{-1}$ and $\overline{\mathcal{R}}$ the subsets of $\mathcal{R}$ composed of invertible and von Neumann regular elements of $\mathcal{R}$, respectively.

The *principal ideals* (also called *image ideals*) generated by $b \in \mathcal{R}$ are defined by $b\mathcal{R} = \{bx : x \in \mathcal{R}\}$ and $\mathcal{R}b = \{xb : x \in \mathcal{R}\}$. The *annihilators* (also called *kernel ideals*) of $b \in \mathcal{R}$ are defined by $b^\circ = \{x \in \mathcal{R} : bx = 0\}$ and $^\circ b = \{x \in \mathcal{R} : xb = 0\}$.

If the ring $\mathcal{R}$ has an involution $x \mapsto x^*$, the element $a \in \mathcal{R}$ is *Moore-Penrose invertible* if there exists $x \in \mathcal{R}$ such that

$$(2.1) \quad (1)\ axa = a, \qquad (2)\ xax = x, \qquad (3)\ (ax)^* = ax, \qquad (4)\ (xa)^* = xa.$$

It can be proved that the set of elements x satisfying the above equations (1)–(4) is either empty, or a singleton. When it is not empty, its unique element is denoted by $a^\dagger$ and $a^\dagger$ is called the *Moore-Penrose inverse* of a. The set of Moore-Penrose invertible elements of $\mathcal{R}$ is denoted by $\mathcal{R}^\dagger$.

Recall that $a \in \mathcal{R}$ is *group invertible* if there exists $x \in \mathcal{R}$ such that

$$(2.2) \quad (1)\ axa = a, \qquad (2)\ xax = x, \qquad (5)\ ax = xa.$$

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Again, it can be proved that $\{x \in \mathcal{R} : axa = a, xax = x, ax = xa\}$ is either empty, or a singleton. When it is not empty, its unique element is denoted by $a^{\#}$ and $a^{\#}$ is called the *group inverse* of a. Observe that for the group invertibility, an involution is not required. The set of group invertible elements of $\mathcal{R}$ is denoted by $\mathcal{R}^{\#}$.

If $I \subset \{1, 2, 3, 4, 5\}$ and $a \in \mathcal{R}$, we say that x is an I-inverse of a if x satisfies the equation j for $j \in I$. We write aI for the set of I-inverses of a. Evidently, if $I \cap \{3, 4\} \neq \emptyset$, an involution in $\mathcal{R}$ is required. For example, if $a \in \mathcal{R}$ is Moore-Penrose invertible, then $a\{1, 2, 3, 4\} = \{a^{\dagger}\}$.

If $\mathcal{R}$ has an involution and $a \in \mathcal{R}^{\dagger}$, then it is simple to prove that

$$a^{\dagger} = (a^*a)^{\dagger}a^* = a^*(aa^*)^{\dagger}, \quad a^* = a^{\dagger}aa^* = a^*aa^{\dagger}.$$

These equalities imply that

$$\mathcal{R}a^* = \mathcal{R}a^{\dagger}, \qquad a^*\mathcal{R} = a^{\dagger}\mathcal{R}, \qquad (a^*)^{\circ} = (a^{\dagger})^{\circ}, \qquad {}^{\circ}(a^*) = {}^{\circ}(a^{\dagger}),$$

while the following equalities follow from the definition of the Moore-Penrose inverse

$$a\mathcal{R} = aa^{\dagger}\mathcal{R}, \qquad \mathcal{R}a = \mathcal{R}a^{\dagger}a, \qquad a^{\dagger}\mathcal{R} = a^{\dagger}a\mathcal{R}, \qquad \mathcal{R}a^{\dagger} = \mathcal{R}aa^{\dagger}.$$

It is evident that $a^* \in \mathcal{R}^{\dagger}$ and $(a^*)^{\dagger} = (a^{\dagger})^*$.

Let $\mathcal{R}$ be a ring with involution. Recall that the *core inverse* [11, Definition 2.3, Theorem 2.14] of $a \in \mathcal{R}$, denoted by $a^{\oplebar}$, is the unique element (if there exists) satisfying

$$aa^{\oplebar}a = a, \qquad a^{\oplebar}\mathcal{R} = a\mathcal{R}, \qquad \mathcal{R}a^{\oplebar} = \mathcal{R}a^*; \tag{2.3}$$

and the *dual core inverse* [11, Definition 2.4, Theorem 2.15] of $a \in \mathcal{R}$, denoted by $a_{\oplebar}$, is the unique element (if there exists) satisfying

$$aa_{\oplebar}a = a, \qquad a_{\oplebar}\mathcal{R} = a^*\mathcal{R}, \qquad \mathcal{R}a_{\oplebar} = \mathcal{R}a. \tag{2.4}$$

The core inverse and dual core inverse are, in some way, between the group and Moore-Penrose inverse. The concept for complex matrices was introduced in [1] by Baksalary and Trenker. Later, this definition was extended to Hilbert spaces [12] and finally to rings with an involution [11]. We denote by $\mathcal{R}^{\oplebar}$ and $\mathcal{R}_{\oplebar}$ the set of all core invertible elements and dual core invertible elements of $\mathcal{R}$, respectively.

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3. The absorption law for the Moore-Penrose, the group inverse, the core inverse, and dual core inverse. The following result will play an important role in the sequel.

LEMMA 3.1. *Let $\mathcal{R}$ be a ring and $a, b, c \in \mathcal{R}$.*

(i) *If $a \in \overline{\mathcal{R}}$ and $a^- \in a\{1,2\}$, then $a^-(a+b)c = a^- + c \iff a^-bc = a^-$ and $a^-ac = c$.*
(ii) *If $b \in \overline{\mathcal{R}}$ and $b^- \in b\{1,2\}$, then $c(a+b)b^- = c + b^- \iff cab^- = b^-$ and $cbb^- = c$.*

Proof. (i) Premultiplying $a^-ac + a^-bc = a^- + c$ by a, we get $aa^-bc = aa^-$. Premultiplying $aa^-bc = aa^-$ by a^-, we obtain $a^-bc = a^-$. Use now $a^-ac + a^-bc = a^- + c$ to get $a^-ac = c$. The reverse implication is trivial.

(ii) The proof of (ii) is similar to the proof of (i). □

This lemma has several consequences for the absorption law. First, we give the absorption law for the Moore-Penrose inverse.

