Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

ELA

MAXIMA OF THE SIGNLESS LAPLACIAN SPECTRAL RADIUS FOR PLANAR GRAPHS*

GUANGLONG YU†, JIANYONG WANG‡, AND SHU-GUANG GUO†

Abstract. The signless Laplacian spectral radius of a graph is the largest eigenvalue of its signless Laplacian. In this paper, it is proved that the graph $K_2 \vee P_{n-2}$ has the maximal signless Laplacian spectral radius among all planar graphs of order $n \geq 456$.

Key words. Signless Laplacian, Spectral radius, Planar graph.

AMS subject classifications. 05C50.

1. Introduction. Recently, the signless Laplacian has attracted the attention of researchers (see [3–6, 9]). Some results on the signless Laplacian spectrum have been reported since 2005 and a new spectral theory called Q-theory is being developed by many researchers. Simultaneously, the application of the Q-theory has been extensively explored [7, 8, 16].

Schwenk and Wilson initiated the study of the eigenvalues of planar graphs [12]. In [2], D. Cao and A. Vince conjectured that $K_2 \vee P_{n-2}$ has the maximum spectral radius among all planar graphs of order n, where $\vee$ denotes the *join* of two graphs obtained from the union of these two graphs by joining each vertex of the first graph to each vertex of the second graph. The conjecture is still open. With the development of the Q-theory, a natural question is: What about the maximum signless Laplacian spectral radius of planar graphs? By some comparisons in [9], it seems plausible that $K_2 \vee P_{n-2}$ also has the maximal signless Laplacian spectral radius among planar graphs. In this paper, we confirm that among planar graphs with order $n \geq 456$, $K_2 \vee P_{n-2}$ has the maximal signless Laplacian spectral radius.

The layout of this paper is as follows. Section 2 gives some notations and some needed lemmas. In Section 3, our results are presented.

*Received by the editors November 25, 2014. Accepted for publication on September 15, 2015. Handling Editor: Bryan L. Shader.

†Department of Mathematics, Yancheng Teachers University, Yancheng, 224002, Jiangsu, P.R. China (yglong01@163.com). Supported by NSFC (11271315, 11171290, 11201417), Jiangsu Qing Lan Project (2014A), and JSNSFC (BK20151295).

‡Department of Mathematics, Linyi University, Linyi, 276005, Shandong, P.R. China.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

2. Preliminaries. All graphs considered in this paper are undirected and simple, i.e., no loops or multiple edges are allowed. Denote by $G = G[V(G), E(G)]$ a graph with vertex set $V(G)$ and edge set $E(G)$. The number of vertices, resp., edges, of G is denoted by $n = |V(G)|$, resp., $m(G) = |E(G)|$. Recall that given a graph G, $Q(G) = D(G) + A(G)$ is the *signless Laplacian matrix* of G, where $D(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ with $d_i = d_G(v_i)$ being the degree of vertex v_i $(1 \leq i \leq n)$, and $A(G)$ being the adjacency matrix of G. The signless Laplacian spectral radius of G, denoted by $q(G)$, is the largest eigenvalue of $Q(G)$. For a connected graph G of order n, the *Perron eigenvector* of $Q(G)$ is the unit (with respect to the Euclidean norm) positive eigenvector corresponding to $q(G)$; the *standard eigenvector* of $Q(G)$ is the positive eigenvector $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ corresponding to $q(G)$ satisfying $\displaystyle\sum_{i=1}^{n} x_i = 1$.

Denote by K_n, C_n, P_n a *complete graph*, a *cycle* and a *path* of order n, respectively. For a graph G, if there is no ambiguity, we use $d(v)$ instead of $d_G(v)$, use δ or $\delta(G)$ to denote the minimum vertex degree, use Δ or $\Delta(G)$ to denote the largest vertex degree, and use $\Delta^{'}$ or $\Delta^{'}(G)$ to denote the second largest vertex degree. In a graph, the notation $v_i \sim v_j$ denotes that vertex v_i is adjacent to v_j. Denote by $K_{s,t}$ a complete bipartite graph with one part of size s and another part of size t. In a graph G, for a vertex $u \in V(G)$, let $N_G(u)$ denote the neighbor set of u, and let $N_G[u] = \{u\} \cup N_G(u)$. $G(u) = G[N_G[u]]$, $G^{\circ}(u) = G[N_G(u)]$ denote the subgraphs induced by $N_G[u]$, $N_G(u)$, respectively.

The reader is referred to [1, 10] for the facts about planar and outer-planar graphs. A graph which can be drawn in the plane in such a way that edges meet only at points corresponding to their common ends is called a planar graph, and such a drawing is called a planar embedding of the graph. A simple planar graph is (edge) *maximal* if no edge can be added to the graph without violating planarity. In the planar embedding of a maximal planar graph G of order $n \geq 3$, each face is triangle. For a planar graph G of order $n \geq 3$, we have $m(G) \leq 3n - 6$ with equality if and only if it is maximal. In a maximal planar graph G of order $n \geq 4$, $\delta(G) \geq 3$. A graph G is *outer-planar* if it has a planar embedding, called *standard embedding*, in which all vertices lie on the boundary of its outer face. A simple outer-planar graph is (edge) *maximal* if no edge can be added to the graph without violating outer-planarity. In a standard embedding of a maximal outer-planar graph G of order $n \geq 3$, the boundary of the outer face is a Hamiltonian cycle (a cycle contains all vertices) of G, and each of the other faces is triangle. For an outer-planar graph G, we have $m(G) \leq 2n - 3$ with equality if and only if it is maximal. In a maximal planar graph G of order $n \geq 4$ and for a vertex $u \in V(G)$, we have that $G^{\circ}(u)$ is an outer-planar graph, and $G(u) = u \vee G^{\circ}(u)$. From a nonmaximal planar graph G, by inserting edges to G, a maximal planar graph $G^{'}$ can be obtained. From spectral graph theory, for a graph

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

ELA

G, it is known that $q(G+e) > q(G)$ if $e \notin E(G)$. Consequently, when we consider the maxima of the signless Laplacian spectral radius among planar graphs, it suffices to consider the maximal planar graphs directly. Note that for $n = 1, 2, 3, 4$, a maximal planar graph G of order n is isomorphic to K_n. Their signless Laplacian spectral radii can be easily determined by some computations. As a result, to consider the maxima of the signless Laplacian spectral radius among planar graphs of order n, we pay more attentions to those of order $n \geq 5$.

