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NONNEGATIVE GENERALIZED DOUBLY STOCHASTIC MATRICES WITH PRESCRIBED ELEMENTARY DIVISORS*

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Abstract. This paper provides sufficient conditions for the existence of nonnegative generalized doubly stochastic matrices with prescribed elementary divisors. These results improve previous results and the constructive nature of their proofs allows for the computation of a solution matrix. In particular, this paper shows how to transform a generalized stochastic matrix into a nonnegative generalized doubly stochastic matrix, at the expense of increasing the Perron eigenvalue, but keeping other elementary divisors unchanged. Under certain restrictions, nonnegative generalized doubly stochastic matrices can be constructed, with spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ for each Jordan canonical form associated with Λ.

Key words. Stochastic matrices, Doubly stochastic matrices, Elementary divisors.

AMS subject classifications. 15A18, 15A51.

1. Introduction.

Let $A \in \mathbb{C}^{n\times n}$ and let

$$J(A) = S^{-1}AS = \begin{bmatrix} J_{n_1(\lambda_1)} & & & \\ & J_{n_2(\lambda_2)} & & \\ & & \ddots & \\ & & & J_{n_k(\lambda_k)} \end{bmatrix}$$

be the *Jordan canonical form* of A (hereafter, the JCF of A). The $n_i \times n_i$ submatrices

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}, \quad i = 1, 2, \ldots, k$$

are the *Jordan blocks* of $J(A)$. The *elementary divisors* of A are the polynomials $(\lambda - \lambda_i)^{n_i}$, that is, the characteristic polynomials of $J_{n_i}(\lambda_i)$, $i = 1, \ldots, k$.

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The *nonnegative inverse elementary divisor problem* (hereafter, the *NIEDP*) is the problem of determining necessary and sufficient conditions under which the polynomials $(\lambda-\lambda_1)^{n_1}, (\lambda-\lambda_2)^{n_2}, \ldots, (\lambda-\lambda_k)^{n_k}$, $n_1+\cdots+n_k=n$, are the *elementary divisors* of an $n\times n$ nonnegative matrix A (see [7, 8, 9]). The *NIEDP* is closely related to another problem, *the nonnegative inverse eigenvalue problem* (hereafter, the *NIEP*), which is the problem of determining necessary and sufficient conditions for a list of complex numbers $\Lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ to be the spectrum of an $n\times n$ entrywise nonnegative matrix. If there exists a nonnegative matrix A with spectrum Λ, we say that Λ is realizable and that A is the realizing matrix. Both, the *NIEDP and the NIEP* remain unsolved; the *NIEP* has been solved only for $n\leq 4$. A number of sufficient conditions or realizability criteria for the existence of a solution for the *NIEP* have been obtained by many authors. In contrast, only a few works are known for the *NIEDP* [3, 4, 5, 7, 8, 14, 15, 16].

A matrix A has *constant row sums* γ if the sum of the entries in each row is γ. The set of all matrices with constant row sums equal to γ is denoted by $\mathcal{CS}_\gamma$. It is clear that $\mathbf{e}=(1,1,\ldots,1)^T$ is an eigenvector of any matrix $A\in\mathcal{CS}_\gamma$, corresponding to the eigenvalue γ. A nonnegative matrix A is called *stochastic* if $A\in\mathcal{CS}_1$ and is called *doubly stochastic* if $A, A^T\in\mathcal{CS}_1$. A matrix A is *generalized stochastic* (respectively *generalized doubly stochastic*) if $A\in\mathcal{CS}_{\lambda_1}$ (respectively $A, A^T\in\mathcal{CS}_{\lambda_1}$). In this paper, we are interested in nonnegative generalized stochastic and doubly stochastic matrices. The relevance of matrices with constant row sums is due to the well known fact that the problem of finding a nonnegative matrix with spectrum $\Lambda=\{\lambda_1,\ldots,\lambda_n\}$ is equivalent to the problem of finding a nonnegative matrix in $\mathcal{CS}_{\lambda_1}$ with spectrum Λ.

In [7], Minc proves the following result:

THEOREM 1.1. [7] *Given a diagonalizable positive (diagonalizable positive doubly stochastic) matrix A, there exists a positive (positive doubly stochastic) matrix with the same spectrum as A, and with arbitrarily prescribed elementary divisors, provided that elementary divisors corresponding to nonreal eigenvalues occur in conjugate pairs.*

Of course Theorem 1.1 also holds for diagonalizable positive generalized stochastic (diagonalizable positive generalized doubly stochastic) matrices. Usually for the *NIEDP,* instead of a diagonalizable positive generalized stochastic (diagonalizable positive generalized doubly stochastic) matrix, we are given a list of polynomials

$$(\lambda-\lambda_1), (\lambda-\lambda_2)^{n_2}, \ldots, (\lambda-\lambda_k)^{n_k}, \;\; n_2+\cdots+n_k=n-1,$$

or a list of complex numbers $\Lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$, from which we want to construct a nonnegative or positive generalized stochastic or generalized doubly stochastic matrix with the given polynomials as its elementary divisors. In this work, we show that under certain restrictions on the spectrum, it is possible to construct a diagonaliz-

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able positive generalized doubly stochastic matrix with prescribed spectrum Λ. Then, in this case, from Theorem 1.1, we may guarantee the existence and construction of a positive generalized doubly stochastic matrix with spectrum Λ and arbitrarily prescribed elementary divisors.

