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ON WILKINSON'S PROBLEM FOR MATRIX PENCILS*

SK. SAFIQUE AHMAD† AND RAFIKUL ALAM‡

Abstract. Suppose that an n-by-n regular matrix pencil $A - \lambda B$ has n distinct eigenvalues. Then determining a defective pencil $E-\lambda F$ which is nearest to $A-\lambda B$ is widely known as Wilkinson's problem. It is shown that the pencil $E - \lambda F$ can be constructed from eigenvalues and eigenvectors of $A - \lambda B$ when $A - \lambda B$ is unitarily equivalent to a diagonal pencil. Further, in such a case, it is proved that the distance from $A - \lambda B$ to $E - \lambda F$ is the minimum "gap" between the eigenvalues of $A - \lambda B$. As a consequence, lower and upper bounds for the "Wilkinson distance" $\mathrm{d}(\mathrm{L})$ from a regular pencil $\mathrm{L}(\lambda)$ with distinct eigenvalues to the nearest non-diagonalizable pencil are derived. Furthermore, it is shown that $\mathrm{d}(\mathrm{L})$ is almost inversely proportional to the condition number of the most ill-conditioned eigenvalue of $\mathrm{L}(\lambda)$.

Key words. Matrix pencil, Pseudospectrum, Backward error, Multiple eigenvalue, Defective pencil, Wilkinson's problem.

AMS subject classifications. 65F15, 15A22, 15A60, 15A18, 65F35.

1. Introduction. Let $\mathrm{L}(\lambda) := A - \lambda B$ be an n-by-n regular matrix pencil with n distinct eigenvalues. Then $\mathrm{L}(\lambda)$ is diagonalizable, that is, there are nonsingular matrices Y and X such that Y^*AX and Y^*BX are diagonal matrices. The columns of Y and X are left and right eigenvectors of $\mathrm{L}(\lambda)$, respectively. A pencil is *defective* if it is not diagonalizable. Define

$$\mathrm{d}(\mathrm{L}) := \inf\{|\!|\!|\Delta\mathrm{L}|\!|\!| : \mathrm{L} + \Delta\mathrm{L} \text{ is defective}\}, \tag{1.1}$$

where $|\!|\!|\cdot|\!|\!|$ is a suitable norm on the vector space of pencils, see [2, 3, 4]. Thus, $\mathrm{d}(\mathrm{L})$ is the radius of the largest open ball centred at $\mathrm{L}(\lambda)$ consisting of pencils which are diagonalizable.

The problem of determining $\mathrm{d}(\mathrm{L})$ and a pencil $\Delta\mathrm{L}(\lambda)$, if it exists, such that the infimum in (1.1) is attained at $\mathrm{L}(\lambda)+\Delta\mathrm{L}(\lambda)$ is known as *Wilkinson's Problem* [14, 26]. Wilkinson's problem for matrices has been studied extensively over the years [5, 6, 7, 8, 11, 12, 13, 15, 17, 18, 21, 22, 23, 24, 25, 26]. See [9] for the existence of a nearest defective matrix and an algorithm that computes a solution to Wilkinson's problem for matrices.

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†Department of Mathematics, Indian Institute of Technology Indore, Indore-452017, INDIA (safique@iiti.ac.in, safique@gmail.com).
‡Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati-781 039, INDIA (rafik@iitg.ernet.in, rafikul@yahoo.com).

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Wilkinson's problem for matrix pencils has been investigated in [4] by considering the norm $\|\mathrm{L}\| := \sqrt{\|A\|_2^2 + \|B\|_2^2}$ and in [10] by considering the norm $\|\mathrm{L}\| := \max(\|A\|_2, \|B\|_2)$, see also [16]. It is shown in [4] that $\mathrm{d(L)}$ is the smallest value of ϵ for which at least two components of the ϵ-pseudospectrum [4] of L coalesce. Further, if $\lambda \in \mathbb{C}$ is a point of coalescence of the components, then λ is a multiple eigenvalue of a pencil $\mathrm{L} + \Delta\mathrm{L}$ (constructed from the SVD of $\mathrm{L}(\lambda)$) such that $\mathrm{d(L)} = \|\Delta\mathrm{L}\|$. Furthermore, $\mathrm{L} + \Delta\mathrm{L}$ is defective when the smallest singular value of $\mathrm{L}(\lambda)$ is simple. Therefore, *generically* the infimum in (1.1) is attained. In the *nongeneric* case, that is, when the smallest singular value of $\mathrm{L}(\lambda)$ is multiple, whether or not the infimum in (1.1) is attained remains inconclusive in [4] and is still an open problem. For example, the nongeneric case always arises in the special case when L is unitarily equivalent to a diagonal pencil.