Next we introduce some needed lemmas.

Lemma 2.1. [13] *Let u be a vertex of a maximal outer-planar graph on $n \geq 2$ vertices. Then* $\sum_{v\sim u} d(v) \leq n + 3d(u) - 4$.

Lemma 2.2. [11] *Let G be a graph. Then*

$$q(G) \leq \max_{u\in V(G)} \left\{ d_G(u) + \frac{1}{d_G(u)} \sum_{v\sim u} d_G(v) \right\}.$$

Lemma 2.3. [5] *Let G be a connected graph containing at least one edge. Then* $q(G) \geq \Delta + 1$ *with equality if and only if* $G \cong K_{1,n-1}$.

3. Main results.

Lemma 3.1. *Let G be a maximal planar graph of order $n \geq 3$. Then*

$$q(G) \leq \max_{u\in V(G)} \left\{ d_G(u) + 2 + \frac{3n-9}{d_G(u)} \right\}.$$

Proof. Let $u \in V(G)$, $N_G(u) = \{v_1, v_2, \ldots, v_t\}$, and $V_1 = V(G)\backslash N_G[u]$. For $1 \leq i \leq t$, let $\alpha_i = d_{G^\circ(u)}(v_i)$. Note that $m(G^\circ(u)) = |E(G^\circ(u))| = \frac{1}{2}\sum_{i=1}^{t} \alpha_i$. Between $N_G(u)$ and V_1, there are $3n - 6 - \frac{1}{2}\sum_{i=1}^{t}\alpha_i - d_G(u)$ edges. Consequently,

$$\sum_{v\sim u} d_G(v) = d_G(u) + \left[3n - 6 - \frac{1}{2}\sum_{i=1}^{t}\alpha_i - d_G(u)\right] + \sum_{i=1}^{t}\alpha_i = 3n - 6 + \frac{1}{2}\sum_{i=1}^{t}\alpha_i.$$

Since $G^\circ(u)$ is an outer-planar graph, $m(G^\circ(u)) \leq 2d_G(u) - 3$. As a result, $\sum_{v\sim u} d_G(v) \leq 3n - 9 + 2d_G(u)$, and thus,

$$d_G(u) + \frac{1}{d_G(u)}\sum_{v\sim u} d_G(v) \leq d_G(u) + 2 + \frac{3n-9}{d_G(u)}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

By Lemma 2.2, $q(G) \leq \max_{u \in V(G)} \left\{ d_G(u) + 2 + \frac{3n-9}{d_G(u)} \right\}$. □

Remark 1. Let $f(x) = x + 2 + \frac{3n-9}{x}$. It can be checked that $f(x)$ is convex when $n \geq 4$. Let G be a maximal planar graph of order n with largest degree $\Delta(G)$. As discussed in section 2, we know that if $n \geq 4$, then $d_G(u) \geq 3$ for any vertex $u \in V(G)$. Thus,

$$d_G(u) + \frac{1}{d_G(u)} \sum_{v \sim u} d_G(v) \leq \max \left\{ 5 + \frac{3n-9}{3}, \Delta(G) + 2 + \frac{3n-9}{\Delta(G)} \right\}.$$

Moreover, if $n \geq 6$ and $\Delta(G) \leq n-3$, then $q(G) \leq n+2$.

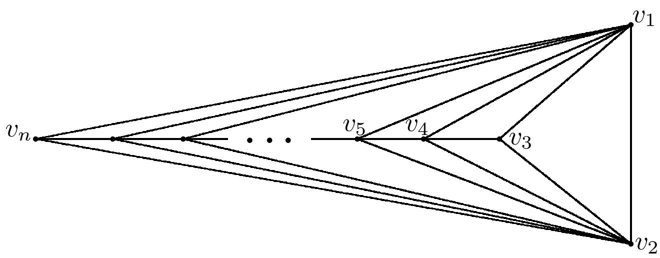

Fig. 3.1. $\mathcal{H}_n$.

Let $\mathcal{H}_1 = K_1$, $\mathcal{H}_2 = K_2$, and $\mathcal{H}_n = k_2 \vee P_{n-2}$ for $n \geq 3$ (see Fig. 3.1).

Lemma 3.2. *If $n \geq 5$, then $q(\mathcal{H}_n) > n+2$.*

Proof. Let $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ be the standard eigenvector of $Q(\mathcal{H}_n)$. By symmetry, $x_1 = x_2$, $x_3 = x_n$. By Lemma 2.3, $q(\mathcal{H}_n) \geq n$.

Note that $q(\mathcal{H}_n)x_1 = (n-1)x_1 + x_2 + \sum_{i=3}^{n} x_i = (n-2)x_1 + 1$. Thus,

$$x_1 = \frac{1}{q(\mathcal{H}_n) - n + 2}. \tag{3.1}$$

Note that $q(\mathcal{H}_n) \sum_{i=3}^{n} x_i = 6 \sum_{i=3}^{n} x_n + 2(n-2)x_1 - 2(x_3 + x_n)$. Thus,

$$\sum_{i=3}^{n} x_i = \frac{2(n-2)x_1 - 4x_3}{q(\mathcal{H}_n) - 6} \quad \text{and} \quad 1 = \sum_{i=1}^{n} x_i = \frac{2(n-2)x_1 - 4x_3}{q(\mathcal{H}_n) - 6} + x_1 + x_2.$$

As a result,

$$x_3 = \frac{2(n-2) - (q(\mathcal{H}_n) - n)(q(\mathcal{H}_n) - 6)}{4(q(\mathcal{H}_n) - n + 2)}. \tag{3.2}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

Note that $q(\mathcal{H}_n)x_3 = 3x_3 + x_1 + x_2 + x_4$. Then

$$(q(\mathcal{H}_n) - 2)(x_1 - x_3) = (n-4)x_1 + \sum_{i=5}^{n} x_i.$$

The fact $n \geq 5$ implies that $x_1 > x_3$. Combining with (3.1) and (3.2), we get

$$\frac{2(n-2) - (q(\mathcal{H}_n) - n)(q(\mathcal{H}_n) - 6)}{4(q(\mathcal{H}_n) - n + 2)} < \frac{1}{q(\mathcal{H}_n) - n + 2}. \tag{3.3}$$

Simplifying (3.3), we get $q^2(\mathcal{H}_n) - (6+n)q(\mathcal{H}_n) + 4n + 8 > 0$. It follows that $q(\mathcal{H}_n) > n+2$. $\square$

From Remark 1 and Lemma 3.2, we see that to consider the maxima of the signless Laplacian spectral radius among planar graphs of order $n \geq 5$, it suffices to consider those with maximum degree $n-1$ or $n-2$.

