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REPRESENTATIONS AND SIGN PATTERN OF THE GROUP INVERSE FOR SOME BLOCK MATRICES*

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Abstract. Let $M=\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ be a complex square matrix where A is square. When $BCB^{\Omega}=0$, $\text{rank}(BC)=\text{rank}(B)$ and the group inverse of $\begin{pmatrix} B^{\Omega}AB^{\Omega} & 0 \\ CB^{\Omega} & 0 \end{pmatrix}$ exists, the group inverse of M exists if and only if $\text{rank}(BC + A\left(B^{\Omega}AB^{\Omega}\right)^{\pi} B^{\Omega}A) = \text{rank}(B)$. In this case, a representation of $M^{\#}$ in terms of the group inverse and Moore-Penrose inverse of its subblocks is given. Let A be a real matrix. The sign pattern of A is a $(0,+,-)$-matrix obtained from A by replacing each entry by its sign. The qualitative class of A is the set of the matrices with the same sign pattern as A, denoted by $Q(A)$. The matrix A is called S^2GI, if the group inverse of each matrix $\widetilde{A}\in Q(A)$ exists and its sign pattern is independent of $\widetilde{A}$. By using the group inverse representation, a necessary and sufficient condition for a real block matrix $\begin{pmatrix} A & \Delta_1 & Y_1 \\ \Delta_2 & 0 & 0 \\ Y_2 & 0 & 0 \end{pmatrix}$ to be an S^2GI-matrix is given, where A is square, Δ_1 and Δ_2 are invertible, Y_1 and Y_2 are sign orthogonal.

Key words. Group inverse, Moore-Penrose inverse, Sign pattern, S^2GI-matrix.

AMS subject classifications. 15A09, 15B35.

1. Introduction.

Let $\mathbb{C}^{m\times n}$ and $\mathbb{R}^{m\times n}$ be the sets of $m\times n$ complex matrices and $m\times n$ real matrices, respectively. For $A\in\mathbb{C}^{n\times n}$, the group inverse of A is a matrix $X\in\mathbb{C}^{n\times n}$ satisfying

$$AXA=A,\quad XAX=X,\quad AX=XA.$$

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It is well-known that the group inverse exists if and only if $\text{rank}(A) = \text{rank}(A^2)$; in this case, the group inverse is unique (see [1]). As is customary, we denote the group inverse of A by $A^{\#}$. When A is nonsingular, $A^{\#} = A^{-1}$. For $A \in \mathbb{C}^{m\times n}$, the matrix $X \in \mathbb{C}^{n\times m}$ is called the Moore-Penrose inverse of A if $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $(XA)^* = XA$, where A^* is the conjugate transpose of A. Let A^+ denote the Moore-Penrose inverse of A. It is well-known that A^+ exists and is unique (see [9]). Throughout this paper, $A^{\Omega} = I - AA^+$, $A^Z = I - A^+A$ and $A^{\pi} = I - AA^{\#}$, where I is the identity matrix.

There are many applications of the group inverse of matrices in algebraic connectivity and algebraic bipartiteness of graphs (see [15, 19]), Markov chains (see [9]), and resistance distance (see [6]). In 1979, Campbell and Meyer proposed the open problem of finding explicit formulas for the Drazin or group inverse of a 2×2 block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in terms of its subblocks, where A and D are square (see [9]). At present, the problem of finding explicit representations for the group inverse of $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ have not been completely solved. Recently, the existence and the representations for the group inverse of block matrices were given under some conditions (see [8, 11, 13, 14, 18]).

Let $\text{sgn}(a)$ be the sign of a real number a, which is defined to be $-, 0$ or $+$ depending on $a < 0, a = 0$ or $a > 0$. The sign pattern of $A \in \mathbb{R}^{m\times n}$ is a $(0, +, -)$-matrix obtained from A by replacing each entry by its sign, denoted by $\text{sgn}(A)$, i.e., for matrix $A = (a_{ij})_{m\times n}$, $\text{sgn}(A) = (\text{sgn}(a_{ij}))_{m\times n}$. The *qualitative class of the real matrix* A is the set of the matrices with the same sign pattern as A, denoted by $Q(A)$ (see [23]). For $A \in \mathbb{R}^{n\times n}$, A is called an SNS-*matrix* if each $\widetilde{A} \in Q(A)$ is nonsingular. The matrix A is called an S^2NS-*matrix* if A is an SNS-matrix and $\text{sgn}(\widetilde{A}^{-1}) = \text{sgn}(A^{-1})$ for each $\widetilde{A} \in Q(A)$ (see [4]). The matrix $A \in \mathbb{R}^{n\times n}$ is called an SGI-*matrix* if $\widetilde{A}^{\#}$ exists for each $\widetilde{A} \in Q(A)$. If A is an SGI-matrix and $\text{sgn}(\widetilde{A}^{\#}) = \text{sgn}(A^{\#})$ for each $\widetilde{A} \in Q(A)$, then A is an S^2GI-*matrix*, sometimes we say A has signed generalized inverse to indicate that A is an S^2GI-matrix (see [25]).

The sign pattern of matrix has important applications in the qualitative economics (see [4, 16, 17, 20, 21, 23]). The monograph of Brualdi and Shader introduces many results on S^2NS-matrices (see [4]). In 1995, Shader gave a description for the structure of matrices with signed Moore-Penrose inverse (see [22]). In 2001, Shao and Shan completely characterized the matrices with signed Moore-Penrose inverse (see [23]). In 2004, Britz, Olesky and Driessche researched the signed Moore-Penrose inverse for the matrices with an acyclic bipartite graph (see [3]). In 2010, M. Catral et al. proved that a nonnegative matrix corresponding to a broom graph has a signed group inverse (see [12]). In 2014, Bapat and Ghorbani gave some results on the zero

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pattern of the inverse of lower triangular matrices (see [2]). In [25], a real block matrix $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ was shown to be an SGI-matrix if $\mathrm{sgn}(B^\top) = \mathrm{sgn}(C)$ and C has signed Moore-Penrose inverse, and M is an S^{2}GI-matrix with an additional condition $A = 0$. In [7, 26], some results on real block matrices $\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ with signed Drazin inverse were given under the condition $\mathrm{sgn}(B^\top) = \mathrm{sgn}(C)$ and other conditions.

Let $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ be a complex square matrix, where A is square. When $BCB^\Omega = 0$, $\mathrm{rank}(BC) = \mathrm{rank}(B)$ and the group inverse of $\begin{pmatrix} B^\Omega AB^\Omega & 0 \\ CB^\Omega & 0 \end{pmatrix}$ exists, we obtain the group inverse of M exists if and only if $\mathrm{rank}(BC + A\left(B^\Omega AB^\Omega\right)^\pi B^\Omega A) = \mathrm{rank}(B)$. In this case, we give the representation of $M^\#$ in terms of the group inverse and Moore-Penrose inverse of its subblocks. By using this representation, we give a necessary and sufficient condition for a real block matrix $\begin{pmatrix} A & \Delta_1 & Y_1 \\ \Delta_2 & 0 & 0 \\ Y_2 & 0 & 0 \end{pmatrix}$ to be an S^{2}GI-matrix, where A is square, Δ_1 and Δ_2 are invertible, $\widetilde{Y_1}\widetilde{Y_2} = 0$ for each $\widetilde{Y_i} \in Q(Y_i)$, $i = 1, 2$.