The main contributions of this paper are as follows. We show that the infimum in (1.1) is attained when $\mathrm{L}(\lambda)$ is unitarily equivalent to a diagonal pencil (we refer to such pencils as nongeneric pencils), that is,

$$U^*AV = \mathrm{diag}(\alpha_i) \quad \text{and} \quad U^*BV = \mathrm{diag}(\beta_i) \tag{1.2}$$

for some unitary matrices U and V. Further, we describe a construction of the nearest defective pencil $\mathrm{L}(\lambda)+\Delta\mathrm{L}(\lambda)$ from eigenvalues and eigenvectors of $\mathrm{L}(\lambda)$. We introduce the notion of a "gap" between eigenvalues of $\mathrm{L}(\lambda)$ and show that $\mathrm{d(L)}$ is the minimum gap between the eigenvalues of $\mathrm{L}(\lambda)$.

Note that $\mathrm{d(L)}$ is the distance from $\mathrm{L}(\lambda)$ to the nearest non-diagonalizable pencil. Consequently, if a pencil $\Delta\mathrm{L}(\lambda)$ is such that $\|\Delta\mathrm{L}\| < \mathrm{d(L)}$ then the perturbed pencil $\mathrm{L}(\lambda) + \Delta\mathrm{L}(\lambda)$ remains diagonalizable. Thus, in a sense, $\mathrm{d(L)}$ is the safety radius for continuous evolution of diagonalizations of $\mathrm{L}(\lambda)$. Hence, a lower bound of $\mathrm{d(L)}$ can be employed for computing an eigendecomposition of $\mathrm{L}(\lambda)$ stably[14].

We derive computable upper and lower bounds for $\mathrm{d(L)}$. We illustrate effectiveness of these bounds by considering a few numerical examples. Further, we show that $\mathrm{d(L)}$ is almost inversely proportional to the condition number of the most sensitive eigenvalue of $\mathrm{L}(\lambda)$ - a fact which is well known for matrices.

2. Preliminaries. We consider nonhomogeneous matrix pencil of the form $\mathrm{L}(z) := A - zB$ as well as homogeneous matrix pencils of the form $\mathrm{L}(c, s) := cA - sB$. A pencil L is said to be regular if $\det(\mathrm{L}(z)) \neq 0$ for some $z \in \mathbb{C}$. The spectrum of a regular pencil L, denoted by $\Lambda(\mathrm{L})$, is given by

$$\Lambda(\mathrm{L}) := \begin{cases} \{(c, s) \in \mathbb{C}^2 \setminus \{0\} : \det(\mathrm{L}(c, s)) = 0\}, & \mathrm{L} \text{ homogeneous}, \\ \{\lambda \in \mathbb{C} : \det(\mathrm{L}(\lambda)) = 0\}, & \mathrm{L} \text{ nonhomogeneous}. \end{cases}$$

If $\mathrm{L}(z) = A - zB$ and B is singular then it is customary to consider $\Lambda(\mathrm{L})$ as a subset of $\mathbb{C}_\infty$, the one-point compactification of $\mathbb{C}$, and add ∞ to $\Lambda(\mathrm{L})$. For completeness, we

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consider both homogeneous and nonhomogeneous matrix pencils. For homogeneous L, normalizing $(c, s) \in \Lambda(\mathrm{L})$ as $|c|^2 + |s|^2 = 1$, we consider $\Lambda(\mathrm{L})$ as a subset of the unit sphere $\mathbb{S}^1 := \{(c, s) \in \mathbb{C}^2 : |c|^2 + |s|^2 = 1\}$. An infinite eigenvalue of L, if any, is then represented by $(0, 1)$.