Lemma 3.3. [14] *Let A be an irreducible nonnegative square real matrix of order n and spectral radius ρ. If there exists a nonnegative real vector $y \neq 0$ and a real coefficient polynomial function f such that $f(A)y \leq ry$ $(r \in \mathbb{R})$, then $f(\rho) \leq r$.*

Lemma 3.4. *Let $1 \leq k \leq 12$ be an integer number, and let G be a maximal planar graph of order $n \geq 115$, where $d_G(v_1) = \Delta(G) = n-2$, for $i = 2, 3, \ldots, k+1$, $\frac{n}{6} + 1 \leq d_G(v_i) \leq n - 61$, and for $k+2 \leq i \leq n$, $d_G(v_i) < \frac{n}{6} + 1$. Then $q(G) \leq n-2$.*

Proof. Let $X = (x_1, x_2, x_3, \ldots, x_n)^T \in \mathbb{R}^n$ be a positive vector, where

$$x_i = \begin{cases} 1, & i = 1; \\ \frac{1}{k}, & 2 \leq i \leq k+1; \\ \frac{3}{n-k-1}, & k+2 \leq i \leq n. \end{cases}$$

For v_1, we have

$$\frac{(n-2)x_1 + \sum_{v_j \sim v_1} x_j}{x_1} \leq n - 2 + 1 + \frac{3(n-k-2)}{n-k-1} < n+2.$$

For v_i $(k+2 \leq i \leq n)$, we have

$$\frac{d_G(v_i)x_i + \sum_{v_j \sim v_i} x_j}{x_i} \leq$$

$$\begin{cases} d_G(v_i) + \frac{\sum_{j=1}^{k+1} x_j + \frac{3(d_G(v_i)-k-1)}{n-k-1}}{\frac{3}{n-k-1}} \leq n+2, & d(v_i) \geq k+1; \\ d_G(v_i) + \frac{\sum_{j=1}^{d_G(v_i)} x_j}{\frac{3}{n-k-1}} \leq d_G(v_i) + \frac{1 + \frac{d_G(v_i)-1}{k}}{\frac{3}{n-k-1}} < n+2, & d_G(v_i) \leq k. \end{cases}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

ELA

For v_i $(2 \le i \le k+1)$, since $n \ge 115$ and $1 \le k \le 12$, we have $d_G(v_i) > k$. Thus,

$$\begin{aligned}\frac{d_G(v_i)x_i + \sum_{v_j \sim v_i} x_j}{x_i} &= \frac{(d_G(v_i)-1)x_i + x_i + \sum_{v_j \sim v_i} x_j}{x_i} \\ &\le d_G(v_i) - 1 + \frac{\sum_{j=1}^{k+1} x_j + \frac{3(d_G(v_i)-k)}{n-k-1}}{\frac{1}{k}} \\ &= \left(1 + \frac{3k}{n-k-1}\right) d_G(v_i) - \frac{3k^2}{n-k-1} + 2k - 1. \end{aligned} \tag{3.4}$$

Let $f(k) = \left(1 + \frac{3k}{n-k-1}\right) d_G(v_i) - \frac{3k^2}{n-k-1} + 2k - 1$. Taking derivation of $f(k)$ with respect to k, we get

$$f^{'}(k) = \frac{(n-k-1)(2n-2+3d_G(v_i)-8k) + 3kd_G(v_i) - 3k^2}{(n-k-1)^2}.$$

Since $d_G(v_i) \ge k$ and $n \ge 115$, we get $f^{'}(k) > 0$. This implies that $f(k)$ is monotone increasing with respect to k. Since $n \ge 115$, $k \le 12$, and $d_G(v_i) \le n-61$, we conclude that

$$\left(1 + \frac{3k}{n-k-1}\right) d_G(v_i) - \frac{3k^2}{n-k-1} + 2k - 1 < n + 2.$$

Thus, from (3.4), we get $\frac{d_G(v_i)x_i + \sum_{v_j \sim v_i} x_j}{x_i} < n + 2$.

By the above discussion, we get $Q(G)X \le (n+2)X$. The proof is now completed by applying Lemma 3.3. $\square$

Lemma 3.5. *Let G be a maximal planar graph of order $n \ge 380$ with $d_G(v_1) = \Delta(G) = n-2$, and $\Delta^{'}(G) \ge n-62$. Then $q(G) \le n+2$.*

Proof. Suppose $d_G(v_2) = \Delta^{'}(G)$. Let $X = (x_1, x_2, x_3, \ldots, x_n)^T \in \mathbb{R}^n$ be a positive vector, where

$$x_i = \begin{cases} 1, & i = 1; \\ 1, & i = 2; \\ \frac{3}{n-2}, & 3 \le i \le n. \end{cases}$$

For v_1, we have

$$\frac{(n-2)x_1 + \sum_{v_j \sim v_1} x_j}{x_1} \le n - 2 + 1 + \frac{3(n-3)}{n-2} < n + 2.$$

Next, there are two cases to consider.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

ELA

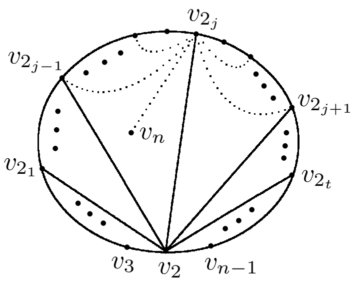

Fig. 3.2. $d_G(v_{2_j})$.

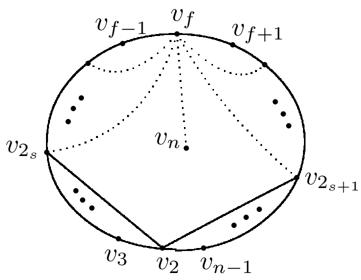

Fig. 3.3. $d_G(v_f)$.

Case 1. $v_2 \in N_G(v_1)$. Suppose $N_G(v_1) = \{v_2, v_3, \ldots, v_{n-2}, v_{n-1}\}$. Without loss of generality suppose that v_1 is in the outer face of $G^{\circ}_{v_1}$. Then v_n is in one of the inner faces of $G^{\circ}_{v_1}$ (see Fig. 3.2).