2. Some lemmas. Before our main results, some lemmas on the group inverse of 2×2 block matrix and the matrix sign pattern are presented. First, we define the notion of sign orthogonality and introduce other notations.

LEMMA 2.1. [5] *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a complex square matrix, where* A *is nonsingular. If the group inverse of* $S = D - CA^{-1}B$ *exists, then*

(i) $M^\#$ *exists if and only if* $R = A^2 + BS^\pi C$ *is nonsingular;* (ii) *If* $M^\#$ *exists, then* $M^\# = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, *where*

$$\begin{aligned}
X &= AR^{-1}(A + BS^\# C)R^{-1}A, \\
Y &= AR^{-1}(A + BS^\# C)R^{-1}BS^\pi - AR^{-1}BS^\#, \\
Z &= S^\pi CR^{-1}(A + BS^\# C)R^{-1}A - S^\# CR^{-1}A, \\
W &= S^\pi CR^{-1}(A + BS^\# C)R^{-1}BS^\pi - S^\# CR^{-1}BS^\pi - S^\pi CR^{-1}BS^\# + S^\#.
\end{aligned}$$

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Lemma 2.2. [10] *Let* $M = \begin{pmatrix} A & 0 \\ B & 0 \end{pmatrix} \in \mathbb{C}^{n\times n}$ *and let* $A \in \mathbb{C}^{r\times r}$. *Then* $M^\#$ *exists if and only if* $A^\#$ *exists and rank* $(A) = \operatorname{rank}\begin{pmatrix} A \\ B \end{pmatrix}$. *If* $M^\#$ *exists, then* $M^\# = \begin{pmatrix} A^\# & 0 \\ B\left(A^\#\right)^2 & 0 \end{pmatrix}$.

Lemma 2.3. [24] *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{n\times n}$. *If* $A - BD^{-1}C$ *is invertible, then*

$$M^{-1} = \begin{pmatrix} (M/D)^{-1} & -(M/D)^{-1}BD^{-1} \\ -D^{-1}C(M/D)^{-1} & D^{-1} + D^{-1}C(M/D)^{-1}BD^{-1} \end{pmatrix},$$

where $M/D = D - CA^{-1}B$.

The term *rank of a matrix*, denoted by $\rho(A)$, is the maximal cardinality of the sets of nonzero entries of A no two of which lie in the same row or same column. It is easy to see that $\rho(A) = n$ when A is an invertible matrix of order n. Let A be an $m \times n$ matrix, and let $[m] = \{1, \ldots, m\}$, $[n] = \{1, \ldots, n\}$. If S and T are the subsets of $[m]$ and $[n]$ respectively, then $A[S|T]$ denotes the submatrix of A, whose rows index set is S and columns index set is T. If $S = [m]$ or $T = [n]$, we abbreviate $A[S|T]$ by $A[:|T]$ or $A[S|:]$. Let $(A)_{i,j}$, $N_r(A)$ and $N_c(A)$ denote the (i, j) entry of matrix A, the number of rows and the number of columns of the matrix A, respectively.

Definition 2.4. If $\operatorname{sgn}(\widetilde{A}\widetilde{B}) = 0$ for all the matrices $\widetilde{A} \in Q(A)$, $\widetilde{B} \in Q(B)$ $(N_c(A) = N_r(B))$, then the matrices A and B are called sign orthogonal.

Lemma 2.5. [23] *Let* A *be a real matrix of order* n *such that* $\rho(A) = n$ *and* A *is not an SNS matrix. Then there exist invertible matrices* A_1 *and* A_2 *in* $Q(A)$*, and integers* p*,* q *with* $1 \le p, q \le n$*, such that* $(A_1^{-1})_{q,p}(A_2^{-1})_{q,p} < 0$.

In [23, Theorem 4.2], Shao and Shan gave a result on $\operatorname{sgn}(\widetilde{A}^+\widetilde{B}\widetilde{C}^+)$ for all matrices $\widetilde{A} \in Q(A)$, $\widetilde{B} \in Q(B)$ and $\widetilde{C} \in Q(C)$. By using the similar methods as [23, Theorem 4.2], we establish a result on $\operatorname{sgn}(ABC)$, for all matrices $\widetilde{A} \in Q(A)$, $\widetilde{B} \in Q(B)$ and $\widetilde{C} \in Q(C)$.

Lemma 2.6. *Let* A, B, C *be real matrices with* $N_c(A) = N_r(B)$ *and* $N_c(B) = N_r(C)$. *If* $\operatorname{sgn}(A\widetilde{B}C) = \operatorname{sgn}(ABC)$ *for all matrices* $\widetilde{B} \in Q(B)$*, then* $\operatorname{sgn}(\widetilde{A}\widetilde{B}\widetilde{C}) = \operatorname{sgn}(ABC)$ *for all matrices* $\widetilde{A} \in Q(A)$*,* $\widetilde{B} \in Q(B)$ *and* $\widetilde{C} \in Q(C)$.

Proof. Let $D = ABC$. Then

$$(D)_{i,j} = \sum_{k_2=1}^{N_c(B)} \sum_{k_1=1}^{N_r(B)} (A)_{i,k_1}(B)_{k_1,k_2}(C)_{k_2,j}. \tag{2.1}$$

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ELA

If there exist matrices $\widetilde{A} \in Q(A)$, $\widetilde{B} \in Q(B)$ and $\widetilde{C} \in Q(C)$ such that $\operatorname{sgn}(\widetilde{A}\widetilde{B}\widetilde{C}) \neq \operatorname{sgn}(ABC)$, then there exist integers i_1, j_1, p_1, p_2, q_1, q_2 and $(p_1, q_1) \neq (p_2, q_2)$ such that

$$\operatorname{sgn}((A)_{i_1,p_1}(B)_{p_1,q_1}(C)_{q_1,j_1}) = + \ ,$$

$$\operatorname{sgn}((A)_{i_1,p_2}(B)_{p_2,q_2}(C)_{q_2,j_1}) = - \ .$$

For $k = 1, 2$, let

$$(B_k)_{p,q} = \begin{cases} \frac{1}{\varepsilon}(B)_{p,q}, & p = p_k, q = q_k \\ (B)_{p,q}, & otherwise \end{cases} ,$$

where $\varepsilon > 0$, $p = 1, \ldots, N_r(B)$ and $q = 1, \ldots, N_c(B)$. Clearly, $B_1, B_2 \in Q(B)$ and

$$\operatorname{sgn}(AB_1C) = \operatorname{sgn}(AB_2C). \tag{2.2}$$

Let $D_1 = AB_1C$, $D_2 = AB_2C$. By (2.1),

$$(D_1)_{i_1,j_1} = \sum_{k_2=1}^{N_c(B)} \sum_{k_1=1}^{N_r(B)} (A)_{i_1,k_1}(B_1)_{k_1,k_2}(C)_{k_2,j_1} + \left(\frac{1}{\varepsilon} - 1\right)(A)_{i_1,p_1}(B_1)_{p_1,q_1}(C)_{q_1,j_1},$$

$$(D_2)_{i_1,j_1} = \sum_{k_2=1}^{N_c(B)} \sum_{k_1=1}^{N_r(B)} (A)_{i_1,k_1}(B_2)_{k_1,k_2}(C)_{k_2,j_1} + \left(\frac{1}{\varepsilon} - 1\right)(A)_{i_1,p_2}(B_2)_{p_2,q_2}(C)_{q_2,j_1}.$$