We equip the vector space of n-by-n matrix pencils with the following norms:

$$\|\mathrm{L}\|_2 := \|[A, B]\|_2, \quad \|\mathrm{L}\|_F := \|[A, B]\|_F \quad \text{and} \quad \|\mathrm{L}\|_{\ell^2} := (\|A\|_2^2 + \|B\|_2^2)^{1/2}, \tag{2.1}$$

where $\|\cdot\|_F$ denotes the Frobenius norm. We write $\|\mathrm{L}\|_M$ for $M = 2$, $M = F$ and $M = \ell^2$. See [3, 4] for various choices of pencil norms. The ϵ-pseudospectrum of L is given by [4]

$$\Lambda_\epsilon(\mathrm{L}) = \bigcup \{\Lambda(\mathrm{L} + \Delta\mathrm{L}) : \|\Delta\mathrm{L}\|_M \leq \epsilon\}.$$

Now defining $\eta(\lambda, \mathrm{L}) := \min\{\|\Delta\mathrm{L}\|_M : \lambda \in \Lambda(\mathrm{L}+\Delta\mathrm{L})\}$ and $\eta(c, s, \mathrm{L}) := \min\{\|\Delta\mathrm{L}\|_M : (c, s) \in \Lambda(\mathrm{L} + \Delta\mathrm{L})\}$ we have [4]

$$\eta(\lambda, \mathrm{L}) = \frac{\sigma_{\min}(\mathrm{L}(\lambda))}{\sqrt{1 + |\lambda|^2}} \quad \text{and} \quad \eta(c, s, \mathrm{L}) = \frac{\sigma_{\min}(\mathrm{L}(c, s))}{\sqrt{|c|^2 + |s|^2}} \tag{2.2}$$

for $\|\cdot\|_F$ and $\|\cdot\|_{\ell^2}$ norms. The equality in (2.2) holds for $\|\cdot\|_2$ as well and follows from Proposition 2.1 which shows that the ϵ-pseudospectrum $\Lambda_\epsilon(\mathrm{L})$ is the same for all three norms.

PROPOSITION 2.1. *The pseudospectrum* $\Lambda_\epsilon(\mathrm{L})$ *is the same for all three norms.*

Proof. We outline the proof for nonhomogeneous pencils. Suppose that $\mathrm{L}(z) = A - zB$. Let $\lambda \in \Lambda(\mathrm{L} + \Delta\mathrm{L})$. Then $\sigma_{\min}(\mathrm{L}(\lambda)) \leq \|\Delta\mathrm{L}(\lambda)\|_2 \leq \|\Delta\mathrm{L}\|_M \sqrt{1 + |\lambda|^2}$ for $M = 2, F, \ell^2$. Hence, $\eta(\lambda, \mathrm{L}) \geq \sigma_{\min}(\mathrm{L}(\lambda))/\sqrt{1 + |\lambda|^2}$. On the other hand, let $\lambda \in \mathbb{C}$. Let u and v, respectively, be left and right singular vectors of the matrix $\mathrm{L}(\lambda)$ corresponding to the smallest singular value $\sigma_{\min}(\mathrm{L}(\lambda))$. Then defining

$$[\Delta A,\ \Delta B] := \sigma_{\min}(\mathrm{L}(\lambda)) \frac{[-1,\ \overline{\lambda}] \otimes uv^*}{1 + |\lambda|^2} \tag{2.3}$$

and considering the pencil $\Delta\mathrm{L}(z) = \Delta A - z\Delta B$, it follows that $\lambda \in \Lambda(\mathrm{L} + \Delta\mathrm{L})$ and $\|\Delta\mathrm{L}\|_M = \sigma_{\min}(\mathrm{L}(\lambda))/\sqrt{1 + |\lambda|^2}$ for $M = 2, F, \ell^2$. Indeed, $\mathrm{L}(\lambda)v = \sigma_{\min}(\mathrm{L}(\lambda))u$ and $\Delta\mathrm{L}(\lambda)v = -\sigma_{\min}(\mathrm{L}(\lambda))u$ show that $(\mathrm{L}(\lambda) + \Delta\mathrm{L}(\lambda))v = 0$. Hence, $\eta(\lambda, \mathrm{L}) = \sigma_{\min}(\mathrm{L}(\lambda))/\sqrt{1 + |\lambda|^2}$ for all three norms in (2.1). Finally, note that $\lambda \in \Lambda_\epsilon(\mathrm{L})$ if and only if $\eta(\lambda, \mathrm{L}) \leq \epsilon$. Hence the result follows. The proof is similar for homogeneous pencils. ∎