For v_2, since $d_G(v_2) \leq n-2$, we have

$$\frac{d_G(v_2)x_2 + \sum_{v_j \sim v_2} x_j}{x_2} \leq d_G(v_2) + 1 + \frac{3(d_G(v_2)-1)}{n-2} \leq n+2-\frac{3}{n-2}.$$

Denote by $C_{v_1} = v_2v_3\cdots v_{n-2}v_{n-1}v_2$ the Hamiltonian cycle in $G^{\circ}_{v_1}$. Suppose that $v_i s$ $(2 \leq i \leq n-1)$ are distributed along the clockwise direction on C_{v_1} and suppose $N_{G^{\circ}_{v_1}}(v_2) = \{v_{2_1}, v_{2_2}, \ldots, v_{2_t}\}$, where for $1 \leq i \leq t-1$, $2_i < 2_{i+1}$, $v_{2_1} = v_3$, $v_{2_t} = v_{n-1}$ (see Fig. 3.2). For $1 \leq j \leq t$, suppose there are l_{j-1} vertices between $v_{2_{j-1}}$ and v_{2_j} along the clockwise direction on C_{v_1}, where if $j = 1$, we let $v_{2_0} = v_2$. Along the clockwise direction on C_{v_1}, suppose there are l_t vertices between v_{2_t} and v_2.

For each v_{2_j} $(1 \leq j \leq t$, see Fig. 3.2), noting that $l_{j-1} + l_j \leq n-3-d_G(v_2)$ and $d_G(v_2) \geq n-62$, we have

$$d_G(v_{2_j}) \leq l_{j-1} + l_j + 5 \leq n+2-d_G(v_2) \leq 64,$$

and

$$\frac{d_G(v_{2_j})x_{2_j} + \sum_{v_k \sim v_{2_j}} x_k}{x_{2_j}} \leq d_G(v_{2_j}) + \frac{2 + \frac{3(d_G(v_{2_j})-2)}{n-2}}{\frac{3}{n-2}} \leq n+2.$$

For each $v_f \in (N_G(v_1) \backslash \{v_2, v_{2_1}, v_{2_2}, \ldots, v_{2_t}\})$, then along the clockwise direction on C_{v_1}, there exists $0 \leq s \leq t$ such that v_f is between v_{2_s} and $v_{2_{s+1}}$, where $v_{2_{t+1}} = v_2$ (see Fig. 3.3). Note that $l_s \leq n-3-d_G(v_2)$. Then $d_G(v_f) \leq l_s + 3 \leq n - d_G(v_2) \leq 62$, and thus,

$$\frac{d_G(v_f)x_f + \sum_{v_k \sim v_f} x_k}{x_f} \leq d_G(v_f) + \frac{2 + \frac{3(d_G(v_f)-2)}{n-2}}{\frac{3}{n-2}} \leq n+2.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

Note that v_n is in one of the inner faces of $G^{\circ}_{v_1}$. Suppose that in $G^{\circ}_{v_1}$, v_n is in a face $v_2v_{2_z}v_{2_z+1}v_{2_z+2}\cdots v_{2_{z+1}}v_2$ (see Fig. 3.4). Note that $l_z \le n-3-d_G(v_2)$ and $d_G(v_n) \le l_z+3$. Then $d_G(v_n) \le n-d_G(v_2) \le 62$, and

$$\frac{d_G(v_n)x_n + \sum_{v_k\sim v_n} x_k}{x_n} \le d_G(v_n) + \frac{1+\frac{3(d_G(v_n)-1)}{n-2}}{\frac{3}{n-2}} < n+2.$$

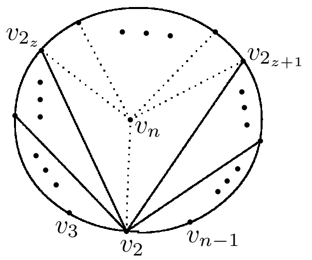

Fig. 3.4. $d_G(v_n)$.

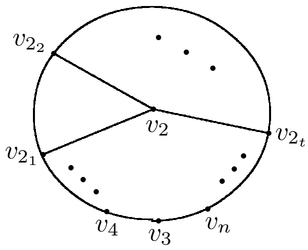

Fig. 3.5. $d_G(v_2)$.

Case 2. $v_2 \notin N_G(v_1)$. Without loss of generality suppose that v_1 is in the outer face of $G^{\circ}_{v_1}$. Then v_2 is in one of the inner faces of $G^{\circ}_{v_1}$. Then $N_G(v_1) = \{v_3, v_4, v_5, \dots, v_{n-1}, v_n\}$. Suppose that $C_{v_1} = v_3v_4\cdots v_{n-1}v_nv_3$ is the Hamiltonian cycle in $G^{\circ}_{v_1}$, $v_i s$ $(3 \le i \le n)$ are distributed along the clockwise direction on C_{v_1}, and suppose $N_{G^{\circ}_{v_1}}(v_2) = \{v_{2_1}, v_{2_2}, \dots, v_{2_t}\}$, where for $1 \le i \le t-1$, $2_i < 2_{i+1}$ (see Fig. 3.5). For $2 \le j \le t$, along the clockwise direction on C_{v_1}, suppose there are l_{j-1} vertices between $v_{2_{j-1}}$ and v_{2_j}. Along the clockwise direction on C_{v_1}, suppose that there are l_t vertices between v_{2_t} and v_{2_1}.

For each v_{2_j} $(1 \le j \le t)$, by an argument similar to Case 1, we have $d_G(v_{2_j}) \le 64$, and

$$\frac{d_G(v_{2_j})x_{2_j} + \sum_{v_k\sim v_{2_j}} x_k}{x_{2_j}} \le d_G(v_{2_j}) + \frac{2+\frac{3(d_G(v_{2_j})-2)}{n-2}}{\frac{3}{n-2}} \le n+2.$$

By an argument similar to Case 1, for each $v_i \in (N_G(v_1)\backslash\{v_{2_1}, v_{2_2}, \dots, v_{2_t}\})$, we have $d_G(v_i) \le n-d_G(v_2) \le 62$, and

$$\frac{d_G(v_i)x_i + \sum_{v_k\sim v_i} x_k}{x_i} \le d_G(v_i) + \frac{1+\frac{3(d_G(v_i)-1)}{n-2}}{\frac{3}{n-2}} < n+2.$$