THEOREM 3.2. *Let $\mathcal{R}$ be a ring with an involution and $a, b \in \mathcal{R}^\dagger$.*

(i) *$a^\dagger(a+b)b^\dagger = a^\dagger + b^\dagger$ if and only if $a\mathcal{R} \subset b\mathcal{R}$ and $\mathcal{R}b \subset \mathcal{R}a$.*
(ii) *If $\mathcal{R}$ is unitary, then $a^\dagger(a+b)b^\dagger = a^\dagger + b^\dagger$ if and only if $a^\circ \subset b^\circ$ and ${}^\circ b \subset {}^\circ a$.*

Proof. By Lemma 3.1, we get $a^\dagger(a+b)b^\dagger = a^\dagger + b^\dagger \iff a^\dagger bb^\dagger = a^\dagger$ and $a^\dagger ab^\dagger = b^\dagger$.

(i) If $a^\dagger bb^\dagger = a^\dagger$, then $a^\dagger \in \mathcal{R}b^\dagger = \mathcal{R}b^*$, which implies $\mathcal{R}a^* = \mathcal{R}a^\dagger \subset \mathcal{R}b^*$, hence $a\mathcal{R} \subset b\mathcal{R}$. If $a^\dagger ab^\dagger = b^\dagger$, then $b^\dagger \in a^\dagger\mathcal{R} = a^*\mathcal{R}$, which implies $b^*\mathcal{R} = b^\dagger\mathcal{R} \subset a^*\mathcal{R}$, hence $\mathcal{R}b \subset \mathcal{R}a$.

If $a\mathcal{R} \subset b\mathcal{R}$, then $a = aa^\dagger a \in b\mathcal{R}$, so there exists $u \in \mathcal{R}$ such that $a = bu$, which leads to $bb^\dagger a = bb^\dagger bu = bu = a$. Therefore,

$$a(a^\dagger bb^\dagger)a = aa^\dagger(bb^\dagger a) = aa^\dagger a = a,$$

$$(a^\dagger bb^\dagger)a(a^\dagger bb^\dagger) = a^\dagger(bb^\dagger a)a^\dagger bb^\dagger = a^\dagger aa^\dagger bb^\dagger = a^\dagger bb^\dagger,$$

$$(a^\dagger bb^\dagger)a = a^\dagger(bb^\dagger a) = a^\dagger a \text{ is selfadjoint},$$

$$\big[a(a^\dagger bb^\dagger)\big]^* = \big[(aa^\dagger)(bb^\dagger)\big]^* = (bb^\dagger)^*(aa^\dagger)^* = bb^\dagger aa^\dagger = aa^\dagger \text{ is selfadjoint}.$$

These four equalities prove that $a^\dagger bb^\dagger = a^\dagger$. If $\mathcal{R}b \subset \mathcal{R}a$, then $b^*\mathcal{R} \subset a^*\mathcal{R}$, and by using the last part of the proof (replacing $b^* \leftrightarrow a$ and $a^* \leftrightarrow b$), we get $(b^*)^\dagger a^*(a^*)^\dagger = (b^*)^\dagger$. Hence, $(b^\dagger)^*a^*(a^\dagger)^* = (b^\dagger)^*$, which taking $*$, reduces to $a^\dagger ab^\dagger = b^\dagger$.

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(ii) Let $\mathbb{1}$ be the unity of $\mathcal{R}$.

Assume in this paragraph that $a^\dagger ab^\dagger = b^\dagger$. If $x \in a^\circ$, then $ax = 0$, which implies $x^*a^* = 0$, i.e., $x^* \in {}^\circ(a^*) = {}^\circ(a^\dagger)$. Now, $x^*b^\dagger = x^*a^\dagger ab^\dagger = 0$. Hence, $x^* \in {}^\circ(b^\dagger) = {}^\circ(b^*)$, and thus, $x \in b^\circ$.

Analogously, we can prove $a^\dagger bb^\dagger = a^\dagger \Rightarrow {}^\circ b \subset {}^\circ a$.

Assume in this paragraph that $a^\circ \subset b^\circ$. Since $a(\mathbb{1} - a^\dagger a) = 0$, we get $\mathbb{1} - a^\dagger a \in a^\circ \subset b^\circ$. Since $\mathbb{1} - a^\dagger a$ is self-adjoint, $\mathbb{1} - a^\dagger a \in {}^\circ(b^*) = {}^\circ(b^\dagger)$. In other words, $(\mathbb{1} - a^\dagger a)b^\dagger = 0$.

The implication ${}^\circ b \subset {}^\circ a \Rightarrow a^\dagger = a^\dagger bb^\dagger$ can be proved in a similar way as in the previous paragraph. □

Next, we give the absorption law for the group inverse. Before doing this, we give a more general result concerning commutative $\{2\}$-inverses. Observe that for an element a of a ring $\mathcal{R}$, the set $a\{2,5\} = \{a^- \in \mathcal{R} : a^-aa^- = a^-, aa^- = a^-a\}$ is always non-empty because 0 belongs to $a\{2,5\}$.

THEOREM 3.3. *Let $\mathcal{R}$ be a ring and $a, b \in \mathcal{R}$. Let $a^- \in a\{2,5\}$ and $b^- \in b\{2,5\}$ be fixed. Then the following statements are equivalent:*

(i) $a^-(a+b)b^- = a^- + b^-$.
(ii) $aa^- = bb^-$.
(iii) $b^-(a+b)a^- = b^- + a^-$.

Proof. (i) $\Rightarrow$ (ii): Premultiplying aa^- on both sides of $a^-(a+b)b^- = a^- + b^-$ gives $a^-bb^- = a^-$. On the other hand, postmultiplying bb^- on the two sides of $a^-(a+b)b^- = a^- + b^-$ gives $a^-ab^- = b^-$. Hence, $aa^- = aa^-bb^- = a^-ab^-b = b^-b = bb^-$.

(ii) $\Rightarrow$ (i) is obvious.

(ii) $\Leftrightarrow$ (iii): This follows by interchanging the roles of a and b in the proof of (i) $\Leftrightarrow$ (ii). □

Observe that we can apply this result to give the absorption law for the Drazin inverse in a ring. The Drazin inverse of an element a in a ring $\mathcal{R}$, which was introduced in [6], is the (unique, if there exists) element $x \in \mathcal{R}$ satisfying the equalities (2), (5) of (2.2), and $axa - a$ is nilpotent.