When ε is sufficiently small,

$$\operatorname{sgn}((D_1)_{i_1,j_1}) = \operatorname{sgn}((A)_{i_1,p_1}(B_1)_{p_1,q_1}(C)_{q_1,j_1}) = + \ ,$$

$$\operatorname{sgn}((D_2)_{i_1,j_1}) = \operatorname{sgn}((A)_{i_1,p_2}(B_2)_{p_2,q_2}(C)_{q_2,j_1}) = - \ .$$

Thus, $\operatorname{sgn}((D_1)_{i_1,j_1}) \neq \operatorname{sgn}((D_2)_{i_1,j_1})$, which contradicts (2.2). So $\operatorname{sgn}(\widetilde{A}\widetilde{B}\widetilde{C}) = \operatorname{sgn}(ABC)$ for all matrices $\widetilde{A} \in Q(A)$, $\widetilde{B} \in Q(B)$ and $\widetilde{C} \in Q(C)$. ∎

3. Main results. In this section, some results on the existence, representations and sign pattern for the group inverse of anti-triangular block matrices are given.

THEOREM 3.1. *Let* $M = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \in \mathbb{C}^{(n+m)\times(n+m)}$ *such that the group inverse of* $\begin{pmatrix} B^\Omega A B^\Omega & 0 \\ CB^\Omega & 0 \end{pmatrix}$ *exists. Let* $\Gamma = BC + A\left(B^\Omega A B^\Omega\right)^\pi B^\Omega A$, *where* $A \in \mathbb{C}^{n\times n}$. *If* $BCB^\Omega = 0$ *and* $\operatorname{rank}(BC) = \operatorname{rank}(B)$, *then*

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(i) *$M^{\#}$ exists if and only if* $\mathrm{rank}(\Gamma) = \mathrm{rank}(B)$;

(ii) *If $M^{\#}$ exists, then* $M^{\#} = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, *where*

$$\begin{aligned}
X &= JAGAH - JAH - GAH - JAG + G + H + J, \\
Y &= \Gamma^{+}B + JAGA\Gamma^{+}B - JA\Gamma^{+}B - GA\Gamma^{+}B, \\
Z &= (C - CGA)\,\Gamma^{+}\,(I + AGAH - AH - AG) + CG^{2}\,(I - AH)\,, \\
W &= (C - CGA)\,\Gamma^{+}A\left(GA\Gamma^{+}B - \Gamma^{+}B\right) - CG^{2}A\Gamma^{+}B, \\
J &= \left(B^{\Omega}AB^{\Omega}\right)^{\pi}B^{\Omega}A\Gamma^{+},\ H = \Gamma^{+}A\left(B^{\Omega}AB^{\Omega}\right)^{\pi}B^{\Omega},\ G = \left(B^{\Omega}AB^{\Omega}\right)^{\#}.
\end{aligned}$$

Proof. By the singular value decomposition (see [1]), there exist unitary matrices $U \in \mathbb{C}^{n\times n}$ and $V^{*} \in \mathbb{C}^{m\times m}$ such that

$$UBV^{*} = \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.1}$$

where Δ is an $r \times r$ invertible diagonal matrix and $r = \mathrm{rank}\,(B)$. Let

$$UAU^{*} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad VCU^{*} = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \tag{3.2}$$

where $A_1, C_1 \in \mathbb{C}^{r\times r}$. Then $M = \Phi\widetilde{M}\Phi^{*}$, where $\Phi = \begin{pmatrix} U^{*} & 0 \\ 0 & V^{*} \end{pmatrix}\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$ is a unitary matrix, and

$$\widetilde{M} = \begin{pmatrix} A_1 & \Delta & A_2 & 0 \\ C_1 & 0 & C_2 & 0 \\ A_3 & 0 & A_4 & 0 \\ C_3 & 0 & C_4 & 0 \end{pmatrix}. \tag{3.3}$$

Hence, if $M^{\#}$ exists, then

$$M^{\#} = \Phi\left(\widetilde{M}\right)^{\#}\Phi^{*}. \tag{3.4}$$

Since $BCB^{\Omega} = 0$, $\mathrm{rank}(BC) = \mathrm{rank}(B)$, (3.1) and (3.2) imply that $C_2 = 0$ and C_1 is invertible. Partition (3.3) into the following form

$$\widetilde{M} = \begin{pmatrix} A_1 & \Delta & A_2 & 0 \\ C_1 & 0 & 0 & 0 \\ A_3 & 0 & A_4 & 0 \\ C_3 & 0 & C_4 & 0 \end{pmatrix} =: \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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where

$$N_1 = \begin{pmatrix} A_1 & \Delta \\ C_1 & 0 \end{pmatrix},\ N_2 = \begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix},\ N_3 = \begin{pmatrix} A_3 & 0 \\ C_3 & 0 \end{pmatrix},\ N_4 = \begin{pmatrix} A_4 & 0 \\ C_4 & 0 \end{pmatrix}.$$

It is easy to see that $(N_1)^{-1} = \begin{pmatrix} 0 & (C_1)^{-1} \\ \Delta^{-1} & -\Delta^{-1}A_1(C_1)^{-1} \end{pmatrix}$. Calculations show that $\widetilde{M}/N_1 = N_4 - N_3(N_1)^{-1}N_2 = N_4$. It follows from (3.1) and (3.2) that $\begin{pmatrix} B^{\Omega}AB^{\Omega} & 0 \\ CB^{\Omega} & 0 \end{pmatrix}$ $= \Phi \begin{pmatrix} 0 & 0 \\ 0 & N_4 \end{pmatrix} \Phi^*$. Note that the group inverse of $\begin{pmatrix} B^{\Omega}AB^{\Omega} & 0 \\ CB^{\Omega} & 0 \end{pmatrix}$ exists, so the group inverse of $\widetilde{M}/N_1$ exists. Since N_1 is invertible and the group inverse of $\widetilde{M}/N_1$ exists, by Lemma 2.1, $\widetilde{M}^{\#}$ exists if and only if $R = (N_1)^2 + N_2(\widetilde{M}/N_1)^{\pi}N_3$ is invertible.