The following perturbation result, due to Rado and introduced by Perfect in [11], shows how to change r eigenvalues of an $n \times n$ matrix, via a *rank* r perturbation, without changing any of the remaining $n - r$ eigenvalues. This result has been employed with success in connection with the *NIEP,* to derive sufficient conditions for the existence and construction of nonnegative matrices with prescribed spectrum (see [11, 17] and the references therein):

THEOREM 1.2. [11] *Let A be an $n \times n$ arbitrary matrix with spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$. Let $X = [\mathbf{x}_1 \mid \cdots \mid \mathbf{x}_r]$ be such that $rank(X) = r$ and $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$, $i = 1, \ldots, r$, $r \leq n$. Let C be an $r \times n$ arbitrary matrix. Then $A + XC$ has eigenvalues $\mu_1, \ldots, \mu_r, \lambda_{r+1}, \ldots \lambda_n$, where $\mu_1, \ldots, \mu_r$ are eigenvalues of the matrix $\Omega + CX$ with $\Omega = diag\{\lambda_1, \ldots, \lambda_r\}$.*

The case $r = 1$ in Theorem 1.2, constitute the well known Brauer Theorem [2, Theorem 27], which has also been successfully employed in connection with the *NIEP* (see [10, 12, 13] and the references therein).

THEOREM 1.3. [2] *Let A be an $n \times n$ arbitrary matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Let $\mathbf{v}$ be an eigenvector of A corresponding to λ_k and let $\mathbf{q}$ be any n-dimensional vector. Then the matrix $A + \mathbf{v}\mathbf{q}^T$ has eigenvalues $\lambda_1, \ldots, \lambda_{k-1}, \lambda_k + \mathbf{v}^T\mathbf{q}, \lambda_{k+1}, \ldots, \lambda_n$.*

Let A and its JCF $J(A)$ be given. The following result, from [14], describes the JCF of $A + \mathbf{e}\mathbf{q}^T$.

LEMMA 1.4. [14] *Let $A \in \mathcal{CS}_{\lambda_1}$ with JCF $J(A) = S^{-1}AS$. Let $\mathbf{q}^T = (q_1, \ldots, q_n)$ and $\lambda_1 + \sum_{i=1}^{n} q_i \neq \lambda_i$, $i = 2, \ldots, n$. Then $A + \mathbf{e}\mathbf{q}^T$ has JCF $J(A) + \left(\sum_{i=1}^{n} q_i\right) E_{11}$.In particular, if $\sum_{i=1}^{n} q_i = 0$, A and $A + \mathbf{e}\mathbf{q}^T$ are similar.*

In [14, 15], by using the Brauer and Rado perturbation results, the authors have obtained constructive sufficient conditions for the existence of nonnegative and positive matrices with constant rows sum and prescribed elementary divisors. The novelty in this paper is that we may also construct, under certain restrictions, nonnegative (positive) generalized doubly stochastic matrices with prescribed elementary divisors. Thus, in this paper, we give sufficient conditions for the existence and construction of a nonnegative (positive) generalized doubly stochastic matrix with certain prescribed elementary divisors. In particular, in Section 2, by applying the Brauer perturbation

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result, we show how to transform a generalized stochastic matrix with given elementary divisors into a nonnegative (positive) generalized doubly stochastic matrix, without changing the elementary divisors. In Section 3, by applying the Rado perturbation result, we show how to construct nonnegative (positive) generalized doubly stochastic matrices with certain prescribed elementary divisors. According to Minc (see [9]), the positivity condition on A is essential for the proof of Theorem 1.1 and it is not known if the result holds without this condition. In Section 4, we show that under certain restrictions on the spectrum, the positivity condition on A can be relaxed to A being nonnegative irreducible with a positive row or column.

2. Generalized doubly stochastic matrices with prescribed elementary divisors I. In this section, we consider a list of complex numbers $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ satisfying certain conditions, and show how to construct a nonnegative generalized doubly stochastic matrix with spectrum Λ and prescribed elementary divisors. To date, the Brauer perturbation result (Theorem 1.3) has been exploited with success to obtain sufficient conditions for the existence and construction of nonnegative generalized stochastic matrices with prescribed elementary divisors (see [14, 15]). The novelty here is that we may also apply the Brauer result to transform a generalized stochastic matrix, not necessarily nonnegative, with given elementary divisors, into a nonnegative (positive) generalized doubly stochastic matrix, at the expense of increasing (or decreasing) the Perron eigenvalue λ_1 to $\lambda_1 + \alpha$, but keeping other elementary divisors unchanged. In particular, under certain conditions, we obtain, from a generalized stochastic matrix A, a nonnegative generalized doubly stochastic matrix with same spectrum and elementary divisors as A (that is, with no increase in the Perron eigenvalue). This is what the following result does. Since the result and its proof are somewhat involved, we start with the following example in order to illustrate the ideas and the constructive procedure followed in the proof.

Example 2.1. Let $\Lambda = \{\lambda_1, 2, 2, -1+i, -1-i, -1+i, -1-i\}$. We want to construct a nonnegative generalized doubly stochastic matrix with elementary divisors $(\lambda-\lambda_1), (\lambda+2)^2, ((\lambda+1)^2+1)^2$. We start with the 7×7 initial matrix A, with constant row sums λ_1 and the desired JCF :

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_1 - 2 - \epsilon & 2 & \epsilon & 0 & 0 & 0 & 0 \\ \lambda_1 - 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ \lambda_1 & 0 & 0 & -1 & 1 & 0 & 0 \\ \lambda_1 + 2 - \epsilon & 0 & 0 & -1 & -1 & \epsilon & 0 \\ \lambda_1 & 0 & 0 & 0 & 0 & -1 & 1 \\ \lambda_1 + 2 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}.$$

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Now we apply the Brauer Theorem (Theorem 1.3) to transform A into a nonnegative matrix $A' = A + \mathbf{e}\mathbf{r}^T \in \mathcal{CS}_{\lambda_1}$, where for $\epsilon = -1$,

$$\begin{aligned}\mathbf{r}^T &= \left(-\sum_{k=2}^{7} r_k, r_2, \ldots, r_7\right)\\ &= (-5, 0, 1, 1, 1, 1, 1).\end{aligned}$$