THEOREM 3.4. *Let $\mathcal{R}$ be a ring. If $a, b \in \mathcal{R}^\#$, then the following statements are equivalent:*

(i) $a^\#(a+b)b^\# = a^\# + b^\#$.
(ii) $aa^\# = bb^\#$.

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(iii) $a\mathcal{R} = b\mathcal{R}$ *and* $\mathcal{R}a = \mathcal{R}b$.
(iv) $a\mathcal{R} \subset b\mathcal{R}$ *and* $\mathcal{R}b \subset \mathcal{R}a$.
(v) $b^{\#}(a+b)a^{\#} = b^{\#} + a^{\#}$.

If in addition, the ring $\mathcal{R}$ *is unitary, then any of the above conditions is equivalent to any of the following conditions:*

(vi) $a^{\circ} = b^{\circ}$ *and* $^{\circ}a = {}^{\circ}b$.
(vii) $b^{\circ} \subset a^{\circ}$ *and* $^{\circ}a \subset {}^{\circ}b$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (v) follow from Theorem 3.3.

(ii) $\Rightarrow$ (iii): Since $a = aa^{\#}a = bb^{\#}a$, we get $a\mathcal{R} \subset b\mathcal{R}$. Since $a = aa^{\#}a = ab^{\#}b$, we get $\mathcal{R}a \subset \mathcal{R}b$. Interchanging the roles of a and b (observe that the roles of a and b are symmetric in the hypothesis), we get $a\mathcal{R} = b\mathcal{R}$ and $\mathcal{R}a = \mathcal{R}b$.

(iii) $\Rightarrow$ (iv) is evident.

(iv) $\Rightarrow$ (ii): From $a = aa^{\#}a \in a\mathcal{R} \subset b\mathcal{R}$, there exists $u \in \mathcal{R}$ such that $a = bu$, and therefore, $bb^{\#}a = bb^{\#}bu = bu = a$. From $b^{\#} = b^{\#}b^2 \in \mathcal{R}b \subset \mathcal{R}a$, there exists $v \in \mathcal{R}$ such that $b^{\#} = va$, and therefore, $b^{\#}aa^{\#} = va^2a^{\#} = va = b^{\#}$. Now, $aa^{\#} = bb^{\#}aa^{\#} = bb^{\#}$.

Now let us assume that $\mathcal{R}$ is unitary and let us denote by $\mathbb{1}$ its unity.

(ii) $\Rightarrow$ (vi): Let $x \in a^{\circ}$. Hence, $ax = 0$, and now, $bx = bb^{\#}bx = ba^{\#}ax = 0$, i.e., $x \in b^{\circ}$. A similar argument proves the remaining inclusions.

(vi) $\Rightarrow$ (vii) is evident.

(vii) $\Rightarrow$ (ii): It is simple to prove that for any $x \in \mathcal{R}^{\#}$ one has that $x^{\circ} = (x^{\#})^{\circ}$ and $^{\circ}x = {}^{\circ}(x^{\#})$. Since $b(\mathbb{1} - bb^{\#}) = 0$, we have $(\mathbb{1} - bb^{\#}) \in b^{\circ} \subset a^{\circ} = (a^{\#})^{\circ}$, and therefore, $a^{\#} = a^{\#}bb^{\#}$. Since $\mathbb{1} - aa^{\#} \in {}^{\circ}a \subset {}^{\circ}b = {}^{\circ}(b^{\#})$, we get $(\mathbb{1} - aa^{\#})b^{\#} = 0$, i.e., $b^{\#} = aa^{\#}b^{\#}$. Now $aa^{\#} = aa^{\#}bb^{\#} = aa^{\#}b^{\#}b = b^{\#}b$. □

We give now the absorption law for the core inverse and dual core inverse.

THEOREM 3.5. *Let* $\mathcal{R}$ *be a ring with an involution and* $a, b \in \mathcal{R}$.

(i) *If* $a, b \in \mathcal{R}^{\oplus}$, *then* $a^{\oplus}(a+b)b^{\oplus} = a^{\oplus} + b^{\oplus}$ *if and only if* $a\mathcal{R} = b\mathcal{R}$.
(ii) *If* $a, b \in \mathcal{R}_{\oplus}$, *then* $a_{\oplus}(a+b)b_{\oplus} = a_{\oplus} + b_{\oplus}$ *if and only if* $\mathcal{R}a = \mathcal{R}b$.

Furthermore, if $\mathcal{R}$ *is unitary, then:*

(iii) *If* $a, b \in \mathcal{R}^{\oplus}$, *then* $a^{\oplus}(a+b)b^{\oplus} = a^{\oplus} + b^{\oplus}$ *if and only if* $^{\circ}a = {}^{\circ}b$.
(iv) *If* $a, b \in \mathcal{R}_{\oplus}$, *then* $a_{\oplus}(a+b)b_{\oplus} = a_{\oplus} + b_{\oplus}$ *if and only if* $a^{\circ} = b^{\circ}$.

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A publication of the International Linear Algebra Society
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Proof. (i) By the definition of the core inverse and [11, Theorem 2.14], one has that $a^{\oplus} \in a\{1,2\}$. From Lemma 3.1, we have

$$a^{\oplus}(a+b)b^{\oplus} = a^{\oplus} + b^{\oplus} \qquad \Leftrightarrow \qquad a^{\oplus}bb^{\oplus} = a^{\oplus}, \quad a^{\oplus}ab^{\oplus} = b^{\oplus}.$$

In the rest of the proof, we will use extensively the equalities (2.13) and $a^{\oplus} = a^{\oplus}aa^{\oplus}$.

Assume in this paragraph $a^{\oplus}bb^{\oplus} = a^{\oplus}$. We have $a^* = (aa^{\oplus}a)^* = (a^{\oplus}a)^*a^* \in \mathcal{R}a^* = \mathcal{R}a^{\oplus} \subset \mathcal{R}b^{\oplus} = \mathcal{R}b^*$, which implies $a \in b\mathcal{R}$, and therefore, $a\mathcal{R} \subset b\mathcal{R}$.

Assume in this paragraph $a^{\oplus}ab^{\oplus} = b^{\oplus}$. We have $b\mathcal{R} = b^{\oplus}\mathcal{R} \subset a^{\oplus}\mathcal{R} = a\mathcal{R}$.