This shows that we could choose any one of the three norms in (2.1) for the pseudospectra of L. As we shall see, d(L) is also the same for all three norms when

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L is a nongeneric pencil. For a block diagonal pencil $\mathrm{L} = \mathrm{diag}(\mathrm{L}_1, \mathrm{L}_2)$, it is easily seen [4] that

$$\Lambda_\epsilon(\mathrm{L}) = \Lambda_\epsilon(\mathrm{L}_1) \cup \Lambda_\epsilon(\mathrm{L}_2). \tag{2.4}$$

Note that $\Lambda_\epsilon(\mathrm{L})$ is invariant under unitary equivalence of L, that is, $\Lambda_\epsilon(U\mathrm{L}V) = \Lambda_\epsilon(\mathrm{L})$ for unitary matrices U and V. Therefore, when L is nongeneric and satisfies (1.2), by (2.2) and (2.4) we have

$$\Lambda_\epsilon(\mathrm{L}) = \bigcup_{j=1}^{n} \Omega_\epsilon(\alpha_j, \beta_j), \tag{2.5}$$

where

$$\Omega_\epsilon(\alpha_j, \beta_j) := \begin{cases} \{z \in \mathbb{C} : |\alpha_j - z\beta_j| \le \epsilon \, \|(1, z)\|_2\}, & \text{L nonhomogeneous,} \\ \{(c, s) \in \mathbb{S}^1 : |c\,\alpha_j - s\,\beta_j| \le \epsilon\}, & \text{L homogeneous,} \end{cases} \tag{2.6}$$

are components of $\Lambda_\epsilon(\mathrm{L})$. Now, supposing that L has n distinct eigenvalues, it follows that for small ϵ the components of $\Lambda_\epsilon(\mathrm{L})$ given in (2.6) are disjoint. Hence, in such a case, we have $\mathrm{d}(\mathrm{L}) > \epsilon$. We require the notion of a gap between two points in $\mathbb{C}^2$ to determine the smallest value of ϵ for which at least two components of $\Lambda_\epsilon(\mathrm{L})$ coalesce.

3. Gap between two points in $\mathbb{C}^2$. Let $\mathbb{T}$ denote the unit circle in $\mathbb{C}$, that is, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Given $x := (\alpha_1, \alpha_2) \in \mathbb{C}^2$ and $y := (\beta_1, \beta_2) \in \mathbb{C}^2$, we define the gap between x and y by

$$\mathrm{Gap}(x, y) := \min_{\zeta \in \mathbb{T}} \frac{|\alpha_1\beta_2 - \alpha_2\beta_1|}{\|x - \zeta y\|_2} = \frac{|\alpha_1\beta_2 - \alpha_2\beta_1|}{\sqrt{\|x\|_2^2 + \|y\|_2^2 + 2\,|\langle x,\, y\rangle|}}, \tag{3.1}$$

where $\langle x,\, y\rangle$ is the usual inner product on $\mathbb{C}^2$, that is, $\langle x,\, y\rangle = y^*x$. The minimum in (3.1) is attained at $\zeta_{\min} := -\mathrm{sign}(\langle y,\, x\rangle)$, where $\mathrm{sign}(z) := \overline{z}/|z|$ if $z \neq 0$ and $\mathrm{sign}(0) := 1$. Indeed, $\|x - \zeta y\|_2^2 = \|x\|_2^2 + \|y\|_2^2 - 2\mathrm{Re}\langle x,\, \zeta y\rangle$ is maximized when $\zeta = \zeta_{\min}$. Thus, $\max_{\zeta \in \mathbb{T}} \|x - \zeta y\|_2 = \sqrt{\|x\|_2^2 + \|y\|_2^2 + 2|\langle x,\, y\rangle|}$, which proves (3.1).