According to Lemma 2.2, it yields that $(\widetilde{M}/N_1)^{\#} = \begin{pmatrix} A_4^{\#} & 0 \\ C_4\left(A_4^{\#}\right)^2 & 0 \end{pmatrix}$. Calculations yield

$$\begin{aligned} R &= (N_1)^2 + N_2(\widetilde{M}/N_1)^{\pi}N_3 \\ &= (N_1)^2 + N_2\left(I - (M/N_1)(M/N_1)^{\#}\right)N_3 \\ &= \begin{pmatrix} A_1^2 + A_2A_3 - A_2A_4A_4^{\#}A_3 + \Delta C_1 & A_1\Delta \\ C_1A_1 & C_1\Delta \end{pmatrix}. \end{aligned}$$

Note that C_1 is invertible, so

$$\begin{aligned} \operatorname{rank}(R) &= \operatorname{rank}\begin{pmatrix} A_1^2 + A_2A_3 - A_2A_4A_4^{\#}A_3 + \Delta C_1 & A_1\Delta \\ C_1A_1 & C_1\Delta \end{pmatrix} \\ &= \operatorname{rank}\begin{pmatrix} A_2A_3 - A_2A_4A_4^{\#}A_3 + \Delta C_1 & A_1\Delta \\ 0 & C_1\Delta \end{pmatrix}. \end{aligned}$$

Hence, R invertible implies that $A_2A_3 - A_2A_4A_4^{\#}A_3 + \Delta C_1$ be invertible, that is

$$\operatorname{rank}(A_2A_3 - A_2A_4A_4^{\#}A_3 + \Delta C_1) = r = \operatorname{rank}(B).$$

By simple computations, we have

$$\Gamma = BC + A\left(B^{\Omega}AB^{\Omega}\right)^{\pi}B^{\Omega}A = \begin{pmatrix} A_2A_3 - A_2A_4A_4^{\#}A_3 + \Delta C_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus, R is invertible if and only if $\operatorname{rank}(\Gamma) = \operatorname{rank}(B)$. Applying Lemma 2.1, we get

$$\left(\widetilde{M}\right)^{\#} = \begin{pmatrix} \widetilde{X} & \widetilde{Y} \\ \widetilde{Z} & \widetilde{W} \end{pmatrix},$$

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where

$$\begin{aligned}
\widetilde{X} =&\ N_1R^{-1}KR^{-1}N_1,\\
\widetilde{Y} =&\ N_1R^{-1}KR^{-1}N_2(\widetilde{M}/N_1)^{\pi} - N_1R^{-1}N_2(\widetilde{M}/N_1)^{\#},\\
\widetilde{Z} =&\ (\widetilde{M}/N_1)^{\pi}N_3R^{-1}KR^{-1}N_1 - (\widetilde{M}/N_1)^{\#}N_3R^{-1}N_1,\\
\widetilde{W} =&\ (\widetilde{M}/N_1)^{\pi}N_3R^{-1}KR^{-1}N_2(\widetilde{M}/N_1)^{\pi} - (\widetilde{M}/N_1)^{\#}N_3R^{-1}N_2(\widetilde{M}/N_1)^{\pi}\\
&- (\widetilde{M}/N_1)^{\pi}N_3R^{-1}N_2(\widetilde{M}/N_1)^{\#}+(\widetilde{M}/N_1)^{\#},\\
K =&\ N_1 + N_2(\widetilde{M}/N_1)^{\#}N_3.
\end{aligned}$$

By (3.4),

$$M^{\#} = \Phi\left(\widetilde{M}\right)^{\#}\Phi^* = \Phi\begin{pmatrix} \widetilde{X} & \widetilde{Y} \\ \widetilde{Z} & \widetilde{W}\end{pmatrix}\Phi^*. \tag{3.5}$$

From (3.1), we get

$$BB^+ = U^*\begin{pmatrix} I & 0 \\ 0 & 0\end{pmatrix}U, \quad B^{\pi} = U^*\begin{pmatrix} 0 & 0 \\ 0 & I\end{pmatrix}U,$$

$$B^+B = V^*\begin{pmatrix} I & 0 \\ 0 & 0\end{pmatrix}V, \quad B^{\Omega} = V^*\begin{pmatrix} 0 & 0 \\ 0 & I\end{pmatrix}V.$$

So by the above $BB^+,\ B^{\pi},\ B^+B,\ B^{\Omega}$ and (3.2), it yields

$$\begin{aligned}
\Phi\begin{pmatrix} N_1 & 0 \\ 0 & 0\end{pmatrix}\Phi^* &= \begin{pmatrix} U^* & 0 \\ 0 & V^*\end{pmatrix}\begin{pmatrix} A_1 & 0 & \Delta & 0 \\ 0 & 0 & 0 & 0 \\ C_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & V\end{pmatrix}\\
&= \begin{pmatrix} BB^+ABB^+ & B \\ B^+BC & 0\end{pmatrix}.
\end{aligned} \tag{3.6}$$

Similarly, we have

$$\Phi\begin{pmatrix} 0 & N_2 \\ 0 & 0\end{pmatrix}\Phi^* = \begin{pmatrix} BB^+AB^{\Omega} & 0 \\ 0 & 0\end{pmatrix}, \tag{3.7}$$

$$\Phi\begin{pmatrix} 0 & 0 \\ N_3 & 0\end{pmatrix}\Phi^* = \begin{pmatrix} B^{\Omega}ABB^+ & 0 \\ B^ZCBB^+ & 0\end{pmatrix}, \tag{3.8}$$

$$\Phi\begin{pmatrix} 0 & 0 \\ 0 & (\widetilde{M}/N_1)\end{pmatrix}\Phi^* = \begin{pmatrix} B^{\Omega}AB^{\Omega} & 0 \\ CB^{\Omega} & 0\end{pmatrix}. \tag{3.9}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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ELA

Note that $BCB^{\Omega} = 0$. Since the group inverse of $\widetilde{M}/N_1$ exists, Lemma 2.2 and (3.9) imply that

$$\Phi \begin{pmatrix} 0 & 0 \\ 0 & (\widetilde{M}/N_1)^{\#} \end{pmatrix} \Phi^* = \Phi \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A_4^{\#} & 0 \\ 0 & 0 & C_4\left(A_4^{\#}\right)^2 & 0 \end{pmatrix} \Phi^*$$

$$(3.10) \quad = \begin{pmatrix} U^* & 0 \\ 0 & V^* \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_4^{\#} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & C_4\left(A_4^{\#}\right)^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} G & 0 \\ CG^2 & 0 \end{pmatrix},$$

where $G = \left(B^{\Omega}AB^{\Omega}\right)^{\#}$. Similarly,

$$(3.11) \quad \Phi \begin{pmatrix} 0 & 0 \\ 0 & (\widetilde{M}/N_1)^{\pi} \end{pmatrix} \Phi^* = \begin{pmatrix} \left(B^{\Omega}AB^{\Omega}\right)^{\pi} B^{\Omega} & 0 \\ -CB^{\Omega}G & B^Z \end{pmatrix}.$$