Assume in this paragraph $a\mathcal{R} \subset b\mathcal{R}$. We have $a^{\oplus} = a^{\oplus}aa^{\oplus} \in \mathcal{R}a^{\oplus} = \mathcal{R}a^*$. There exists $u \in \mathcal{R}$ such that $a^{\oplus} = ua^*$. Therefore, $(a^{\oplus})^* = au^* \in a\mathcal{R} \subset b\mathcal{R}$. Hence, $(a^{\oplus})^* = bv$ for some $v \in \mathcal{R}$. Thus, $a^{\oplus} = v^*b^* \in \mathcal{R}b^* = \mathcal{R}b^{\oplus}$, hence $a^{\oplus} = wb^{\oplus}$ for some $w \in \mathcal{R}$. Finally, $a^{\oplus}bb^{\oplus} = wb^{\oplus}bb^{\oplus} = wb^{\oplus} = a^{\oplus}$.

Assume in this paragraph $b\mathcal{R} \subset a\mathcal{R}$. We have $b^{\oplus} = b^{\oplus}bb^{\oplus} \in b^{\oplus}\mathcal{R} = b\mathcal{R} \subset a\mathcal{R} = a^{\oplus}\mathcal{R}$. Therefore, there exists $u \in \mathcal{R}$ such that $b^{\oplus} = a^{\oplus}u$. Now, $a^{\oplus}ab^{\oplus} = a^{\oplus}aa^{\oplus}u = a^{\oplus}u = b^{\oplus}$.

(ii) Since $a \in \mathcal{R}_{\oplus}$, by taking $*$ on the equalities of (2.4), one has $a^*(a_{\oplus})^*a^* = a^*$, $\mathcal{R}(a_{\oplus})^* = \mathcal{R}a$, and $(a_{\oplus})^*\mathcal{R} = a^*\mathcal{R}$, which means that $a^* \in \mathcal{R}^{\oplus}$ and $(a_{\oplus})^* = (a^*)^{\oplus}$. The same happens to b. Then, (ii) follows from (i) by replacing $b^* \leftrightarrow a$ and $a^* \leftrightarrow b$.

(iii) It is sufficient to prove $a\mathcal{R} = b\mathcal{R} \Leftrightarrow {}^\circ a = {}^\circ b$ when $\mathcal{R}$ is unitary. If $a\mathcal{R} \subset b\mathcal{R}$, is evident that ${}^\circ b \subset {}^\circ a$. If ${}^\circ b \subset {}^\circ a$, then $(bb^{\oplus} - \mathbb{1})b = 0$ implies $(bb^{\oplus} - \mathbb{1})a = 0$, which leads to $a = bb^{\oplus}a \in b\mathcal{R}$, and therefore, $a\mathcal{R} \subset b\mathcal{R}$. By exchanging the roles of a and b, one has $b\mathcal{R} \subset a\mathcal{R} \Leftrightarrow {}^\circ a \subset {}^\circ b$ and the conclusion follows.

(iv) The proof is similar as in (ii). □

By Theorem 3.4 and Theorem 3.5, the following statement, which shows the relation of the absorption law for the group inverse, the core inverse and dual core inverse, is true. We recall (see [11, Remark 2.16]) that $\mathcal{R}^{\#} \cap \mathcal{R}^{\dagger} = \mathcal{R}^{\oplus} \cap \mathcal{R}_{\oplus}$.

COROLLARY 3.6. *Let $\mathcal{R}$ be a ring with an involution and $a \in \mathcal{R}^{\#} \cap \mathcal{R}^{\dagger}$. The following statements are equivalent:*

(i) $a^{\#}(a+b)b^{\#} = a^{\#} + b^{\#}$.
(ii) $a^{\oplus}(a+b)b^{\oplus} = a^{\oplus} + b^{\oplus}$ *and* $a_{\oplus}(a+b)b_{\oplus} = a_{\oplus} + b_{\oplus}$.

4. The absorption law for $\{1\}$, $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$-inverses. Next we state the absorption law for the $\{1\}$-inverses. Let us recall that if $\mathcal{R}$ is a ring, then $\overline{\mathcal{R}}$ denotes the set of von Neumann regular elements in $\mathcal{R}$, i.e., $a \in \overline{\mathcal{R}}$ if and only if there

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A publication of the International Linear Algebra Society
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exists $x \in a\{1\}$. For such a, x we can easily see that $xax \in a\{1,2\}$. In particular, if $a\{1\} \neq \emptyset$, then $a\{1,2\} \neq \emptyset$.

THEOREM 4.1. *Let $\mathcal{R}$ be a unitary ring and $a, b \in \overline{\mathcal{R}}$. The following statements are equivalent:*

(i) $a^-(a+b)b^- = a^- + b^-$ *for any* $a^- \in a\{1\}$, $b^- \in b\{1\}$.
(ii) *If* $s \in a\{1,2\}$ *and* $t \in b\{1,2\}$, *then* $sa = \mathbb{1} = bt$.
(iii) *If* $a^- \in a\{1\}$ *and* $b^- \in b\{1\}$, *then* $a^-a = bb^- = \mathbb{1}$.

Proof. (i) $\Rightarrow$ (ii): Let $s \in a\{1,2\}$ and $t \in b\{1,2\}$. By Lemma 3.1, we get $sbt = s$ and $sat = t$. It is easy to see that $s + \mathbb{1} - as \in a\{1\}$. By hypothesis, we have

$$(s + \mathbb{1} - as)(a+b)t = s + \mathbb{1} - as + t.$$

This equality yields $sat + sbt + (\mathbb{1} - as)bt = s + \mathbb{1} - as + t$, which, by using $sbt = s$ and $sat = t$, reduces to $bt = \mathbb{1}$. Also, it is easy to check that $t + \mathbb{1} - tb \in b\{1\}$. By hypothesis we get $s(a+b)(t + \mathbb{1} - tb) = s + t + \mathbb{1} - tb$, which reduces to $sa = \mathbb{1}$.

(ii) $\Rightarrow$ (iii): Let $a^- \in a\{1\}$ and $b^- \in b\{1\}$. By the previous paragraph of the theorem, $a^-aa^- \in a\{1,2\}$ and $b^-bb^- \in b\{1,2\}$. By hypothesis we have $a^-aa^-a = bb^-bb^- = \mathbb{1}$, which reduces to $a^-a = bb^- = \mathbb{1}$.

(iii) $\Rightarrow$ (i): If $a^- \in a\{1\}$ and $b^- \in b\{1\}$, then $a^-(a+b)b^- = a^-ab^- + a^-bb^- = a^- + b^-$. □

To prove the following corollary, it suffices to observe that if $a \in \mathcal{R}^\#$, then $a^\# \in a\{1,2\}$ and $aa^\# = a^\#a$.