For v_2, since $d_G(v_2) \le n-2$, we have

$$\frac{d_G(v_2)x_2 + \sum_{v_k\sim v_2} x_k}{x_2} \le d_G(v_2) + \frac{3d_G(v_2)}{n-2} \le n+1.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

By the above discussion, we get $Q(G)X \leq (n+2)X$. The proof is now completed by applying Lemma 3.3. $\square$

LEMMA 3.6. *Let G be a maximal planar graph of order $n \geq 4$, where $d(v_1) = \Delta(G) = n-2$, and for $2 \leq i \leq n$, $d_G(v_i) < 1 + \frac{n}{6}$. Then $q(G) \leq n-2$.*

Proof. Let $X = (x_1, x_2, x_3, \ldots, x_n)^T \in \mathbb{R}^n$ be a positive vector, where

$$x_i = \begin{cases} 1, & i = 1; \\ \\ \frac{4}{n-1}, & 2 \leq i \leq n. \end{cases}$$

For v_1, we have

$$\frac{(n-2)x_1 + \sum_{v_j \sim v_1} x_j}{x_1} \leq n - 2 + \frac{4(n-2)}{n-1} < n+2.$$

For v_i $(2 \leq i \leq n)$, we have

$$\frac{d_G(v_i)x_i + \sum_{v_j \sim v_i} x_j}{x_i} \leq d_G(v_i) + \frac{1 + \frac{4(d_G(v_i)-1)}{n-1}}{\frac{4}{n-1}} < n+2.$$

By the above discussion, we get $Q(G)X \leq (n+2)X$. Applying Lemma 3.3 completes the proof. $\square$

THEOREM 3.7. *Let G be a maximal planar graph of order $n \geq 380$ with $\Delta(G) = n-2$. Then $q(G) \leq n+2$.*

Proof. This theorem follows from Lemmas 3.4–3.6. $\square$

LEMMA 3.8. *Let $1 \leq k \leq 13$ be an integer number, and let G be a maximal planar graph of order $n \geq 91$, where $d_G(v_1) = \Delta(G) = n-1$, for $i = 2, 3, \ldots, k+1$, $\frac{n}{7} + \frac{19}{7} \leq d(v_i) \leq n - 75$, and for $k+2 \leq i \leq n$, $d_G(v_i) < \frac{n}{7} + \frac{19}{7}$. Then $q(G) \leq n+2$.*

Proof. Let $X = (x_1, x_2, x_3, \ldots, x_n)^T \in \mathbb{R}^n$ be a positive vector, where

$$x_i = \begin{cases} 1, & i = 1; \\ \\ \frac{2}{3k}, & 2 \leq i \leq k+1; \\ \\ \frac{7}{3(n-k-1)}, & k+2 \leq i \leq n. \end{cases}$$

For v_1, we have $\dfrac{(n-1)x_1 + \sum_{v_j \sim v_1} x_j}{x_1} = n+2.$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

For v_i $(k+2 \leq i \leq n)$, we have

$$\frac{d_G(v_i)x_i + \sum_{v_j\sim v_i} x_j}{x_i} \leq \begin{cases} d_G(v_i) + \frac{\frac{5}{3}+\frac{7(d_G(v_i)-k-1)}{3(n-k-1)}}{\frac{7}{3(n-k-1)}} \leq n+2, & d_G(v_i) \geq k+1; \\ d_G(v_i) + \frac{\frac{5}{3}}{\frac{7}{3(n-k-1)}} < n+2, & d_G(v_i) \leq k. \end{cases}$$

For v_i $(2 \leq i \leq k+1)$, since $n \geq 91$ and $1 \leq k \leq 13$, we have $d_G(v_i) > k$. Thus,

$$\begin{aligned} \frac{d_G(v_i)x_i + \sum_{v_j\sim v_i} x_j}{x_i} &= \frac{(d_G(v_i)-1)x_i + x_i + \sum_{v_j\sim v_i} x_j}{x_i} \\ &\leq d_G(v_i) - 1 + \frac{\sum_{j=1}^{k+1} x_j + \frac{7(d_G(v_i)-k)}{3(n-k-1)}}{\frac{2}{3k}} \\ &= d_G(v_i) - 1 + \frac{\frac{5}{3} + \frac{7(d_G(v_i)-k)}{3(n-k-1)}}{\frac{2}{3k}} \\ &= d_G(v_i) - 1 + \frac{5}{2}k + \frac{\frac{7}{2}k(d_G(v_i)-k)}{n-k-1}. \end{aligned} \tag{3.5}$$

As the proof of Lemma 3.4, since $n \geq 91$ and $k < d_G(v_i) \leq n-75$, we can prove that

$$d_G(v_i) - 1 + \frac{5}{2}k + \frac{\frac{7}{2}k(d_G(v_i)-k)}{n-k-1} \leq n+2.$$

Thus, (3.5) implies $\dfrac{d_G(v_i)x_i + \sum_{v_j\sim v_i} x_j}{x_i} \leq n+2$.

By the above discussion, we get $Q(G)X \leq (n+2)X$. Applying Lemma 3.3 completes the proof. □

LEMMA 3.9. *Let G be a maximal planar graph of order $n \geq 6$, where $d(v_1) = \Delta(G) = n-1$, and for $2 \leq i \leq n$, $d_G(v_i) < \frac{n}{7} + \frac{19}{7}$. Then $q(G) \leq n+2$.*

Proof. Let $X = (x_1, x_2, x_3, \ldots, x_n)^T \in \mathbb{R}^n$ be a positive vector, where

$$x_i = \begin{cases} 1, & i = 1; \\ \frac{3}{n-1}, & 2 \leq i \leq n. \end{cases}$$

For v_1, we have

$$\frac{(n-1)x_1 + \sum_{v_j\sim v_1} x_j}{x_1} = n+2.$$

For v_i $(2 \leq i \leq n)$, we have

$$\frac{d_G(v_i)x_i + \sum_{v_j\sim v_i} x_j}{x_i} \leq d_G(v_i) + \frac{1 + \frac{3(d_G(v_i)-1)}{n-1}}{\frac{3}{n-1}} \leq n+2.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

By the above discussion, we get $Q(G)X \leq (n+2)X$. The proof is now completed by applying Lemma 3.3. $\square$

LEMMA 3.10. *Let G be a maximal planar graph of order $n \geq 461$ with $d_G(v_1) = \Delta(G) = n-1$ and $n-81 \leq \Delta^{'}(G) \leq n-4$. Then $q(G) \leq n+2$.*