Then we obtain

$$A' = A + \mathbf{e}\mathbf{r}^T = \begin{bmatrix} \lambda_1 - 5 & 0 & 1 & 1 & 1 & 1 & 1 \\ \lambda_1 - 6 & 2 & 0 & 1 & 1 & 1 & 1 \\ \lambda_1 - 7 & 0 & 3 & 1 & 1 & 1 & 1 \\ \lambda_1 - 5 & 0 & 1 & 0 & 2 & 1 & 1 \\ \lambda_1 - 2 & 0 & 1 & 0 & 0 & 0 & 1 \\ \lambda_1 - 5 & 0 & 1 & 1 & 1 & 0 & 2 \\ \lambda_1 - 3 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix},$$

which, by Lemma 1.4 is a nonnegative, generalized stochastic matrix for all $\lambda_1 \geq 7$, with the same elementary divisors as A. We now describe how to perturb A' to make it nonnegative generalized doubly stochastic. We define

$$\begin{aligned}\mathbf{q}^T &= \left(-\sum_{k=2}^{7} q_k, q_2, \ldots, q_7\right), \quad \text{with}\\ q_k &= \frac{1}{7}(\lambda_1 - Re\lambda_k - Im\lambda_k - 7r_k - \epsilon) \;\; \text{for } k = 3, 6 \;\text{ in } A,\\ q_k &= \frac{1}{7}(\lambda_1 - Re\lambda_k - Im\lambda_k - 7r_k) \;\; \text{for } k = 2, 4, 5, 7 \;\text{ in } A.\end{aligned}$$

Then we have

$$\begin{aligned}q_2 &= \frac{1}{7}(\lambda_1 - 2),\; q_3 = \frac{1}{7}(\lambda_1 - 8),\; q_4 = \frac{1}{7}(\lambda_1 - 5)\\ q_5 &= \frac{1}{7}(\lambda_1 - 7),\; q_6 = \frac{1}{7}(\lambda_1 - 4),\; q_7 = \frac{1}{7}(\lambda_1 - 7)\\ q_1 &= -\sum_{k=2}^{7} q_k = -\frac{1}{7}(6\lambda_1 - 33),\end{aligned}$$

and

$$B = A' + \mathbf{e}\mathbf{q}^T$$

will be nonnegative generalized doubly stochastic, with same elementary divisors as A, provided

$$\lambda_1 - 7 - \frac{1}{7}(6\lambda_1 - 33) \geq 0, \quad \text{that is} \quad \lambda_1 \geq 16.$$

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The result illustrated above is general; namely, if the Perron eigenvalue is large enough, then $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ can be the spectrum of a nonnegative generalized doubly stochastic matrix. Before stating the result, we define

$$\begin{aligned} M &= \max_{2\leq k\leq n}\{0, Re\lambda_k + Im\lambda_k\} \\ m_k &= -\min\{0, Re\lambda_k, Im\lambda_k, \epsilon\}, \quad k = 2, \ldots, n \end{aligned} \tag{2.1}$$

Theorem 2.2. *Let* $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ *be a list of complex numbers, with* $\overline{\Lambda} = \Lambda$, $\sum\limits_{i=1}^{n} \lambda_i > 0$, $\lambda_1 > |\lambda_i|$, $i = 2, \ldots, n$, *where* $\lambda_1, \ldots, \lambda_p$ *are real and* $\lambda_{p+1}, \ldots, \lambda_n$ *are complex nonreal numbers. Let*

$$(\lambda - \lambda_1), (\lambda - \lambda_2)^{n_2}, \ldots, (\lambda - \lambda_k)^{n_k}, \;\; n_2 + \cdots + n_k = n - 1,$$

the prescribed elementary divisors (prescribed JCF). If each of the following statements hold

i) $\lambda_1 \geq M + \sum\limits_{k=2}^{n} m_k$, *with* M *and* m_k *as in* (2.1),

ii) $\lambda_1 > Re\lambda_k + Im\lambda_k + n(m_k)$, $k = 2, \ldots, n$,

iii) $Re\lambda_k + Im\lambda_k - \frac{t}{n}\epsilon < \frac{1}{n}\sum\limits_{k=1}^{n} \lambda_k$, $k = 2, \ldots, n$,

where $\epsilon < 0$, *with*

$$|\epsilon| \leq \min\left\{\min_{2\leq k\leq p}|\lambda_k|, \min_{p+1\leq k\leq n}|Im\lambda_k|, \frac{1}{t}\sum_{i=1}^{n}\lambda_i\right\}, \tag{2.2}$$

for real $\lambda_k \neq 0$, $k = 2, \ldots, p$, *or*

$$|\epsilon| \leq \min\left\{\min_{p+1\leq k\leq n}|Im\lambda_k|, \frac{1}{t}\sum_{i=1}^{n}\lambda_i\right\}, \tag{2.3}$$

if $\lambda_k = 0$, *with* λ_k *being the zero of an elementary divisor* $(\lambda - \lambda_k)^{n_k}$, $n_k \geq 2$, *and* t *being the total number of times that* ϵ *appears in the prescribed JCF, in certain positions* $(i, i+1)$, $i = 2, \ldots, n-1$, *then there exists a nonnegative generalized doubly stochastic matrix with the prescribed elementary divisors.*

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Proof. Let λ_i be real, $i = 2,\ldots,p$, and let $x_j = Re\lambda_j$ and $y_j = Im\lambda_j$, $j = p+1,\ldots,n-1$. Consider the initial matrix $A \in \mathcal{CS}_{\lambda_1}$,

$$A = \left[\begin{array}{ccccccccccc}
\lambda_1 & & & & & & & & & & \\
\vdots & \ddots & & & & & & & & & \\
\lambda_1-\lambda_l-\epsilon & & \lambda_l & \epsilon & & & & & & & \\
\lambda_1-\lambda_l & & & \lambda_l & \ddots & & & & & & \\
\vdots & & & & \ddots & \ddots & & & & & \\
\lambda_1-x_h-y_h & & & & & x_h & y_h & & & & \\
\lambda_1-x_h+y_h-\epsilon & & & & & -y_h & x_h & \epsilon & & & \\
\lambda_1-x_h-y_h & & & & & & & x_h & y_h & & \\
\lambda_1-x_h+y_h & & & & & & & -y_h & x_h & & \\
\vdots & & & & & & & & & \ddots & \\
\lambda_1-x_{n-1}+y_{n-1} & & & & & & & & & -y_{n-1} & x_{n-1}
\end{array}\right],$$

which has the desired *JCF*. First, from Theorem 1.3 we can transform A into a nonnegative matrix $A' \in \mathcal{CS}_{\lambda_1}$. Let

$$\begin{aligned} r_k &= m_k,\ k = 2,\ldots,n, \quad \text{and} \\ \mathbf{r}^T &= \left(-\sum_{k=2}^{n} r_k, r_2, r_3, \ldots, r_n\right). \end{aligned}$$