COROLLARY 4.2. *Let $\mathcal{R}$ be a unitary ring and $a, b \in \mathcal{R}^\#$. The following statements are equivalent:*

(i) $a^-(a+b)b^- = a^- + b^-$ *for any* $a^- \in a\{1\}$, $b^- \in b\{1\}$.
(ii) $a, b \in \mathcal{R}^{-1}$.

COROLLARY 4.3. *Let $\mathcal{R}$ be a unitary ring with an involution and $a, b \in \mathcal{R}^\dagger$. The following statements are equivalent:*

(i) $a^-(a+b)b^- = a^- + b^-$ *for any* $a^- \in a\{1\}$, $b^- \in b\{1\}$.
(ii) $a^\dagger a = bb^\dagger = \mathbb{1}$.

Proof. (i) $\Rightarrow$ (ii) follows from Theorem 4.1. To prove the reciprocal implication, let $a^- \in a\{1\}$ and $b^- \in b\{1\}$. Now $aa^-a = a$ implies $a^\dagger aa^-a = a^\dagger a$, and by hypothesis, $a^-a = \mathbb{1}$. Similarly, $bb^- = \mathbb{1}$. Now, $a^-(a+b)b^- = a^-ab^- + a^-bb^- = b^- + a^-$. □

Next, we give the absorption law for $\{1,2\}$-inverses. Let us recall that a ring $\mathcal{R}$ is said to be *prime* if for any two elements a and b of $\mathcal{R}$, the condition "$arb = 0$ for all

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A publication of the International Linear Algebra Society
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$r \in \mathcal{R}$" implies that either $a = 0$ or $b = 0$. It is known that any matrix ring over an integral domain is a prime ring. Before giving the absorption law for $\{1,2\}$-inverses, we establish a useful result, interesting in its own right.

THEOREM 4.4. *Let $\mathcal{R}$ be a prime ring and $a \in \overline{\mathcal{R}}$. The following affirmations are equivalent:*

(i) *a^-a is invariant on the choice of $a^- \in a\{1,2\}$.*
(ii) *$a^-a = \mathbb{1}$ for any $a^- \in a\{1,2\}$ or $a = 0$.*

Proof. (i) $\Rightarrow$ (ii): Pick any $a^- \in a\{1,2\}$. Let us suppose that there exists $z \in (\mathbb{1} - a^-a)\mathcal{R}aa^-$ such that $z \neq 0$. Let $u \in \mathcal{R}$ be such that $z = (\mathbb{1} - a^-a)uaa^-$. Evidently, we have $az = 0$ and $zaa^- = z$, which imply $a^- + z \in a\{1,2\}$. In view of the hypothesis of the invariance, we get $a^-a = (a^- + z)a$. Therefore, $0 = za$, which implies $0 = zaa^- = z$. This is a contradiction, and thus, $(\mathbb{1} - a^-a)\mathcal{R}aa^- = 0$. Since $\mathcal{R}$ is prime, we get $a^-a = \mathbb{1}$ or $aa^- = 0$.

(ii) $\Rightarrow$ (i) is obvious. □

The following result is analogous to the former theorem and it has a similar proof.

THEOREM 4.5. *Let $\mathcal{R}$ be a prime ring and $a \in \overline{\mathcal{R}}$. The following affirmations are equivalent:*

(i) *aa^- is invariant on the choice of $a^- \in a\{1,2\}$.*
(ii) *$aa^- = \mathbb{1}$ for any $a^- \in a\{1,2\}$ or $a = 0$.*

THEOREM 4.6. *Let $\mathcal{R}$ be a unitary ring and $a, b \in \overline{\mathcal{R}}$. The following statements are equivalent:*

(i) *$a^-(a+b)b^- = a^- + b^-$ for any $a^- \in a\{1,2\}$ and $b^- \in b\{1,2\}$.*
(ii) *$a^-a = bb^-$ for any $a^- \in a\{1,2\}$ and $b^- \in b\{1,2\}$. In particular, a^-a and bb^- are invariant on the choice of $a^- \in a\{1,2\}$ and $b^- \in b\{1,2\}$.*

If the ring $\mathcal{R}$ is prime, then any of the above conditions is equivalent to

(iii) *$a = b = 0$ or $bb^- = a^-a = \mathbb{1}$ for any $a^- \in a\{1,2\}$, $b^- \in b\{1,2\}$.*

Proof. (i) $\Rightarrow$ (ii): By Lemma 3.1, one has

$$a^-bb^- = a^- \text{ and } a^-ab^- = b^- \text{ for any } a^- \in a\{1,2\}, b^- \in b\{1,2\}. \tag{4.1}$$

Let $s \in a\{1,2\}$ and $t \in b\{1,2\}$ be fixed. It is simple to prove that $s + sa(\mathbb{1} - as) \in a\{1,2\}$. By hypothesis we get

$$[s + sa(\mathbb{1} - as)]\,(a+b)t = s + sa(\mathbb{1} - as) + t. \tag{4.2}$$

The left hand of (4.2) simplifies to $sat + sbt + sa(\mathbb{1} - as)bt$. From (4.1), we get $sat = t$

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and $sbt = s$. Therefore, (4.2) reduces to $sa(\mathbb{1} - as)bt = sa(\mathbb{1} - as)$, or equivalently, $sabt - sa^2sbt = sa - sa^2s$, and by using again $sbt = s$, we get $sabt = sa$.

Now, it is simple to prove $t + (\mathbb{1} - tb)bt \in b\{1, 2\}$. By hypothesis we get

$$s(a + b)\,[t + (\mathbb{1} - tb)bt] = s + t + (\mathbb{1} - tb)bt.$$

Reasoning similarly as we did in the previous paragraph, we get $sabt = bt$. Therefore, we get $bt = sabt = sa$.