It follows from (3.6)-(3.8) and (3.10) that

$$\Phi \begin{pmatrix} K & 0 \\ 0 & 0 \end{pmatrix} \Phi^*$$

$$= \Phi \begin{pmatrix} N_1 & 0 \\ 0 & 0 \end{pmatrix} \Phi^* + \Phi \begin{pmatrix} 0 & N_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & (M/N_1)^{\#} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ N_3 & 0 \end{pmatrix} \Phi^*$$

$$(3.12) \quad = \begin{pmatrix} BB^+AGABB^+ + BB^+ABB^+ & B \\ B^+BC & 0 \end{pmatrix}.$$

Note that

$$R = \begin{pmatrix} A_1^2 + A_2A_3 - A_2A_4A_4^{\#}A_3 + \Delta C_1 & A_1\Delta \\ C_1A_1 & C_1\Delta \end{pmatrix}.$$

Computations yield that the Schur complement of R is

$$R/(C_1\Delta) = A_2A_3 - A_2A_4A_4^{\#}A_3 + \Delta C_1.$$

Since $R/(C_1\Delta)$ is invertible, by Lemma 2.3, we have

$$R^{-1} = \begin{pmatrix} S^{-1} & -S^{-1}A_1C_1^{-1} \\ -\Delta^{-1}A_1S^{-1} & \Delta^{-1}C_1^{-1} + \Delta^{-1}A_1S^{-1}A_1C_1^{-1} \end{pmatrix},$$

where $S = R/(C_1\Delta)$. Computation shows that

$$(3.13) \quad \Phi \begin{pmatrix} S^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Phi^* = \begin{pmatrix} \Gamma^+ & 0 \\ 0 & 0 \end{pmatrix},$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 744-759, November 2015

$$\Phi\begin{pmatrix} 0 & -S^{-1}A_1C_1^{-1} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\Phi^*$$

$$=\begin{pmatrix} -\Gamma^+ & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix}\left[\begin{pmatrix} 0 & 0\\ 0 & B^+B\end{pmatrix}\begin{pmatrix} 0 & 0\\ C & 0\end{pmatrix}\right]^+$$

$$=\begin{pmatrix} 0 & -\Gamma^+A\left(B^+BC\right)^+\\ 0 & 0\end{pmatrix}, \tag{3.14}$$

$$\Phi\begin{pmatrix} 0 & 0 & 0 & 0\\ -\Delta^{-1}A_1S^{-1} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\Phi^* = \begin{pmatrix} 0 & 0\\ B^+ & 0\end{pmatrix}\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix} -\Gamma^+ & 0\\ 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix} 0 & 0\\ -B^+A\Gamma^+ & 0\end{pmatrix}. \tag{3.15}$$

Similarly, we have

$$\Phi\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & \Delta^{-1}C_1^{-1}+\Delta^{-1}A_1S^{-1}A_1C_1^{-1} & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\Phi^* \tag{3.16}$$

$$=\begin{pmatrix} 0 & 0\\ 0 & B^+A\Gamma^+A\left(B^+BC\right)^+ + B^+\left(B^+BC\right)^+\end{pmatrix}.$$

Adding (3.13)-(3.16) yields

$$\Phi\begin{pmatrix} R^{-1} & 0\\ 0 & 0\end{pmatrix}\Phi^* = \begin{pmatrix} \Gamma^+ & -\Gamma^+A\left(B^+BC\right)^+\\ -B^+A\Gamma^+ & B^+A\Gamma^+A\left(B^+BC\right)^+ + B^+\left(B^+BC\right)^+\end{pmatrix}. \tag{3.17}$$

Substituting the equations (3.6)-(3.12) and (3.17) into (3.5) gives that

$$M^\# = \begin{pmatrix} X & Y\\ Z & W\end{pmatrix},$$

where

$$\begin{aligned}
X &= JAGAH - JAH - GAH - JAG + G + H + J,\\
Y &= \Gamma^+B + JAGA\Gamma^+B - JA\Gamma^+B - GA\Gamma^+B,\\
Z &= (C - CGA)\,\Gamma^+\left(I + AGAH - AH - AG\right) + CG^2\left(I - AH\right),\\
W &= (C - CGA)\,\Gamma^+A\left(GA\Gamma^+B - \Gamma^+B\right) - CG^2A\Gamma^+B,\\
J &= \left(B^\Omega AB^\Omega\right)^\pi B^\Omega A\Gamma^+, \qquad H = \Gamma^+A\left(B^\Omega AB^\Omega\right)^\pi B^\Omega. \quad \square
\end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 744-759, November 2015

For the matrix $M=\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$ in Theorem 3.1, when $C = B^*$, we have the following result.

Corollary 3.2. *Let* $M=\begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}$*, where* $A \in \mathbb{C}^{n\times n}$ *and* $B \in \mathbb{C}^{n\times m}$*. If the group inverse of* $B^\Omega AB^\Omega$ *exists, and let* $\Gamma = BB^* + A\left(B^\Omega AB^\Omega\right)^\pi B^\Omega A$*. Then*

(i) $M^\#$ *exists if and only if* $\operatorname{rank}(\Gamma) = \operatorname{rank}(B)$*;*
(i) *If* $M^\#$ *exists, then* $M^\# = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$*, where*

$$\begin{aligned}
X &= JAGAH - JAH - GAH - JAG + G + H + J,\\
Y &= \Gamma^+B + JAGA\Gamma^+B - JA\Gamma^+B - GA\Gamma^+B,\\
Z &= B^*\Gamma^+ + B^*\Gamma^+AGAH - B^*\Gamma^+AH - B^*\Gamma^+AG,\\
W &= B^*\Gamma^+AGA\Gamma^+B - B^*\Gamma^+A\Gamma^+B,\ G = \left(B^\Omega AB^\Omega\right)^\#,\\
J &= \left(B^\Omega AB^\Omega\right)^\pi B^\Omega A\Gamma^+,\ H = \Gamma^+A\left(B^\Omega AB^\Omega\right)^\pi B^\Omega.
\end{aligned}$$

Let Δ be a nonsingular matrix. If $\operatorname{sgn}(\widetilde{\Delta}^{-1}\widetilde{Y}_1) = \operatorname{sgn}(\Delta^{-1}Y_1)$ and $\operatorname{sgn}(\widetilde{Y}_2\widetilde{\Delta}^{-1}) = \operatorname{sgn}(Y_2\Delta^{-1})$ for all the matrices $\widetilde{\Delta} \in Q(\Delta)$, $\widetilde{Y}_1 \in Q(Y_1)$ and $\widetilde{Y}_2 \in Q(Y_2)$ ($N_c(\Delta) = N_r(Y_1)$, $N_c(Y_2) = N_r(\Delta)$), then $\Delta^{-1}Y_1$ and $Y_2\Delta^{-1}$ are called *sign unique*.