Then from condition $i)$, the matrix $A' = A + \mathbf{e}\mathbf{r}^T$ is nonnegative with constant rows sum equal to λ_1, and from Lemma 1.4, it has same elementary divisors as A. We now describe how to perturb A' to make it nonnegative generalized doubly stochastic. Define

$$\mathbf{q}^T = \left(-\sum_{k=2}^{n} q_k, q_2, q_3, \ldots, q_n\right),$$

with

$$q_k = \frac{1}{n}(\lambda_1 - Re\lambda_k - Im\lambda_k - nr_k - \epsilon)$$

if the k^{th} column of A has ϵ above $Im\lambda_k$ or above of a real λ_k, and

$$q_k = \frac{1}{n}(\lambda_1 - Re\lambda_k - Im\lambda_k - nr_k),\ k = 2,\ldots,n,$$

otherwise. Let $B = A' + \mathbf{e}\mathbf{q}^T$. Again by Lemma 1.4, B has the same elementary divisors as A' (which are the same of A). From $ii)$ it is clear that all entries of B, on columns 2 to n, are nonnegative. For the $(1,1)$ entry of B, we have, from (2.2) or

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(2.3), and assuming that ϵ appears a total of t times in positions $(i, i+1)$, that

$$\lambda_1 - \sum_{k=2}^{n} r_k - \sum_{k=2}^{n} q_k = \lambda_1 - \sum_{k=2}^{n} r_k - \frac{1}{n}\sum_{k=2}^{n}(\lambda_1 - Re\lambda_k - Im\lambda_k - nr_k) + \frac{1}{n}t\epsilon$$
$$= \frac{1}{n}\left(\sum_{k=1}^{n}\lambda_k + t\epsilon\right) \geq 0.$$

In position $(k, 1)$, $k = 2, \ldots, n$, we have from condition $iii)$ that

$$\lambda_1 - Re\lambda_k - Im\lambda_k - \sum_{k=2}^{n} r_k - \sum_{k=2}^{n} q_k = \frac{1}{n}\left(\sum_{k=1}^{n}\lambda_k + t\epsilon\right) - Re\lambda_k - Im\lambda_k \geq 0.$$

Thus, all entries in B are nonnegative. Now we show that $B, B^T \in \mathcal{CS}_{\lambda_1}$. It is clear that $B \in \mathcal{CS}_{\lambda_1} : B\mathbf{e} = (A' + \mathbf{e}\mathbf{q}^T)\mathbf{e} = \lambda_1\mathbf{e}$. Then, since the row sums are each λ_1, and from the way in which the q_k were defined, each of columns $2, \ldots, n$ have column sum λ_1, the first column sum is also λ_1. $\square$

REMARK 2.3. Observe that if the starting matrix A, in the proof of Theorem 2.2, is diagonalizable ($\epsilon = 0$ in this case), then from Theorem 2.2 (with strict inequality in $i)$), we may construct a positive diagonalizable generalized doubly stochastic matrix with spectrum Λ, and then from Theorem 1.1 we may guarantee the existence of a positive generalized doubly stochastic matrix with arbitrarily prescribed elementary divisors. In other words, any list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, which is realizable from Theorem 2.2, by a diagonalizable positive generalized doubly stochastic matrix, gives rise to a positive generalized doubly stochastic matrix for each one of the possible JCF associated with Λ.

For a list $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ of real numbers, we have the following Corollary. Observe that in this case,

$$M = \max_{2\leq k\leq n}\{0, \lambda_k\};\ m_k = -\min\{0, \lambda_k, \epsilon\},\ k = 2, \ldots, n.$$

COROLLARY 2.4. *Let* $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ *be a list of real numbers, with* $\lambda_1 > \lambda_2 \geq \ldots \geq \lambda_n$, $\sum_{i=1}^{n}\lambda_i > 0$, $\lambda_1 > |\lambda_i|$, $i = 2, \ldots, n$. *If*

i) $\lambda_1 \geq \max_{2\leq k\leq n}\{0, \lambda_k\} - \sum_{2\leq k\leq n}\min\{0, \lambda_k, \epsilon\}$

ii) $\lambda_1 > \max_{2\leq k\leq n}\{\lambda_k + nm_k\}$

iii) $\lambda_k < \frac{1}{n}\left(\sum_{i=1}^{n}\lambda_i + t\epsilon\right), \quad k = 2, \ldots, n,$

then there exist a nonnegative generalized doubly stochastic, with spectrum Λ, *for each possible* JCF *associated with* Λ.

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COROLLARY 2.5. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list real numbers of Suleimanova type, that is, a list with*

$$\lambda_1 > 0 \geq \lambda_2 \geq \ldots \geq \lambda_n.$$

If $\sum\limits_{i=1}^{n} \lambda_i > 0$ and $\lambda_1 + (n-1)\lambda_n \geq 0$, then there exists a nonnegative generalized doubly stochastic matrix with spectrum Λ, for each possible JCF associated with Λ.

The following two results allow us to estimate, from Theorem 2.2 point of view, what is the increase in the Perron eigenvalue λ_1 of a given nonnegative generalized stochastic matrix A with given elementary divisors, needed to obtain a nonnegative generalized doubly stochastic matrix B with the elementary divisors associated to $\lambda_k \neq \lambda_1$ unchanged. We point out that the estimated increase is not necessarily optimal.