(ii) $\Rightarrow$ (i): Let $a^- \in a\{1, 2\}$ and $b^- \in b\{1, 2\}$. As in the proof of Theorem 4.4, we can prove $(\mathbb{1} - a^-a)\mathcal{R}aa^- = \{0\}$. Therefore,

$$0 = (\mathbb{1} - a^-a)(b^-a^-)aa^- = (\mathbb{1} - a^-a)b^-a^-,$$

so $b^-a^- = a^-ab^-a^-$. Using $bb^- = a^-a$ we obtain

$$a^-ab^- = a^-ab^-bb^- = a^-ab^-a^-a = b^-a^-a = b^-bb^- = b^-. \tag{4.3}$$

Similarly, from the assumption that bb^- is invariant on the choice of $b^- \in b\{1, 2\}$, one can show that $b^-br(\mathbb{1} - bb^-) = 0$ for any $r \in \mathcal{R}$. Choosing $r = b^-a^-$ we obtain $b^-a^- = b^-a^-bb^-$. From $bb^- = a^-a$, we derive

$$a^-bb^- = a^-aa^-bb^- = bb^-a^-bb^- = bb^-a^- = a^-aa^- = a^-. \tag{4.4}$$

Now, the proof follows from (4.3) and (4.4).

Now, let us assume that $\mathcal{R}$ is prime. (ii) $\Rightarrow$ (iii) follows from Theorem 4.4 and Theorem 4.5, while (iii) $\Rightarrow$ (ii) is evident. □

Example 4.7. The hypothesis "$\mathcal{R}$ is a prime ring" in Theorem 4.6 cannot be removed as the next example shows. Let $\mathcal{R} = \mathbb{Z}_6$ and $a = b = [2]$. It is simple to see (by trial and error) that $a\{1\} = \{[2], [5]\}$ and $a\{1, 2\} = \{[2]\}$. Also, $[2]([2] + [2])[2] = [4] = [2] + [2]$, i.e., the $\{1, 2\}$ absorption law holds for this choice of a and b. However, the condition (iii) of Theorem 4.6 does not hold.

Next corollary has the same proof as Corollaries 4.2 and 4.3.

Corollary 4.8. *Let $\mathcal{R}$ be a unitary and prime ring.*

(i) *If $\mathcal{R}$ has an involution and $a, b \in \mathcal{R}^\dagger$, then $a^-(a + b)b^- = a^- + b^-$ for any $a^- \in a\{1, 2\}$, $b^- \in b\{1, 2\}$ if and only if $a^\dagger a = bb^\dagger = \mathbb{1}$ or $a = b = 0$.*
(ii) *If $a, b \in \mathcal{R}^\#$, then $a^-(a + b)b^- = a^- + b^-$ for any $a^- \in a\{1, 2\}$, $b^- \in b\{1, 2\}$ if and only if $a, b \in \mathcal{R}^{-1}$ or $a = b = 0$.*

To deal with the absorption law for $\{1, 3\}$-inverses, we need the following result. In fact it was established in [2, Chapter 2, Corollary 3] (in terms of complex matrices);

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but we give the proof for the sake of completeness (and for showing that it can be established in rings with an involution).

THEOREM 4.9. *Let $\mathcal{R}$ be a ring, $a \in \overline{\mathcal{R}}$, and $a^- \in a\{1\}$.*

(i) *Let $d \in \mathcal{R}$. The equation $ax = d$ has a solution if and only if $aa^-d = d$. In this case, $\{x \in \mathcal{R} : ax = d\} = \{a^-d + y - a^-ay : y \in \mathcal{R}\}$.*
(ii) *If $\mathcal{R}$ has an involution and $a^- \in a\{1,3\}$, then $a\{1,3\} = \{x \in R : ax = aa^-\}$.*
(iii) *If $\mathcal{R}$ has an involution and $a^- \in a\{1,3\}$, then $a\{1,3\} = \{a^- + z - a^-az : z \in \mathcal{R}\}$.*

Proof. The proof of the first affirmation of (i) is easy and is left to the reader. If $x \in \mathcal{R}$ satisfies $ax = d$, then $x = a^-d + x - a^-ax \in \{a^-d + y - a^-ay : y \in \mathcal{R}\}$. Also, if $y \in \mathcal{R}$ is arbitrary, then

$$a(a^-d + y - a^-ay) = aa^-d + ay - aa^-ay = d.$$

(ii) First, let us note that $aa^-a = a$ implies $a^* = (aa^-a)^* = a^*(aa^-)^* = a^*aa^-$.

If $u \in a\{1,3\}$, then $aa^- = auaa^- = (au)^*aa^- = u^*a^*aa^- = u^*a^* = (au)^* = au$. If $x \in \mathcal{R}$ satisfies $ax = aa^-$, then $axa = aa^-a = a$ and ax is selfadjoint because $ax = aa^-$; and therefore, $x \in a\{1,3\}$.

(iii) Evidently, the equation $ax = aa^-$ has a solution (namely, a^-). By item (ii), the set of solutions is $a\{1,3\}$. Using item (i) leads to

$$\begin{aligned} a\{1,3\} &= \{a^-aa^- + y - a^-ay : y \in \mathcal{R}\} \\ &= \{a^-aa^- + (z + a^-) - a^-a(z + a^-) : z \in \mathcal{R}\} \\ &= \{a^- + z - a^-az : z \in \mathcal{R}\}. \end{aligned}$$

Observe that we cannot assure $a^-aa^- = a^-$ because we do not know whether $a^- \in a\{2\}$. The proof is completed. □

In fact, only the third item of the previous result will be utilized. Next, we give the absorption law for $\{1,3\}$-inverses. Observe that for $a \in \mathcal{R}$, being $\mathcal{R}$ a ring with an involution, if $a^- \in a\{1,3\}$, then $a^-aa^- \in a\{1,2,3\}$, which in particular leads to $a\{1,3\} \neq \emptyset \Rightarrow a\{1,2,3\} \neq \emptyset$.

THEOREM 4.10. *Let $\mathcal{R}$ be a unitary ring with an involution. Let $a, b \in \mathcal{R}$ such that $a\{1,3\} \neq \emptyset$ and $b\{1,3\} \neq \emptyset$. Let $s \in a\{1,2,3\}$ and $t \in b\{1,3\}$ be fixed. Then the following statements are equivalent:*

(i) *$a^-(a+b)b^- = a^- + b^-$ for any $a^- \in a\{1,3\}$ and $b^- \in b\{1,3\}$.*
(ii) *$s(a+b)b^- = s + b^-$ for any $b^- \in b\{1,3\}$.*
(iii) *$sa = \mathbb{1}$ and $sbb^- = s$ for any $b^- \in b\{1,3\}$.*
(iv) *$sa = \mathbb{1}$ and $sbt = s$.*

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Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are obvious.