Theorem 3.3. *Let* $N = \begin{pmatrix} A & \Delta_1 & Y_1 \\ \Delta_2 & 0 & 0 \\ Y_2 & 0 & 0 \end{pmatrix}$ *be a real square matrix, where* A *is square,* Δ_1 *and* Δ_2 *are invertible,* Y_1 *and* Y_2 *are sign orthogonal. Then* N *is an* $\mathrm{S}^2\mathrm{GI}$*-matrix if and only if the following hold:*

(i) $\Delta_1^{-1}Y_1$ *and* $Y_2\Delta_2^{-1}$ *are sign unique;*
(ii) $U = \begin{pmatrix} I & Y_2\Delta_2^{-1} & 0 & 0 \\ 0 & \Delta_1 & A & 0 \\ 0 & 0 & \Delta_2 & \Delta_1^{-1}Y_1 \\ 0 & 0 & 0 & I \end{pmatrix}$ *is* $\mathrm{S}^2\mathrm{NS}$*-matrix.*

Proof. Let $B = \begin{pmatrix} \Delta_1 & Y_1 \end{pmatrix}$ and $C = \begin{pmatrix} \Delta_2 \\ Y_2 \end{pmatrix}$. Then $N = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$. Since Δ_1 and Δ_2 are invertible, Y_1 and Y_2 are sign orthogonal, we get that $B^\Omega = 0$ and $\operatorname{rank}(BC) = \operatorname{rank}(B)$. By computing, we get $BCB^\Omega = 0$, $\begin{pmatrix} B^\Omega AB^\Omega & 0 \\ CB^\Omega & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\operatorname{rank}(BC + A(B^\Omega AB^\Omega)^\pi B^\Omega A) = \operatorname{rank}(BC) = \operatorname{rank}(B)$. By Theo-

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 30, pp. 744-759, November 2015

rem 3.1, the group inverse of N exists and

$$N^{\#} = \begin{pmatrix} 0 & \Delta_2^{-1} & \Delta_2^{-1}\Delta_1^{-1}Y_1 \\ \Delta_1^{-1} & -X_1 & -X_2 \\ Y_2\Delta_2^{-1}\Delta_1^{-1} & -X_3 & -X_4 \end{pmatrix}, \tag{3.18}$$

where

$$X_1 = \Delta_1^{-1}A\Delta_2^{-1}, \qquad X_2 = \Delta_1^{-1}A\Delta_2^{-1}\Delta_1^{-1}Y_1,$$
$$X_3 = Y_2\Delta_2^{-1}\Delta_1^{-1}A\Delta_2^{-1}, \qquad X_4 = Y_2\Delta_2^{-1}\Delta_1^{-1}A\Delta_2^{-1}\Delta_1^{-1}Y_1.$$

Next, we show that the condition is necessary. Since Δ_1 is invertible, $\rho(\Delta_1) = N_r(\Delta_1)$.

Next, we show that Δ_1 and Δ_2 are SNS-matrices. If Δ_1 is not an SNS-matrix. By Lemma 2.5, there exist matrices $\widetilde{\Delta}_1, \widetilde{\widetilde{\Delta}}_1 \in Q(\Delta_1)$ such that $(\widetilde{\Delta}_1^{-1})_{q,p}(\widetilde{\widetilde{\Delta}}_1^{-1})_{q,p} < 0$, where the integers p, q with $1 \le p, q \le N_r(\Delta_1)$. Let

$$N_1 = \begin{pmatrix} A & \widetilde{\Delta}_1 & Y_1 \\ \Delta_2 & 0 & 0 \\ Y_2 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} A & \widetilde{\widetilde{\Delta}}_1 & Y_1 \\ \Delta_2 & 0 & 0 \\ Y_2 & 0 & 0 \end{pmatrix}.$$

Clearly, $N_1, N_2 \in Q(N)$ and $N_1{}^{\#}$, $N_2{}^{\#}$ exist. From (3.18) and $(\widetilde{\Delta}_1^{-1})_{q,p}(\widetilde{\widetilde{\Delta}}_1^{-1})_{q,p} < 0$, we have $(N_1^{\#})_{N_r(A)+q,p}(N_2^{\#})_{N_r(A)+q,p} < 0$. This is contrary to N being an S^{2}GI-matrix. Thus, we have Δ_1 is an SNS-matrix. Similarly, Δ_2 is an SNS-matrix.

Therefore, for each matrix $\widehat{N} = \begin{pmatrix} \widehat{A} & \widehat{\Delta}_1 & \widehat{Y}_1 \\ \widehat{\Delta}_2 & 0 & 0 \\ \widehat{Y}_2 & 0 & 0 \end{pmatrix} \in Q(N)$, we have

$$\widehat{N}^{\#} = \begin{pmatrix} 0 & \widehat{\Delta}_2^{-1} & \widehat{\Delta}_2^{-1}\widehat{\Delta}_1^{-1}\widehat{Y}_1 \\ \widehat{\Delta}_1^{-1} & -\widehat{X}_1 & -\widehat{X}_2 \\ \widehat{Y}_2\widehat{\Delta}_2^{-1}\widehat{\Delta}_1^{-1} & -\widehat{X}_3 & -\widehat{X}_4 \end{pmatrix},$$

where

$$\widehat{X}_1 = \widehat{\Delta}_1^{-1}\widehat{A}\widehat{\Delta}_2^{-1}, \qquad \widehat{X}_2 = \widehat{\Delta}_1^{-1}\widehat{A}\widehat{\Delta}_2^{-1}\widehat{\Delta}_1^{-1}\widehat{Y}_1,$$
$$\widehat{X}_3 = \widehat{Y}_2\widehat{\Delta}_2^{-1}\widehat{\Delta}_1^{-1}\widehat{A}\widehat{\Delta}_2^{-1}, \qquad \widehat{X}_4 = \widehat{Y}_2\widehat{\Delta}_2^{-1}\widehat{\Delta}_1^{-1}\widehat{A}\widehat{\Delta}_2^{-1}\widehat{\Delta}_1^{-1}\widehat{Y}_1.$$