THEOREM 2.6. *Let $A = (a_{ij})_{i,j=1}^{n}$ be a nonnegative generalized stochastic matrix with spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, λ_1 being the Perron eigenvalue, and elementary divisors*

$$(\lambda - \lambda_1), (\lambda - \lambda_2)^{n_2}, \ldots, (\lambda - \lambda_k)^{n_k}, \;\; n_2 + \cdots + n_k = n - 1.$$

Let

$$\alpha = \max_{1 \leq k \leq n} \sum_{i=1}^{n} a_{ik} - \lambda_1.$$

Then $\Lambda_\alpha = \{\lambda_1 + \alpha, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of a nonnegative generalized doubly stochastic matrix with the elementary divisors associated to $\lambda_k \neq \lambda_1$ unchanged.

Proof. Let $A = (a_{ij})_{i,j=1}^{n} \in \mathcal{CS}_{\lambda_1}$ and

$$c_k = \sum_{i=1}^{n} a_{ik} \;\; \text{with} \;\; c_0 = \max_{1 \leq k \leq n} c_k.$$

Observe that $\sum\limits_{k=1}^{n} c_k = n\lambda_1$ and $c_0 \geq \lambda_1$. Then we may assume, without loss of generality, that $c_0 > \lambda_1$, otherwise A is already a nonnegative generalized doubly stochastic matrix. Let

$$\mathbf{q}^T = (q_1, q_2, \ldots, q_n)\,, \;\; \text{with} \;\; q_k = \frac{1}{n}\left(c_0 - c_k\right), \; k = 1, \ldots, n.$$

Since $q_k \geq 0$, $A + \mathbf{e}\mathbf{q}^T \geq 0$. It is clear that all columns of $A + \mathbf{e}\mathbf{q}^T$ sum to c_0, and all the rows of $A + \mathbf{e}\mathbf{q}^T$ sum to $\lambda_1 + \sum\limits_{k=1}^{n} q_k = \lambda_1 + c_0 - \lambda_1 = c_0$. Then $\alpha = c_0 - \lambda_1$. Finally,

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from Lemma 1.4, $A+\mathbf{e}\mathbf{q}^T$ has the same elementary divisors associated to $\lambda_k \neq \lambda_1$ as A. ☐

As an example, let

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 6 & 0 & 0 \end{bmatrix} \in \mathcal{CS}_6 \text{ with spectrum } \left\{6, -\frac{1}{2}+\frac{\sqrt{7}}{2}i, -\frac{1}{2}-\frac{\sqrt{7}}{2}i\right\}.$$

Then $c_0 = 11$, $\alpha = c_0 - \lambda_1 = 5$, and

$$A+\mathbf{e}\mathbf{q}^T = A + \mathbf{e}\left(0, \frac{7}{3}, \frac{8}{3}\right) = \frac{1}{3}\begin{bmatrix} 9 & 13 & 11 \\ 6 & 13 & 14 \\ 12 & 7 & 8 \end{bmatrix}$$

is doubly stochastic with Perron eigenvalue 11. Next result shows that it is also possible to decrease λ_1, and still to obtain a nonnegative generalized doubly stochastic matrix.

THEOREM 2.7. *Let $A = (a_{ij})_{i,j=1}^n$ be a nonnegative generalized stochastic matrix with spectrum $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, λ_1 being the Perron eigenvalue, and elementary divisors*

$$(\lambda-\lambda_1), (\lambda-\lambda_2)^{n_2}, \ldots, (\lambda-\lambda_k)^{n_k}, \quad n_2+\cdots+n_k = n-1.$$

Let $\alpha = \max\{\alpha_1, \alpha_2\}$, with

$$\begin{aligned} &i)\ \ \alpha_1 \geq \max_{2\leq k\leq n} \sum_{i=1}^n a_{ik} - \lambda_1 \\ &ii)\ \alpha_2 \geq \sum_{k=2}^n \left(\lambda_1 - \sum_{i=1}^n a_{ik}\right) - n \min_{1\leq i\leq n} a_{i1}. \end{aligned} \tag{2.4}$$

Then $\Lambda_\alpha = \{\lambda_1+\alpha, \lambda_2, \ldots, \lambda_n\}$ is the spectrum of a nonnegative generalized doubly stochastic matrix with the elementary divisors associated to $\lambda_k \neq \lambda_1$ unchanged.

Proof. Let $A \in \mathcal{CS}_{\lambda_1}$ and

$$B = A + \alpha\mathbf{e}\mathbf{e}_1^T = \begin{bmatrix} \alpha+a_{11} & a_{12} & \cdots & a_{1n} \\ \alpha+a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha+a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

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It is clear that $B \in \mathcal{CS}_{\lambda_1+\alpha}$ is nonnegative. Let

$$q_k = \frac{1}{n}\left(\lambda_1 + \alpha - \sum_{i=1}^{n} a_{ik}\right), \; k = 2, \ldots, n$$
$$\mathbf{q}^T = \left(-\sum_{k=2}^{n} q_k, q_2, \ldots, q_n\right).$$

Then

$$B + \mathbf{e}\mathbf{q}^T = \begin{bmatrix} \alpha + a_{11} - \sum\limits_{k=2}^{n} q_k & a_{12} + q_2 & \cdots & a_{1n} + q_n \\ \alpha + a_{21} - \sum\limits_{k=2}^{n} q_k & a_{22} + q_2 & \cdots & a_{2n} + q_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha + a_{n1} - \sum\limits_{k=2}^{n} q_k & a_{n2} + q_2 & \cdots & a_{nn} + q_n \end{bmatrix}.$$

From (2.4) i), $q_k \geq 0$ for $k = 2, \ldots, n$, and from (2.4) ii), we have

$$\alpha + \min_{1 \leq i \leq n} a_{i1} - \sum_{k=2}^{n} q_k \geq 0.$$

Thus, $B + \mathbf{e}\mathbf{q}^T$ is nonnegative and from Lemma 1.4 it has the same elementary divisors as B. Besides $(B + \mathbf{e}\mathbf{q}^T)\mathbf{e} = (\lambda_1 + \alpha)\mathbf{e}$, and from the way in which the q_k were defined the entries of $B + \mathbf{e}\mathbf{q}^T$ on each column from 2 to n sum to $\lambda_1 + \alpha$. Then the entries on the first column of $B + \mathbf{e}\mathbf{q}^T$ also sum to $\lambda_1 + \alpha$. ∎