We also write $\mathrm{Gap}(x, y)$ as $\mathrm{Gap}(\alpha_1, \alpha_2; \beta_1, \beta_2)$. If x and y are normalized, that is, $\|x\|_2 = \|y\|_2 = 1$ then it follows that

$$\mathrm{Gap}(x, y) = \frac{1}{2}|\alpha_1\beta_2 - \alpha_2\beta_1| \sec(\theta/2), \tag{3.2}$$

where $\theta \in [0, \pi/2]$ is such that $\cos(\theta) = |\langle x,\, y\rangle|$. Some essential properties of $\mathrm{Gap}(x, y)$ are as follows:

- We have $\mathrm{Gap}(x, y) = \mathrm{Gap}(y, x)$ and $\mathrm{Gap}(\alpha_1, \alpha_2; \beta_1, \beta_2) = \mathrm{Gap}(\alpha_2, \alpha_1; \beta_2, \beta_1)$.
- We have $\mathrm{Gap}(x, y) = 0 \iff x = ty$ for some $t \in \mathbb{C}$. Equivalently, $\mathrm{Gap}(x, y) = 0 \iff \alpha_1/\alpha_2 = \beta_1/\beta_2$ in $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$.

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- We have $\mathrm{Gap}(tx, ty) = |t| \,\mathrm{Gap}(x, y)$ for $t \in \mathbb{C}$.

The chordal metric [19] on $\mathbb{CP}^1$, the complex projective space, or on $\mathbb{C}_\infty$ is closely related to gap. Indeed, treating $x := (\alpha_1, \alpha_2)$ and $y := (\beta_1, \beta_2)$ as points in $\mathbb{CP}^1$ the chordal distance between x and y is given by

$$\mathrm{chord}(x, y) := \frac{|\alpha_1\beta_2 - \alpha_2\beta_1|}{\|x\|_2 \,\|y\|_2}.$$

The chordal distance between α_1/α_2 and β_1/β_2 as points in $\mathbb{C}_\infty$ is given by

$$\mathrm{chord}(\alpha_1/\alpha_2,\, \beta_1/\beta_2) = \frac{|\alpha_1\beta_2 - \alpha_2\beta_1|}{\|x\|_2 \,\|y\|_2} = \frac{|\alpha_1/\alpha_2 - \beta_1/\beta_2|}{\sqrt{1 + |\alpha_1/\alpha_2|^2}\sqrt{1 + |\beta_1/\beta_2|^2}}.$$

Since $\|x\|_2^2+\|y\|_2^2+2\,|\langle x,\, y\rangle| \geq 2\|x\|_2\|y\|_2$ and $\|x\|_2^2+\|y\|_2^2+2\,|\langle x,\, y\rangle| \leq (\|x\|_2+\|y\|_2)^2$, we have

$$\frac{\|x\|_2\|y\|_2}{\|x\|_2 + \|y\|_2}\,\mathrm{chord}(x, y) \leq \mathrm{Gap}(x, y) \leq \sqrt{\frac{\|x\|_2\|y\|_2}{2}}\,\mathrm{chord}(x, y). \tag{3.3}$$

When x and y are normalized, by (3.2) we have $\mathrm{Gap}(x, y) = \dfrac{1}{2}\mathrm{chord}(x, y)\sec(\theta/2)$ and

$$\frac{1}{2}\mathrm{chord}(x, y) \leq \mathrm{Gap}(x, y) \leq \frac{1}{\sqrt{2}}\mathrm{chord}(x, y). \tag{3.4}$$

By identifying $\lambda \in \mathbb{C}$ with $(1, \lambda) \in \mathbb{C}^2$ and ∞ with $(0, 1) \in \mathbb{C}^2$, we obtain “gap” between two points in $\mathbb{C}_\infty$. Indeed, for $\lambda, \mu \in \mathbb{C}_\infty$, we have

$$\mathrm{Gap}(\lambda, \mu) = \frac{|\lambda - \mu|}{\sqrt{(1 + |\lambda|^2 + 1 + |\mu|^2 + 2|1 + \lambda\bar{\mu}|)}} \tag{3.5}$$

and $\mathrm{Gap}(\lambda, \infty) = \dfrac{1}{\sqrt{1 + (1 + |\lambda|)^2}} \leq \dfrac{1}{\sqrt{1 + |\lambda|^2}} = \mathrm{chord}(\lambda, \infty)$.