Since N is an S^{2}GI-matrix, we have

$$\operatorname{sgn}(\widehat{\Delta}_1^{-1}) = \operatorname{sgn}(\Delta_1^{-1}), \qquad \operatorname{sgn}(\widehat{\Delta}_2^{-1}) = \operatorname{sgn}(\Delta_2^{-1}),$$
$$\operatorname{sgn}(\widehat{\Delta}_2^{-1}\widehat{\Delta}_1^{-1}\widehat{Y}_1) = \operatorname{sgn}(\Delta_2^{-1}\Delta_1^{-1}Y_1), \qquad \operatorname{sgn}(\widehat{Y}_2\widehat{\Delta}_2^{-1}\widehat{\Delta}_1^{-1}) = \operatorname{sgn}(Y_2\Delta_2^{-1}\Delta_1^{-1}),$$
$$\operatorname{sgn}(\widehat{X}_1) = \operatorname{sgn}(X_1), \qquad \operatorname{sgn}(\widehat{X}_2) = \operatorname{sgn}(X_2),$$
$$\operatorname{sgn}(\widehat{X}_3) = \operatorname{sgn}(X_3), \qquad \operatorname{sgn}(\widehat{X}_4) = \operatorname{sgn}(X_4),$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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for all matrices $\widehat{\Delta}_1 \in Q(\Delta_1)$, $\widehat{\Delta}_2 \in Q(\Delta_2)$, $\widehat{Y}_1 \in Q(Y_1)$ and $\widehat{Y}_2 \in Q(Y_2)$. Since $\mathrm{sgn}(\widehat{\Delta}_1^{-1}) = \mathrm{sgn}(\Delta_1^{-1})$, $\mathrm{sgn}(\widehat{\Delta}_2^{-1}) = \mathrm{sgn}(\Delta_2^{-1})$, both Δ_1 and Δ_2 are S^{2}NS-matrices. Hence, there exists permutation matrix P_1 such that $(P_1\Delta_2)_{i,i} \neq 0$ $(i = 1, 2, \ldots, N_r(P_1\Delta_2))$.

Let $Q_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & P_1 & 0 \\ 0 & 0 & I \end{pmatrix}$ and $W_1 = Q_1NQ_1^T = \begin{pmatrix} A & \Delta_1P_1^T & Y_1 \\ P_1\Delta_2 & 0 & 0 \\ Y_2 & 0 & 0 \end{pmatrix}$. It follows from Theorem 3.1 that

$$W_1^{\#} = \begin{pmatrix} 0 & \Delta_2^{-1}P_1^T & \Delta_2^{-1}P_1^TP_1\Delta_1^{-1}Y_1 \\ P_1\Delta_1^{-1} & -P_1X_1P_1^T & -P_1X_2 \\ Y_2\Delta_2^{-1}P_1^TP_1\Delta_1^{-1} & -X_3P_1^T & -X_4 \end{pmatrix}.$$

Note that N is S^{2}GI-matrix. Thus, W_1 is an S^{2}GI-matrix. So $\mathrm{sgn}(\widetilde{\Delta}_2^{-1}P_1^TP_1\widetilde{\Delta}_1^{-1}\widetilde{Y}_1) = \mathrm{sgn}(\Delta_2^{-1}P_1^TP_1\Delta_1^{-1}Y_1)$ for all the matrices $\widetilde{\Delta}_1 \in Q(\Delta_1)$, $\widetilde{\Delta}_2 \in Q(\Delta_2)$ and $\widetilde{Y}_1 \in Q(Y_1)$.

Next, we prove $\Delta_1^{-1}Y_1$ and $Y_2\Delta_2^{-1}$ are sign unique. If $\Delta_1^{-1}Y_1$ is not sign unique. Let $Z_1 = \Delta_1P_1^T$ and let$Z_2 = P_1\Delta_2$. Then $Z_1^{-1}Y_1 = P_1\Delta_1^{-1}Y_1$ is not sign unique, i.e., there exist integers i_1, i_2 and matrices $\widetilde{Y}_1, \widetilde{\widetilde{Y}}_1 \in Q(Y_1)$ such that $(Z_1^{-1}\widetilde{Y}_1)_{i_1,i_2} > 0$ and $(Z_1^{-1}\widetilde{\widetilde{Y}}_1)_{i_1,i_2} < 0$. For $1 \le i \le N_r(Z_2), \varepsilon > 0$, let

$$\widetilde{Z}_2[i|:] = \begin{cases} Z_2[i|:] & i \neq i_1, \\ \varepsilon Z_2[i|:] & i = i_1. \end{cases}$$

Clearly, $\widetilde{Z}_2 \in Q(Z_2)$. It is easy to see that

$$(\widetilde{Z}_2^{-1})_{i_1,i} = \begin{cases} (Z_2^{-1})_{i_1,i} & i \neq i_1 \\ \frac{1}{\varepsilon}(Z_2^{-1})_{i_1,i} & i = i_1 \end{cases} \quad (1 \le i \le N_r(Z_2)).$$

Note that

$$(\widetilde{Z}_2^{-1}Z_1^{-1}\widetilde{Y}_1)_{i_1,i_2} = \frac{1}{\varepsilon}(Z_2^{-1})_{i_1,i_1}(Z_1^{-1}\widetilde{Y}_1)_{i_1,i_2} + \sum_{i=1,i\neq i_1}^{N_c(Z_2)} (Z_2^{-1})_{i_1,i}(Z_1^{-1}\widetilde{Y}_1)_{i,i_2},$$

$$(\widetilde{Z}_2^{-1}Z_1^{-1}\widetilde{\widetilde{Y}}_1)_{i_1,i_2} = \frac{1}{\varepsilon}(Z_2^{-1})_{i_1,i_1}(Z_1^{-1}\widetilde{\widetilde{Y}}_1)_{i_1,i_2} + \sum_{i=1,i\neq i_1}^{N_c(Z_2)} (Z_2^{-1})_{i_1,i}(Z_1^{-1}\widetilde{\widetilde{Y}}_1)_{i,i_2}.$$

When ε is sufficiently small, we get

$$\mathrm{sgn}((\widetilde{Z}_2^{-1}Z_1^{-1}\widetilde{Y}_1)_{i_1,i_2}) = \mathrm{sgn}(\frac{1}{\varepsilon}(Z_2^{-1})_{i_1,i_1}(Z_1^{-1}\widetilde{Y}_1)_{i_1,i_2}),$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 30, pp. 744-759, November 2015

$$\operatorname{sgn}((\widetilde{Z}_2^{-1}Z_1^{-1}\widetilde{\widetilde{Y}}_1)_{i_1,i_2}) = \operatorname{sgn}(\frac{1}{\varepsilon}(Z_2^{-1})_{i_1,i_1}(Z_1^{-1}\widetilde{\widetilde{Y}}_1)_{i_1,i_2}).$$

Since

$$\operatorname{sgn}((Z_1^{-1}\widetilde{Y}_1)_{i_1,i_2}) = -\operatorname{sgn}((Z_1^{-1}\widetilde{\widetilde{Y}}_1)_{i_1,i_2}),$$

we have

$$\operatorname{sgn}((\widetilde{Z}_2^{-1}Z_1^{-1}\widetilde{Y}_1)_{i_1,i_2}) = -\operatorname{sgn}((\widetilde{Z}_2^{-1}Z_1^{-1}\widetilde{\widetilde{Y}}_1)_{i_1,i_2}).$$

This contradicts the assumption that W_1 is an S^2GI-matrix. So $\Delta_1^{-1}Y_1$ is sign unique and $\operatorname{sgn}(\widetilde{\Delta}_2^{-1}\widetilde{H}_1) = \operatorname{sgn}(\Delta_2^{-1}\Delta_1^{-1}Y_1)$ for all matrices $\widetilde{\Delta}_2 \in Q(\Delta_2)$, $\widetilde{H}_1 \in Q(\Delta_1^{-1}Y_1)$. Similarly, $Y_2\Delta_2^{-1}$ is sign unique, and $\operatorname{sgn}(\widetilde{H}_2\widetilde{\Delta}_1^{-1}) = \operatorname{sgn}(Y_2\Delta_2^{-1}\Delta_1^{-1})$ for all matrices $\widetilde{H}_2 \in Q(Y_2\Delta_2^{-1})$, $\widetilde{\Delta}_1 \in Q(\Delta_1)$.