Proof. Without loss of generality suppose that $d_G(v_2) = \Delta^{'}(G)$. Let $X = (x_1, x_2, x_3, \dots, x_n)^T \in \mathbb{R}^n$ be a positive vector, where

$$x_i = \begin{cases} 1, & i = 1; \\ \frac{4}{7}, & i = 2; \\ \frac{17}{7(n-2)}, & 3 \leq i \leq n. \end{cases}$$

For v_1, we have

$$\frac{(n-1)x_1 + \sum_{v_j \sim v_1} x_j}{x_1} = n+2.$$

For v_2, since $d_G(v_2) \leq n-4$, we have

$$\frac{d_G(v_2)x_2 + \sum_{v_j \sim v_2} x_j}{x_2} \leq d_G(v_2) + \frac{1 + \frac{17(d_G(v_2)-1)}{7(n-2)}}{\frac{4}{7}} < n+2.$$

Without loss of generality suppose that v_1 is in the outer face of $G^{\circ}_{v_1}$, $C_{v_1} = v_2 v_3 \cdots v_{n-1} v_n v_2$ is the Hamiltonian cycle in $G^{\circ}_{v_1}$, $v_i s$ $(2 \leq i \leq n)$ are distributed along the clockwise direction on C_{v_1}, and suppose $N_{G^{\circ}_{v_1}}(v_2) = \{v_{2_1}, v_{2_2}, \dots, v_{2_t}\}$, where for $1 \leq i \leq t-1$, $2_i < 2_{i+1}$, $v_{2_1} = v_3$, $v_{2_t} = v_n$. On C_{v_1}, along the clockwise direction, for $1 \leq j \leq t$, suppose that there are l_{j-1} vertices between $v_{2_{j-1}}$ and v_{2_j}, where if $j = 1$, we let $v_{2_0} = v_2$. Along the clockwise direction on C_{v_1}, suppose that there are l_t vertices between v_{2_t} and v_2.

For each v_{2_j} $(1 \leq j \leq t)$, noting that $l_{j-1} + l_j \leq n-2-d_G(v_2)$ and $d_G(v_2) \geq n-81$, we have

$$d_G(v_{2_j}) \leq l_{j-1} + l_j + 4 \leq n+2-d_G(v_2) \leq 83$$

and

$$\frac{d_G(v_{2_j})x_{2_j} + \sum_{v_k \sim v_{2_j}} x_k}{x_{2_j}} \leq d_G(v_{2_j}) + \frac{\frac{11}{7} + \frac{17(d_G(v_{2_j})-2)}{7(n-2)}}{\frac{17}{7(n-2)}} \leq n+2.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

For each $v_f \in (N_G(v_1)\backslash\{v_2, v_{2_1}, v_{2_2}, \ldots, v_{2_t}\})$, along the clockwise direction, there exists $0 \leq s \leq t$ such that v_f is between v_{2_s} and $v_{2_{s+1}}$ on C_{v_1}. Then

$$d_G(v_f) \leq l_s + 2 \leq n - d_G(v_2) \leq 81$$

and

$$\frac{d_G(v_f)x_f + \sum_{v_k \sim v_f} x_k}{x_f} \leq d_G(v_f) + \frac{\frac{11}{7} + \frac{17(d_G(v_f)-2)}{7(n-2)}}{\frac{17}{7(n-2)}} \leq n + 2.$$

By the above discussion, we get $Q(G)X \leq (n+2)X$. Applying Lemma 3.3 completes the proof. □

LEMMA 3.11. *Let G be a maximal planar graph of order $n \geq 15$ with $d_G(v_1) = \Delta(G) = n-1$.*
(i) *If $\Delta^{'}(G) = n-2$, then $q(G) < q(\mathcal{H}_n)$;*
(ii) *If $\Delta^{'}(G) = n-3$, then $q(G) < q(\mathcal{H}_n)$.*

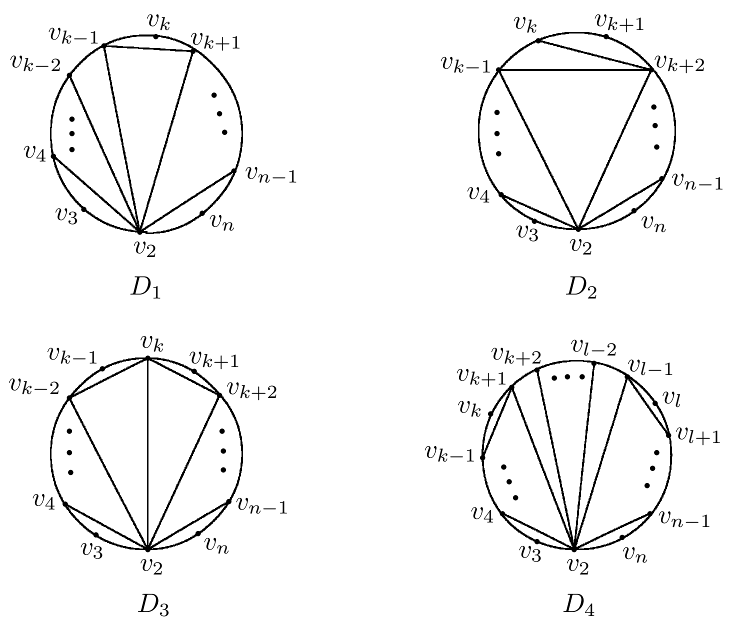

Fig. 3.6. $D_1 - D_4$.

Proof. Without loss of generality suppose that $d_G(v_2) = \Delta^{'}(G)$, v_1 is in the outer face of $G^{\circ}_{v_1}$, and suppose that $C_{v_1} = v_2v_3\cdots v_{n-1}v_nv_2$ is the Hamiltonian cycle in $G^{\circ}_{v_1}$ (see Fig. 3.6).

(i) $d_G(v_2) = n-2$ and $v_k \notin N_G(v_2)$ $(4 \leq k \leq n-1)$. Then $G^{\circ}_{v_1} \cong D_1$ (see Fig. 3.6). For convenience, we suppose $G^{\circ}_{v_1} = D_1$. By Lemma 2.3, we get that $q(G) > 15$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

Let $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ be the Perron eigenvector of $Q(G)$.