Consider the matrix

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 6 & 0 & 0 \end{bmatrix} \text{ from the previous example.}$$

Since $\alpha_1 = -2$, $\alpha_2 = -1$, then $\alpha = -1$ and

$$B = A - \mathbf{e}\mathbf{e}_1^T = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 5 & 0 & 0 \end{bmatrix}, \; \mathbf{q}^T = \left(-1, \frac{1}{3}, \frac{2}{3}\right), \text{ with}$$

$$B + \mathbf{e}\mathbf{q}^T = \frac{1}{3}\begin{bmatrix} 3 & 7 & 5 \\ 0 & 7 & 8 \\ 12 & 1 & 2 \end{bmatrix} \in \mathcal{CS}_5 \text{ is doubly stochastic with}$$

spectrum $\left\{5, -\frac{1}{2} + \frac{\sqrt{7}}{2}i, -\frac{1}{2} - \frac{\sqrt{7}}{2}i\right\}$.

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3. Generalized doubly stochastic matrices with prescribed elementary divisors II. In this section, we extend Theorem 2.2 to setting that there exists a partition of the given list $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ into sublists Λ_k, along with suitably chosen Perron eigenvalues, which are realizable by nonnegative (positive) generalized doubly stochastic matrices A_k with certain of the prescribed elementary divisors, and (ii) an $r \times r$, $r < n$, nonnegative (positive) generalized doubly stochastic matrix exists with diagonal entries being the Perron eigenvalues of the matrices A_k, and with certain of the prescribed elementary divisors. Our result is built on the basis of Rado Theorem to generate sufficient conditions, which are more general than the conditions of Theorem 2.1. The constructive nature of the proof allows us to compute a matrix solution.

THEOREM 3.1. *Let* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *be a list of complex numbers with* $\overline{\Lambda} = \Lambda$, $\sum\limits_{i=1}^{n} \lambda_i > 0$, $\lambda_1 > |\lambda_i|$, $i = 2, \ldots, n$. *Suppose there exists a partition* $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_{p_0}$ *with*

$$\begin{aligned} \Lambda_0 &= \{\lambda_{01}, \lambda_{02}, \ldots, \lambda_{0p_0}\}, \;\; \lambda_{01} = \lambda_1 \\ \Lambda_k &= \{\lambda_{k1}, \lambda_{k2}, \ldots, \lambda_{kp_k}\}, \;\; p_k = p, \; k = 1, \ldots, p_0, \end{aligned}$$

where the lists Λ_k, $k = 1, \ldots, p_0$, *have cardinality* p, *in such a way that:*

i) For each $k = 1, \ldots, p_0$, *there exists a list*

$$\Gamma_k = \{\omega_k, \lambda_{k1}, \ldots, \lambda_{kp_k}\}, \quad 0 < \omega_k < \lambda_1$$

which is realizable by a nonnegative (positive) matrix A_k, *with* $A_k, A_k^T \in \mathcal{CS}_{\omega_k}$, *and with prescribed elementary divisors*

$$(\lambda - \omega_k), (\lambda - \lambda_{k1})^{n_{k1}}, \ldots, (\lambda - \lambda_{kj})^{n_{kj}}, \;\; n_{k1} + \cdots + n_{kj} = p_k,$$

and

ii) There exists a $p_0 \times p_0$ *nonnegative (positive) matrix* $B = (b_{ij})_{i,j=1}^{n}$ *such that* $B, B^T \in \mathcal{CS}_{\lambda_1}$, *with spectrum* Λ_0 *and diagonal entries* $\omega_1, \omega_2, \ldots, \omega_{p_0}$, *and with certain of the prescribed elementary divisors.*

Then there exists a nonnegative (positive) matrix A, *such that* $A, A^T \in \mathcal{CS}_{\lambda_1}$, *with spectrum* Λ *and with the prescribed elementary divisors associated to the lists* Λ_k.

Proof. From $i)$ let

$$G = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_{p_0} \end{bmatrix},$$

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where A_k with $A_k, A_k^T \in \mathcal{CS}_{\omega_k}$ is nonnegative (positive), with spectrum Γ_k and prescribed elementary divisors associated to the list Γ_k,. From ii), let $C = B - \Omega$, where $\Omega = diag = \{\omega_1, \omega_2, \ldots, \omega_{p_0}\}$, and let $X = [\mathbf{x}_1 \mid \cdots \mid \mathbf{x}_{p_0}]$ be the matrix whose columns are the normalized eigenvectors of G corresponding to the eigenvalues $\omega_1, \omega_2, \ldots, \omega_{p_0}$, respectively. Observe that $\mathbf{x}_k$, $k = 1, \ldots, p_0$, is nonnegative with entries $\frac{1}{\sqrt{p}}$ and zeros,

$$\mathbf{x}_k^T = \Big(0, \ldots, 0, \underbrace{\frac{1}{\sqrt{p}}, \ldots, \frac{1}{\sqrt{p}}}_{p \text{ times}}, 0, \ldots, 0\Big).$$