Next, we prove that part (ii) of the theorem holds. Let

$$L_1 = \Delta_1^{-1}, \quad L_2 = \Delta_2^{-1}, \quad L_3 = \Delta_2^{-1}\Delta_1^{-1}Y_1, \quad L_4 = Y_2\Delta_2^{-1}\Delta_1^{-1}.$$

Then

$$X_1 = L_1AL_2, \quad X_2 = L_1AL_3, \quad X_3 = L_4AL_2, \quad X_4 = L_4AL_3.$$

Since $\operatorname{sgn}(\widehat{\Delta}_1^{-1}\widehat{A}\widehat{\Delta}_2^{-1}) = \operatorname{sgn}(X_1)$ for all matrices $\widehat{\Delta}_1 \in Q(\Delta_1)$, $\widehat{A} \in Q(A)$ and $\widehat{\Delta}_2 \in Q(\Delta_2)$, we have $\operatorname{sgn}(L_1\widetilde{A}L_2) = \operatorname{sgn}(L_1AL_2)$ for each matrix $\widetilde{A} \in Q(A)$. It follows from Lemma 2.6 that $\operatorname{sgn}(\widehat{L}_1\widehat{A}\widehat{L}_2) = \operatorname{sgn}(L_1AL_2)$ for all matrices $\widehat{L}_1 \in Q(L_1)$, $\widehat{A} \in Q(A)$ and $\widehat{L}_2 \in Q(L_2)$. Similarly, $\operatorname{sgn}(\widehat{L}_1\widehat{A}\widehat{L}_3) = \operatorname{sgn}(L_1AL_3)$, $\operatorname{sgn}(\widehat{L}_4\widehat{A}\widehat{L}_2) = \operatorname{sgn}(L_4AL_2)$, $\operatorname{sgn}(\widehat{L}_4\widehat{A}\widehat{L}_3) = \operatorname{sgn}(L_4AL_3)$ for all matrices $\widehat{A} \in Q(A)$, $\widehat{L}_1 \in Q(L_1)$, $\widehat{L}_2 \in Q(L_2)$, $\widehat{L}_3 \in Q(L_3)$, $\widehat{L}_4 \in Q(L_4)$.

Let

$$U = \begin{pmatrix} I & Y_2\Delta_2^{-1} & 0 & 0 \\ 0 & \Delta_1 & A & \\ 0 & 0 & \Delta_2 & \Delta_1^{-1}Y_1 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Since Δ_1 and Δ_2 are SNS-matrices, U is an SNS-matrix. By calculation, we have

$$U^{-1} = \begin{pmatrix} I & -L_4 & L_4AL_2 & -L_4AL_3 \\ 0 & L_1 & -L_1AL_2 & L_1AL_3 \\ 0 & 0 & L_2 & -L_3 \\ 0 & 0 & 0 & I \end{pmatrix}. \tag{3.19}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 30, pp. 744-759, November 2015

Clearly, $\operatorname{sgn}(\widehat{U}^{-1}) = \operatorname{sgn}(U^{-1})$ for all matrices $\widehat{U} = \begin{pmatrix} \widehat{I} & \widehat{H}_2 & 0 & 0 \\ 0 & \widehat{\Delta}_1 & \widehat{A} & 0 \\ 0 & 0 & \widehat{\Delta}_2 & \widehat{H}_1 \\ 0 & 0 & 0 & \widehat{\widehat{I}} \end{pmatrix} \in Q(U)$,

where $\widehat{I}, \widehat{\widehat{I}} \in Q(I)$, $\widehat{\Delta}_1 \in Q(\Delta_1)$, $\widehat{\Delta}_2 \in Q(\Delta_2)$, $\widehat{A} \in Q(A)$, $\widehat{H}_1 \in Q(\Delta_1^{-1}Y_1)$, $\widehat{H}_2 \in Q(Y_2\Delta_2^{-1})$. Hence, U is an S^{2}NS-matrix. So (i) and (ii) hold.

If (i) and (ii) hold, then by (3.18) and (3.19), N is an S^{2}GI matrix. □

Theorem 3.4. *Let* $N = \begin{pmatrix} A & I & Y_1 \\ I & 0 & 0 \\ Y_2 & 0 & 0 \end{pmatrix}$ *be a real square matrix, where* A *is square,* Y_1 *and* Y_2 *are sign orthogonal. Then* N *is an* S^{2}GI-*matrix if and only if* $\operatorname{sgn}(\widetilde{Y_2}\widetilde{A}) = \operatorname{sgn}(Y_2A)$, $\operatorname{sgn}(\widetilde{Y_2}\widetilde{A}\widetilde{Y_1}) = \operatorname{sgn}(Y_2AY_1)$ *and* $\operatorname{sgn}(\widetilde{A}\widetilde{Y_1}) = \operatorname{sgn}(AY_1)$ *for each* $\widetilde{A} \in Q(A), \widetilde{Y_1} \in Q(Y_1)$ *and* $\widetilde{Y_2} \in Q(Y_2)$.

Proof. From Theorem 3.3, we have N is an S^{2}GI-matrix if and only if $U = \begin{pmatrix} I & Y_2 & 0 & 0 \\ 0 & I & A & 0 \\ 0 & 0 & I & Y_1 \\ 0 & 0 & 0 & I \end{pmatrix}$ is an S^{2}NS-matrix.

Clearly, U is an SNS-matrix. Calculations gives

$$U^{-1} = \begin{pmatrix} I & -Y_2 & Y_2A & -Y_2AY_1 \\ 0 & I & -A & AY_1 \\ 0 & 0 & I & -Y_1 \\ 0 & 0 & 0 & I \end{pmatrix}.$$

Since $\operatorname{sgn}(\widetilde{Y_2}\widetilde{A}) = \operatorname{sgn}(Y_2A)$, $\operatorname{sgn}(\widetilde{Y_2}\widetilde{A}\widetilde{Y_1}) = \operatorname{sgn}(Y_2AY_1)$ and $\operatorname{sgn}(\widetilde{A}\widetilde{Y_1}) = \operatorname{sgn}(AY_1)$ for each $\widetilde{A} \in Q(A), \widetilde{Y_1} \in Q(Y_1)$ and $\widetilde{Y_2} \in Q(Y_2)$, we have U is an S^{2}NS-matrix. Hence, N is an S^{2}GI-matrix. □

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