Note that

$$q(G)x_k = 3x_k + x_{k-1} + x_{k+1} + x_1, \tag{3.6}$$

$$q(G)x_2 = (n-2)x_2 + x_1 + \sum_{3\leq i\leq n, i\neq k} x_i. \tag{3.7}$$

Equations (3.6) and (3.7) imply that $(q(G)-3)(x_2 - x_k) = (n-5)x_2 + \sum_{3\leq i\leq k-2} x_i + \sum_{k+2\leq i\leq n} x_i$. Because $n \geq 15$, it follows immediately that $x_2 > x_k$.

Note that

$$q(G)x_{k-1} = 5x_{k-1} + x_1 + x_2 + x_{k-2} + x_k + x_{k+1},$$

$$q(G)x_{k+1} = 5x_{k+1} + x_1 + x_2 + x_{k-1} + x_k + x_{k+2}.$$

Thus,

$$q(G)(x_{k-1} + x_{k+1}) = 6(x_{k-1} + x_{k+1}) + 2(x_1 + x_k + x_2) + x_{k-2} + x_{k+2}. \tag{3.8}$$

From (3.6) and (3.7), we also get that

$$q(G)(x_2 + x_k) = (n-2)x_2 + 3x_k + 2x_1 + 2x_{k-1} + 2x_{k+1} + \sum_{3\leq i\leq k-2} x_i + \sum_{k+2\leq i\leq n} x_i. \tag{3.9}$$

From (3.8) and (3.9), we have

$$\begin{aligned} &(q(G)-4)[x_2 + x_k - (x_{k-1} + x_{k+1})] \\ &\quad = (n-11)x_2 + 3(x_2 - x_k) + \sum_{3\leq i\leq k-3} x_i + \sum_{k+3\leq i\leq n} x_i. \end{aligned} \tag{3.10}$$

The fact that $n \geq 15$ and (3.10) holding imply that $x_2 + x_k > x_{k-1} + x_{k+1}$.

Let $F = G - v_{k-1}v_{k+1} + v_2v_k$. Note the relation between the Rayleigh quotient and the largest eigenvalue of a non-negative real symmetric matrix, and note that

$$X^TQ(F)X - X^TQ(G)X = (x_2 + x_k)^2 - (x_{k-1} + x_{k+1})^2.$$

It follows that when $n \geq 15$, then $q(F) > X^TQ(F)X > X^TQ(G)X = q(G)$. Because $F \cong \mathcal{H}_n$, it follows that $q(\mathcal{H}_n) > q(G)$. Then (i) follows.

(ii) $d_G(v_2) = n-3$. Since v_1 is adjacent to v_2, among $v_2, v_3, \ldots, v_{n-1}, v_n$, there must be two vertices nonadjacent to v_2. Thus, there are three cases for G, that is,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

ELA

$G \cong D_2$, $G \cong D_3$ or $G \cong D_4$ (see Fig. 3.6). Without loss of generality suppose that in D_2, neither v_k nor v_{k+1} are adjacent to v_2; in D_3, neither v_{k-1} nor v_{k+1} are adjacent to v_2; in D_4, neither v_k nor v_l are adjacent to v_2. By Lemma 2.3, we know that $q(G) > 15$.

Case 1. $G \cong D_2$. For convenience, we suppose that $G = D_2$. Because $d_G(v_2) = n-3$, it follows that $4 \le k \le n-2$. Let $X = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ be the Perron eigenvector of $Q(G)$.

Note that

$$q(G)x_{k+1} = 3x_{k+1} + x_1 + x_k + x_{k+2}, \tag{3.11}$$

$$q(G)x_k = 4x_k + x_1 + x_{k-1} + x_{k+1} + x_{k+2}. \tag{3.12}$$

From (3.11) and (3.12), we get

$$(q(G)-2)(x_k - x_{k+1}) = x_k + x_{k-1}. \tag{3.13}$$

Since $n \ge 15$ and (3.13) holds, we conclude that $x_k > x_{k+1}$.

Note that

$$q(G)x_2 = (n-3)x_2 + x_1 + \sum_{3\le i\le k-1} x_i + \sum_{k+2\le i\le n} x_i. \tag{3.14}$$

From (3.12) and (3.14), we get

$$\begin{aligned} &q(G)(x_2 + x_k) \\ &= (n-3)x_2 + 2x_1 + 4x_k + 2x_{k-1} + x_{k+1} + 2x_{k+2} + \sum_{3\le i\le k-2} x_i + \sum_{k+3\le i\le n} x_i. \end{aligned} \tag{3.15}$$

Note that

$$q(G)x_{k-1} = 5x_{k-1} + x_1 + x_2 + x_{k-2} + x_k + x_{k+2}. \tag{3.16}$$

From (3.11) and (3.16), we get that

$$q(G)(x_{k-1} + x_{k+1}) = 5x_{k-1} + x_{k-2} + 2x_1 + x_2 + 2x_k + 3x_{k+1} + 2x_{k+2}. \tag{3.17}$$

From (3.12) and (3.17), we get that

$$(q(G)-2)(x_{k-1} + x_{k+1} - x_k) = x_1 + x_2 + x_{k-2} + 2x_{k-1} + x_{k+2} > 0. \tag{3.18}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

Since $n \geq 15$ and (3.18) holds, we conclude that $x_{k-1} + x_{k+1} > x_k$.

From (3.11) and (3.14), we get that

$$(3.19)\quad (q(G)-4)(x_2 - x_{k+1}) = (n-7)x_2 + x_{k-1} + x_{k+1} - x_k + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+3 \leq i \leq n} x_i > 0.$$

Since $n \geq 15$ and (3.19) holds, we conclude that $x_2 > x_{k+1}$.

From (3.12) and (3.14), we get that

$$(3.20)\quad (q(G)-4)(x_2 - x_k) = (n-8)x_2 + x_2 - x_{k+1} + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+3 \leq i \leq n} x_i > 0.$$

Since $n \geq 15$ and (3.20) holds, we conclude that $x_2 > x_k$.

From (3.14) and (3.16), we get that

$$(3.21)\quad (q(G)-4)(x_2 - x_{k-1}) = (n-9)x_2 + x_2 - x_k + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+3 \leq i \leq n} x_i.$$

Since $n \geq 15$ and (3.21) holds, we conclude that $x_2 > x_{k-1}$.