Let $\widetilde{C} = CX^T$. Then

$$X\widetilde{C} = XCX^T = \frac{1}{p}\begin{bmatrix} 0 & B_{12} & \cdots & B_{1p_0} \\ B_{21} & 0 & \ddots & B_{2p_0} \\ \vdots & \ddots & \ddots & \vdots \\ B_{p_0 1} & \cdots & B_{p_0,p_0-1} & 0 \end{bmatrix}$$

where

$$B_{ij} = \begin{bmatrix} b_{ij} & b_{ij} & b_{ij} & \ddots & b_{ij} \\ b_{ij} & b_{ij} & b_{ij} & \ddots & b_{ij} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{ij} & b_{ij} & b_{ij} & \cdots & b_{ij} \end{bmatrix}$$

is positive with $\frac{1}{p}B_{ij} \in \mathcal{CS}_{b_{ij}}$. Then from Theorem 1.2, $A = G + X\widetilde{C}$ is nonnegative (positive) with spectrum Λ (the Rado perturbation $G + X\widetilde{C}$ changes $\omega_1, \ldots, \omega_{p_0}$ by the eigenvalues of $\Omega + \widetilde{C}X = \Omega + CX^TX = \Omega + C = B$). Moreover,

$$A\mathbf{e} = (G + XCX^T)\mathbf{e} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_1 \\ \vdots \\ \omega_{p_0} \\ \vdots \\ \omega_{p_0} \end{bmatrix} + \begin{bmatrix} \lambda_1 - \omega_1 \\ \vdots \\ \lambda_1 - \omega_1 \\ \vdots \\ \lambda_1 - \omega_{p_0} \\ \vdots \\ \lambda_1 - \omega_{p_0} \end{bmatrix} = A^T\mathbf{e} = \lambda_1\mathbf{e}.$$

Then $A = G + X\widetilde{C}$ is an $n \times n$ nonnegative (positive) generalized doubly stochastic matrix, with the prescribed elementary divisors. $\square$

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In order to apply Theorem 3.1, we need a $p_0 \times p_0$ nonnegative (positive) generalized doubly stochastic matrix B with spectrum Λ_0 and diagonal entries $\omega_1, \omega_2, \ldots, \omega_{p_0}$. The existence and construction of such a matrix is an open and hard problem, for which so far we have only partial responses. For $p_0 = 2$, the nonnegative (positive) generalized doubly stochastic matrix is necessarily symmetric, with $\omega_1 = \omega_2$ and eigenvalues $\lambda_1, \lambda_2 = 2\omega_1 - \lambda_1$,

$$B = \begin{bmatrix} \omega_1 & \lambda_1 - \omega_1 \\ \lambda_1 - \omega_1 & \omega_1 \end{bmatrix}.$$

For the case $p_0 = 3$, we have a sufficient condition:

Lemma 3.2. *Let* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$ *and* $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ *with* $\lambda_1 > |\lambda_i|$, $i = 2, 3$, *and* $\lambda_1 \geq \omega_1 \geq \omega_2 \geq \omega_3 \geq 0$. *If*

i) $(\lambda_2 - \lambda_3)^2 \leq 4\left[(\omega_1 - \omega_2)^2 + (\omega_1 - \omega_2)(\omega_2 - \omega_3) + (\omega_2 - \omega_3)^2\right]$
ii) $\lambda_1 - \omega_1 - y \geq 0$
iii) $\omega_3 - \omega_2 + y \geq 0$,

where

$$y = \frac{1}{2}(\lambda_1 - \omega_1 + \omega_2 - \omega_3) + \frac{1}{6}\sqrt{3}\sqrt{\alpha - (\lambda_2 - \lambda_3)^2}, \quad \textit{with}$$
$$\alpha = 4\left[(\omega_1 - \omega_2)^2 + (\omega_1 - \omega_2)(\omega_2 - \omega_3) + (\omega_2 - \omega_3)^2\right],$$

then there exists a 3×3 *nonnegative generalized doubly stochastic matrix with eigenvalues* $\lambda_1, \lambda_2, \lambda_3$ *and diagonal entries* $\omega_1, \omega_2, \omega_3$.

Proof. Consider the matrix

$$B = \begin{bmatrix} \omega_1 & \lambda_1 - \omega_1 - y & y \\ \omega_3 - \omega_2 + y & \omega_2 & \lambda_1 - \omega_3 - y \\ \lambda_1 - \omega_1 + \omega_2 - \omega_3 - y & \omega_1 - \omega_2 + y & \omega_3 \end{bmatrix}.$$

Clearly $B, B^T \in \mathcal{CS}_{\lambda_1}$ with the prescribed diagonal entries. From $ii)$ it follows that $\lambda_1 - \omega_3 - y \geq 0$ and $\lambda_1 - \omega_1 + \omega_2 - \omega_3 - y \geq \omega_2 - \omega_3 \geq 0$. From $iii)$ $\omega_3 - \omega_2 + y \geq 0$, and from $i)$ $y \geq 0$ and then $\omega_1 - \omega_2 + y \geq 0$. Thus, B is nonnegative. Finally a simple algebraic calculation shows that B has the prescribed eigenvalues. ◻

Example 3.3. Let $\Lambda = \{12, 5, 2, 2, 0, -1, -1, -2 + i, -2 - i\}$. To construct a positive generalized doubly stochastic matrix A with elementary divisors

$$(\lambda - 12),\ (\lambda - 5),\ (\lambda - 2)^2,\ \lambda,\ (\lambda + 1)^2,\ \lambda^2 + 4\lambda + 5$$

we take the partition

$$\Lambda_0 = \{12, 5, 0\},\ \ \Gamma_1 = \{7, 2, 2\},\ \ \Gamma_2 = \{6, -2 + i, -2 - i\},\ \ \Gamma_3 = \{4, -1, -1\}$$

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and from Theorem 2.2, we compute the positive generalized doubly stochastic matrices

$$A_1 = \frac{1}{3}\begin{bmatrix} 10 & 5 & 6 \\ 7 & 11 & 3 \\ 4 & 5 & 12 \end{bmatrix}, \ A_2 = \frac{1}{3}\begin{bmatrix} 2 & 9 & 7 \\ 5 & 3 & 10 \\ 11 & 6 & 1 \end{bmatrix}, \ A_3 = \frac{1}{3}\begin{bmatrix} 1 & 5 & 6 \\ 7 & 2 & 3 \\ 4 & 5 & 3 \end{bmatrix}$$

with spectra $\Gamma_1, \Gamma_2, \Gamma_3$, and elementary divisors $(\lambda-7), (\lambda-2)^2$; $(\lambda-6), \lambda^2+4\lambda+5$; and $(\lambda-4), (\lambda+1)^2$, respectively. Moreover, from Lemma 3.2 we compute

$$B = \begin{bmatrix} 7 & 1 & 4 \\ 2 & 6 & 4 \\ 3 & 5 & 4 \end{bmatrix} \text{ with spectrum } \Lambda_0, \text{ and } C = \begin{bmatrix} 0 & 1 & 4 \\ 2 & 0 & 4 \\ 3 & 5 & 0 \end{bmatrix}.$$