(ii) $\Rightarrow$ (iii): Pick any $b^- \in b\{1,3\}$. By Lemma 3.1 (i), we have $sab^- = b^-$ and $sbb^- = s$. Observe that $b^- + \mathbb{1} - b^-b \in b\{1,3\}$ since $b^- \in b\{1,3\}$. By hypothesis, $s(a+b)(b^- + \mathbb{1} - b^-b) = s + b^- + \mathbb{1} - b^-b$, which again by Lemma 3.1, implies $sa(b^- + \mathbb{1} - b^-b) = b^- + \mathbb{1} - b^-b$, which by using $sab^- = b^-$, reduces to $sa = \mathbb{1}$.

(iv) $\Rightarrow$ (i): By Theorem 4.9, we have $a\{1,3\} = \{s\}$ and $b\{1,3\} = \{t + z - tbz : z \in \mathcal{R}\}$. Now

$$s(a+b)(t+z-tbz) = sa(t+z-tbz) + sbt + sbz - sbtbz = t + z - tbz + s$$

leads to (i). □

Observe that, by the proof of Theorem 3.2, we have $a^\dagger bb^\dagger = a^\dagger \iff a\mathcal{R} \subset b\mathcal{R}$ (provided that $\mathcal{R}$ is a ring with an involution and $a, b \in \mathcal{R}^\dagger$). This observation leads to the following corollary.

COROLLARY 4.11. *Let $\mathcal{R}$ be a unitary ring with an involution and $a, b \in \mathcal{R}^\dagger$. Then the following statements are equivalent:*

(i) $a^-(a+b)b^- = a^- + b^-$ *for any* $a^- \in a\{1,3\}$ *and* $b^- \in b\{1,3\}$.
(ii) $a^\dagger a = \mathbb{1}$ *and* $a\mathcal{R} \subset b\mathcal{R}$.

The absorption law for the $\{1,4\}$-inverse is stated as follows and the proof is similar as in Theorem 4.10 (but now, using Lemma 3.1 (ii) and if a is a Von Neumann regular element of a ring $\mathcal{R}$ with an involution, then $a\{1,4\} = \{a^- + y - yaa^- : y \in \mathcal{R}\}$, where a^- is a fixed element of $a\{1,4\}$).

THEOREM 4.12. *Let $\mathcal{R}$ be a unitary ring with an involution. Let $a, b \in \mathcal{R}$ be such that $a\{1,4\} \neq \emptyset$ and $b\{1,4\} \neq \emptyset$. Let $s \in a\{1,4\}$ and $t \in b\{1,2,4\}$ be fixed. Then the following statements are equivalent:*

(i) $a^-(a+b)b^- = a^- + b^-$ *for any* $a^- \in a\{1,4\}$, $b^- \in b\{1,4\}$.
(ii) $a^-(a+b)t = a^- + t$ *for any* $a^- \in a\{1,4\}$.
(iii) $bt = \mathbb{1}$ *and* $a^-at = t$ *for any* $a^- \in a\{1,4\}$.
(iv) $bt = \mathbb{1}$ *and* $sat = t$.

Furthermore, if $a, b \in \mathcal{R}^\dagger$, then any of the above conditions is equivalent to

(v) $\mathcal{R}b \subset \mathcal{R}a$ *and* $bb^\dagger = \mathbb{1}$.

5. The mixed absorption law. In this section, we consider various types of mixed absorption law for the generalized inverses.

THEOREM 5.1. *Let $\mathcal{R}$ be a unitary ring with an involution, $a, b \in \mathcal{R}$, and let $\beta, \gamma \in \{\{1,2\},\{1,3\},\{1,4\}\}$ and $\beta \neq \gamma$. Fix $s \in a\beta$ and $t \in b\gamma$. Then the following*

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statements are equivalent:

(i) $a^-(a+b)b^- = a^- + b^-$ *for any* $a^- \in a\beta$ *and* $b^- \in b\gamma$.
(ii) $sa = \mathbb{1} = bt$.

Proof. (ii) $\Rightarrow$ (i): Pick any $a^- \in a\{1\}$. From $aa^-a = a$ and $\mathbb{1} = sa$ (by multiplying by s on the left), we have $a^-a = \mathbb{1}$. Similarly, if $b^- \in b\{1\}$, then $bb^- = \mathbb{1}$. Now, $a^-(a+b)b^- = a^-ab^- + a^-bb^- = a^- + b^-$.

(i) $\Rightarrow$ (ii): By hypothesis, we have $s(a+b)t = s + t$, or equivalently,

$$sat + sbt = s + t. \tag{5.1}$$

which, by multiplying by a on the left and by b on the right, we get, respectively

$$asbt = as, \qquad satb = tb. \tag{5.2}$$

There are six cases to consider.

Case 1. $\beta = \{1,2\}$, $\gamma = \{1,3\}$.

Since $s \in a\{2\}$, multiplying (5.2) by s on the left leads to $sbt = s$, and using (5.1) we get $sat = t$. Observe that $s \in a\{1,2\}$ implies $s + sa(\mathbb{1} - as) \in a\{1,2\}$. Again by hypothesis we have

$$[s + sa(\mathbb{1} - as)]\,(a+b)t = s + sa(\mathbb{1} - as) + t,$$

which reduces to $sat + sbt + sa(\mathbb{1} - as)bt = s + sa(\mathbb{1} - as) + t$. By $sat = t$ and $sbt = s$, this reduces to $sabt = sa$.

Now, it is simple to see that $t + \mathbb{1} - tb \in b\{1,3\}$. By hypothesis, we get $s + t + \mathbb{1} - tb = s(a+b)(t + \mathbb{1} - tb)$, which by using (5.1) leads to $\mathbb{1} - tb = sa - satb$, and therefore by (5.2), $\mathbb{1} = sa$. By $sabt = sa$ (this was proven in the previous paragraph), we get $bt = sa = \mathbb{1}$.

Case 2. $\beta = \{1,2\}$, $\gamma = \{1,4\}$.

As we did in the Case 1, we get $sat = t$, $sbt = s$, and $sabt = sa$.

We know that $b\{1,4\} = \{t + y - ybt : y \in \mathcal{R}\}$. By hypothesis, we get

$$s(a+b)(t + y - ybt) = s + t + y - ybt \quad \text{for all } y \in \mathcal{R},$$

which by using $sbt = s$ and $sat = t$ reduces to

$$say + sby - saybt - sbybt = y - ybt \quad \text{for all } y \in \mathcal{R}. \tag{5.3}$$

Choose $y = tb$ in (5.3) and use $sbt = s$, $sat = t$ to get $tb + sb - tb^2t - sb^2t = tb - tb^2t$, which obviously simplifies to $sb = sb^2t$.