Note that

$$(3.22)\quad q(G)x_{k+2} = 6x_{k+2} + x_{k+1} + x_k + x_{k-1} + x_{k+3} + x_2 + x_1.$$

From (3.14) and (3.22), we get that

$$(3.23)\quad (q(G)-5)(x_2 - x_{k+2}) = (n-11)x_2 + x_2 - x_k + x_2 - x_{k+1} + \sum_{3 \leq i \leq k-2} x_i + \sum_{k+4 \leq i \leq n} x_i.$$

Since $n \geq 15$ and (3.23) holds, we conclude that $x_2 > x_{k+2}$.

From (3.16) and (3.22), we get that

$$(3.24)\quad q(G)(x_{k-1} + x_{k+2}) = 2x_1 + 2x_2 + x_{k-2} + 6x_{k-1} + 2x_k + x_{k+1} + 7x_{k+2} + x_{k+3}.$$

From (3.15) and (3.24), we get that

$$\begin{aligned} & q(G)(x_2 + x_k) - q(G)(x_{k-1} + x_{k+2}) \\ (3.25)\quad & = (n-14)x_2 + 4x_2 - 4x_{k-1} + 2x_k + 5x_2 - 5x_{k+2} + \sum_{3 \leq i \leq k-3} x_i + \sum_{k+4 \leq i \leq n} x_i. \end{aligned}$$

Since $n \geq 15$ and (3.25) holds, we conclude that $x_2 + x_k > x_{k-1} + x_{k+2}$.

Let $\mathbb{F} = G - v_{k-1}v_{k+2} + v_2v_k$. Note that

$$X^T Q(\mathbb{F})X - X^T Q(G)X = (x_2 + x_k)^2 - (x_{k-1} + x_{k+2})^2.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

It follows that when $n \geq 15$, then $q(\mathbb{F}) > X^T Q(\mathbb{F})X > X^T Q(G)X = q(G)$. By (i), it follows immediately that $q(\mathcal{H}_n) > q(\mathbb{F}) > q(G)$.

Case 2. $G \cong D_3$. For convenience, we suppose that $G = D_3$. Because $d_G(v_2) = n-3$, it follows that $5 \leq k \leq n-2$. Let $\mathbb{F} = G - v_k v_{k+2} + v_2 v_{k+1}$. By an argument similar to Case 1, it can be proved that $q(G) < q(\mathbb{F})$. By (i), we get that $q(\mathbb{F}) < q(\mathcal{H}_n)$. Then $q(G) < q(\mathcal{H}_n)$.

Case 3. $G \cong D_4$. For convenience, we suppose that $G = D_4$. Because $d_G(v_2) = n-3$, it follows that $4 \leq k \leq l-2$, $l \leq n-1$. Let $\mathbb{F} = G - v_{l-1}v_{l+1} + v_2 v_l$. By an argument similar to Case 1, it can be proven that $q(G) < q(\mathbb{F})$. By (i), we get that $q(\mathbb{F}) < q(\mathcal{H}_n)$. Then $q(G) < q(\mathcal{H}_n)$.

From the above three cases, (ii) follows. □

THEOREM 3.12. *Let G be a planar graph of order $n \geq 456$. Then $q(G) \leq q(\mathcal{H}_n)$ with equality if and only if $G \cong \mathcal{H}_n$.*

Proof. This theorem follows from the discussions in Section 2, Lemmas 3.1, 3.2, 3.8–3.11 and Theorem 3.7. □

REMARK 2. As for the planar graphs of order $n \leq 455$, perhaps by computations with computer, one can check and find which ones have the maximal signless Laplacian spectral radius. By some computations and comparisons with computer, for the planar graphs of order $n \leq 10$, we find $\mathcal{H}_n$ has the maximal signless Laplacian spectral radius. Based on this, for the planar graphs of order $n \leq 455$, we conjecture that the graph $\mathcal{H}_n$ still has the maximal signless Laplacian spectral radius.

Acknowledgment. We offer many thanks to the referees for their kind reviews and helpful suggestions.

REFERENCES

[1] J. Bondy and U. Murty. *Graph Theory.* Springer, London, 2008.
[2] D. Cao and A. Vince. The spectral radius of a planar graph. *Linear Algebra Appl.*, 187:251–257, 1993.
[3] D. Cvetković, P. Rowlinson, and S. Simić. Signless Laplacians of finite graphs, *Linear Algebra Appl.*, 423:155–171, 2007.
[4] D. Cvetković and S. Simić. Towards a spectral theory of graphs based on the signless Laplacian, I. *Publications De ĺ institut Mathématique Nouvelle série*, 85(99):19–33, 2009.
[5] D. Cvetković and S. Simić. Towards a spectral theory of graphs based on the signless Laplacian, II. *Linear Algebra Appl.*, 432:2257–2272, 2010.
[6] D. Cvetković and S. Simić. Towards a spectral theory of graphs based on the signless Laplacian, III. *Discrete Math.*, 4:156–166, 2010.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 795-811, November 2015

ELA

[7] D. Cvetković and S. Simić. Graph spectra in Computer Science. *Linear Algebra Appl.*, 434:1545–1562, 2011.

[8] E. van Dam and W. Haemers. Which graphs are determined by their spectrum. *Linear Algebra Appl.*, 373:241–272, 2003.

[9] L. Feng, G. Yu, A. Ilić, and D. Stevanović. The signless Laplacian spectral radius of graphs on surfaces. *Linear Multilinear Algebra*, 61:573–581, 2013.

[10] F. Harary. *Graph Theory*. Addison-Wesley, New York, 1969.

[11] R. Merris. A note on Laplacian graph eigenvalues. *Linear Algebra Appl.*, 285:33–35, 1998.

[12] A. Schwenk and R. Wilson. On the eigenvalues of a graph. In: L.W. Beineke and R.J. Wilson (editors), *Selected Topics in Graph Theory*, Academic Press, New York, 307–336, 1978.

[13] J. Shu and Y. Hong. Upper bounds of the spectral radius for outer-planar graphs and Halin graphs. *Chin. Ann. Math., Ser. A*, 21:677–682, 2001.

[14] G. Yu, J. Shu, and Y. Hong. Bounds of the spectral radii of $K_{2,3}$ -minor free graphs. *Electron. J. Linear Algebra*, 23:171–179, 2012.

[15] L. Zhang, H. Jiang, and S. Guo. Cluster anticonsensus of multiagent systems based on the Q-Theory. *Discrete Dyn. Nat. Soc.*, Article ID 254749, 7 pages, 2014.