Finally,

$$XCX^T = \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 4 & 4 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 & 4 & 4 & 4 \\ 0 & 0 & 0 & 1 & 1 & 1 & 4 & 4 & 4 \\ 2 & 2 & 2 & 0 & 0 & 0 & 4 & 4 & 4 \\ 2 & 2 & 2 & 0 & 0 & 0 & 4 & 4 & 4 \\ 2 & 2 & 2 & 0 & 0 & 0 & 4 & 4 & 4 \\ 3 & 3 & 3 & 5 & 5 & 5 & 0 & 0 & 0 \\ 3 & 3 & 3 & 5 & 5 & 5 & 0 & 0 & 0 \\ 3 & 3 & 3 & 5 & 5 & 5 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & A_3 \end{bmatrix} + XCX^T$$

is a positive generalized doubly stochastic matrix with the prescribed elementary divisors.

As we did in Section 2 with Theorem 2.2, we may also estimate the increase α in the Perron eigenvalue λ_1, needed to obtain, from Theorem 3.1 point of view, a nonnegative generalized doubly stochastic matrix without changing the elementary divisors associated to $\lambda_k \neq \lambda_1$. In this case, $\alpha = \sum_{k=1}^{p_0} \alpha_k$, where α_k is the increase in the Perron eigenvalue ω_k of the nonnegative generalized doubly stochastic submatrix A_k, $k = 1, \ldots, p_0$.

4. More on stochastic and doubly stochastic matrices.

We start this section with the following lemma:

LEMMA 4.1. *Let $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a list of complex numbers, which is realizable by a nonnegative generalized doubly stochastic matrix. Then for all $\epsilon > 0$,*

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the list

$$\Lambda_\epsilon = \{\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n\}$$

is also realizable by a positive generalized doubly stochastic matrix.

Proof. Let A be nonnegative generalized doubly stochastic with Perron eigenvalue λ_1. Let $\mathbf{q}^T = (\frac{\epsilon}{n}, \ldots, \frac{\epsilon}{n})$. Then it is clear that $A + \mathbf{e}\mathbf{q}^T$ is positive. Moreover,

$$(A + \mathbf{e}\mathbf{q}^T)\mathbf{e} = A\mathbf{e} + \mathbf{q}^T\mathbf{e}\mathbf{e} = \lambda_1\mathbf{e}+\epsilon\mathbf{e} = (\lambda_1+\epsilon)\mathbf{e} \text{ and}$$
$$(A + \mathbf{e}\mathbf{q}^T)^T\mathbf{e} = A^T\mathbf{e} + \mathbf{q}\mathbf{e}^T\mathbf{e} = \lambda_1\mathbf{e}+n\mathbf{q} = \lambda_1\mathbf{e}+\epsilon\mathbf{e} = (\lambda_1+\epsilon)\mathbf{e},$$

and the result follows from Theorem 1.1. ∎

In [1] and [6], the authors show that if A is a nonnegative irreducible matrix with a positive column or row, then A is similar to a positive matrix. The following result gives a sufficient condition on a list of complex numbers $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$, for the existence and construction of a nonnegative generalized stochastic matrix with spectrum Λ, for each possible JCF associated with Λ.

THEOREM 4.2. *Let* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *be a list of complex numbers with* $\overline{\Lambda} = \Lambda$, $\sum_{i=1}^{n} \lambda_i > 0$, $\lambda_1 > |\lambda_i|$, $i = 2, \ldots, n$. *Let*

$$M = \max_{2\le k\le n}\{0, Re\lambda_k + Im\lambda_k\};$$
$$m_k = -\min\{0, Re\lambda_k, Im\lambda_k,\}, \quad k = 2, \ldots, n.$$

Then, if

$$\lambda_1 > M + \sum_{k=2}^{n} m_k, \tag{4.1}$$

there exists a positive generalized stochastic matrix with spectrum Λ, *for each possible* JCF *associated with* Λ.

Proof. Let $A \in \mathcal{CS}_{\lambda_1}$ be the initial matrix in Theorem 2.2, with diagonal JCF. Since (4.1) holds, then we can take $r_k > 0$, $k = 2, \ldots, n$, with $r_1 = -\sum_{k=2}^{n} r_k$, in such a way that the entries on the first row and first column of $A + \mathbf{e}\mathbf{r}^T$ are all positive. Then $A+\mathbf{e}\mathbf{r}^T$ is diagonalizable irreducible nonnegative generalized stochastic. Hence, from the result in [1], mentioned above, $B = A + \mathbf{e}\mathbf{r}^T$ is similar to a diagonalizable positive generalized stochastic matrix, and from Theorem 1.1, there exists a positive generalized stochastic matrix with spectrum Λ and arbitrarily prescribed elementary divisors associated to Λ. ∎

COROLLARY 4.3. *Let* $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *be a list of complex numbers with* $\overline{\Lambda} = \Lambda$, $\sum_{i=1}^{n} \lambda_i > 0$, $\lambda_1 > |\lambda_i|$, $i = 2, \ldots, n$. *Then, if the conditions* (4.1) *and* (2.4),

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with strict inequality, are satisfied, there exists a positive generalized doubly stochastic matrix with spectrum Λ *and arbitrarily prescribed elementary divisors associated to* Λ.

Proof. From the proof of Theorem 4.2, there exists a diagonalizable irreducible nonnegative generalized stochastic matrix $B = A + \mathbf{e}\mathbf{r}^T$, which is similar to a diagonalizable positive generalized stochastic matrix C. Since (2.4) holds, with strict inequality, then $C + \mathbf{e}\mathbf{q}^T$, for an appropriate vector $\mathbf{q}$, is a diagonalizable positive generalized doubly stochastic matrix, and from Theorem 1.1 the result follows. □

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