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Choose $y = \mathbb{1}$ in (5.3) to get $sa + sb - sabt - sb^2t = \mathbb{1} - bt$. From $sabt = sa$ and $sb^2t = sb$ we get $bt = \mathbb{1}$.

Since $s \in a\{1,2\}$, it is simple to observe that $s+(\mathbb{1}-sa)uas \in a\{1,2\}$ for arbitrary $u \in \mathcal{R}$. Hence, by hypothesis,

$$[s + (\mathbb{1} - sa)uas]\,(a + b)t = s + (\mathbb{1} - sa)uas + t \quad \text{for all } u \in \mathcal{R}.$$

Use $a = asa$, $sat = t$, and $bt = \mathbb{1}$ to get

$$(\mathbb{1} - sa)uat = 0 \quad \text{for all } u \in \mathcal{R}.$$

Choose $u = bs$ and use $sat = t$, $bt = \mathbb{1}$ to get $0 = (\mathbb{1} - sa)bsat = \mathbb{1} - sa$.

Case 3. $\beta = \{1,3\}, \gamma = \{1,2\}$.

It is simple to observe that $x \in a\{1,3\} \Leftrightarrow x^* \in a^*\{1,4\}$ and $y \in b\{1,2\} \Leftrightarrow y^* \in b^*\{1,2\}$. Therefore, this case reduces to the Case 2 exchanging $a \leftrightarrow b^*$ and $b \leftrightarrow a^*$ and recalling $\mathbb{1}^* = \mathbb{1}$.

Case 4. $\beta = \{1,3\}, \gamma = \{1,4\}$.

By Theorem 4.9, $a\{1,3\} = \{s + z - saz : z \in \mathcal{R}\}$. By hypothesis, we have

$$(s + z - saz)(a + b)t = s + z - saz + t \quad \text{for all } z \in \mathcal{R},$$

which by (5.1) reduces to

$$zat + zbt - sazat - sazbt = z - saz \quad \text{for all } z \in \mathcal{R}. \tag{5.4}$$

Choose $z = as$ in (5.4) and use $asa = a$ and (5.2) to obtain

$$as = sa^2t. \tag{5.5}$$

Choose $z = s$ in (5.4) and use $asa = a$ and (5.2) to obtain $sbt = s$. From this last equality and (5.1) we obtain $sat = t$.

Next, choose $z = bs$ in (5.4) and use $sat = t$ and $sbt = s$ to derive

$$bt = sabt. \tag{5.6}$$

Finally, when $z = 1$, from (5.4), we obtain $at + bt - sa^2t - sabt = \mathbb{1} - sa$. From (5.5) and (5.6), we have $sa = \mathbb{1}$.

Therefore, $s(a + b)b^- = s + b^-$ for any $b^- \in b\{1,4\}$, which (by $sa = \mathbb{1}$) is equivalent to $sbb^- = s$ for any $b^- \in b\{1,4\}$. Since $b\{1,4\} = \{t + u - ubt : u \in \mathcal{R}\}$ we have $sb(t+u-ubt) = s$ for any $u \in \mathcal{R}$, which in view of $sbt = s$ reduces to $sbu = sbubt$ for any $u \in \mathcal{R}$. Choose $u = ta$ and use $sbt = s$ and $sa = \mathbb{1}$ to get $\mathbb{1} = bt$.

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Case 5. $\beta = \{1,4\}, \gamma = \{1,2\}$.

This case reduces to the Case 1 exchanging $a \leftrightarrow b^*$ and $b \leftrightarrow a^*$.

Case 6. $\beta = \{1,4\}, \gamma = \{1,3\}$.

By Theorem 4.9, we know that $t + \mathbb{1} - tb \in b\{1,3\}$. By hypothesis, we have $s(a+b)(t+\mathbb{1}-tb) = s+t+\mathbb{1}-tb$, which in view of (5.1) and $b = btb$ reduces to $sa - satb = \mathbb{1} - tb$. Use (5.2) to obtain $\mathbb{1} = sa$.

Next, it is simple to see that $s + \mathbb{1} - as \in a\{1,4\}$. By hypothesis, we obtain $(s+\mathbb{1}-as)(a+b)t = s+\mathbb{1}-as+t$. From (5.1) and $asa = a$ we get $bt - asbt = \mathbb{1} - as$, which taking into account (5.2) reduces to $bt = \mathbb{1}$. □

It is noteworthy that when the involved two elements are Moore-Penrose invertible, the mixed absorption law takes the following form.

THEOREM 5.2. *Let $\mathcal{R}$ be a unitary ring with an involution and $a, b \in \mathcal{R}^\dagger$. The following conditions are equivalent:*

(i) $a^- + b^- = a^-(a+b)b^-$ *for any* $a^- \in a\{1,4\}$ *and* $b^- \in b\{1,3\}$.
(ii) $a^- + b^- = a^-(a+b)b^-$ *for any* $a^- \in a\{1,3\}$ *and* $b^- \in b\{1,4\}$.
(iii) $a^- + b^- = a^-(a+b)b^-$ *for any* $a^- \in a\{1,2\}$ *and* $b^- \in b\{1,3\}$.
(iv) $a^- + b^- = a^-(a+b)b^-$ *for any* $a^- \in a\{1,3\}$ *and* $b^- \in b\{1,2\}$.
(v) $a^- + b^- = a^-(a+b)b^-$ *for any* $a^- \in a\{1,2\}$ *and* $b^- \in b\{1,4\}$.
(vi) $a^- + b^- = a^-(a+b)b^-$ *for any* $a^- \in a\{1,4\}$ *and* $b^- \in b\{1,2\}$.
(vii) $a^\dagger a = \mathbb{1}$, $bb^\dagger = \mathbb{1}$.

Proof. Notice that $a, b \in \mathcal{R}^\dagger$. If we choose $s = a^\dagger$ and $t = b^\dagger$ in Theorem 5.1, then the proof follows immediately. □

Acknowledgment. The authors wish to thank the referee for his/her suggestions to improve the quality of this article. The first author is grateful to China Scholarship Council for giving him a purse for his further study in Universidad Politécnica de Valencia, Spain.

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Volume 30, pp. 827-842, December 2015

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