Note that Gap is not scale invariant whereas chord is scale invariant. The non-invariance of Gap under scaling is important for perturbation analysis of eigenvalues of L. We mention here that neither chordal metric nor any other measure of distance used in perturbation analysis of eigenvalues of matrix pencils is helpful in determining d(L).

4. Construction of nearest defective pencil. Let L be a nongeneric pencil of size n with n distinct eigenvalues. Since by (1.1), d(L) is invariant under unitary equivalence of L, that is, $\mathrm{d(L)} = \mathrm{d}(U\mathrm{L}V)$ for unitary matrices U and V, without loss of generality, for the rest of this section we assume that L is a diagonal pencil given by $\mathrm{L}(z) = \mathrm{diag}(\alpha_i) - z\mathrm{diag}(\beta_i)$ or $\mathrm{L}(c, s) = c\,\mathrm{diag}(\alpha_i) - s\,\mathrm{diag}(\beta_i)$. Note that

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$(\beta_j, \alpha_j), j = 1 : n$, are (unnormalized) eigenvalues of $\mathrm{L}(c, s) := c\,\mathrm{diag}(\alpha_i) - s\,\mathrm{diag}(\beta_i)$. Recall that by (2.5) we have $\Lambda_\epsilon(\mathrm{L}) = \cup_{j=1}^n \Omega_\epsilon(\alpha_j, \beta_j)$, where $\Omega_\epsilon(\alpha_j, \beta_j)$ is given in (2.6). The following result characterizes coalescence of components of $\Lambda_\epsilon(\mathrm{L})$.

THEOREM 4.1. *Consider* $\Lambda_\epsilon(\mathrm{L}) = \cup_{j=1}^n \Omega_\epsilon(\alpha_j, \beta_j)$ *and set* $x_j := (\beta_j, \alpha_j)$ *for* $j = 1 : n$. *Suppose that the components* $\Omega_\epsilon(\alpha_i, \beta_i)$ *and* $\Omega_\epsilon(\alpha_j, \beta_j)$ *contain only the (unnormalized) eigenvalues* x_i *and* x_j *of* L, *respectively. Then* $\Omega_\epsilon(\alpha_i, \beta_i) \cap \Omega_\epsilon(\alpha_j, \beta_j) = \emptyset$ *if and only if* $\epsilon < \mathrm{Gap}(x_i, x_j)$. *Set* $\zeta := \mathrm{sign}(\langle x_j, x_i \rangle)$ *and define*

$$\lambda_\zeta := \frac{\alpha_i + \zeta\,\alpha_j}{\beta_i + \zeta\,\beta_j}, \quad c_\zeta := \frac{\beta_i + \zeta\beta_j}{\|x_i + \zeta x_j\|_2} \quad \textit{and} \quad s_\zeta := \frac{\alpha_i + \zeta\alpha_j}{\|x_i + \zeta x_j\|_2}.$$

Then λ_ζ *(resp.,* $(c_\zeta, s_\zeta) \in \mathbb{S}^1$*) is a common boundary point of* $\Omega_\epsilon(\alpha_i, \beta_i)$ *and* $\Omega_\epsilon(\alpha_j, \beta_j)$ *when* $\epsilon = \mathrm{Gap}(x_i, x_j)$ *and* L *is nonhomogeneous (resp., homogeneous).*

Proof. Suppose that L is nonhomogeneous, that is, $\mathrm{L}(z) = \mathrm{diag}(\alpha_i) - z\mathrm{diag}(\beta_i)$. Note that if $\Omega_\epsilon(\alpha_i, \beta_i) \cap \Omega_\epsilon(\alpha_j, \beta_j) \neq \emptyset$ then there is a complex number $z \in \mathbb{C}$ such that $\frac{|\alpha_i - z\beta_i|}{\|(1,z)\|_2} = \epsilon = \frac{|\alpha_j - z\beta_j|}{\|(1,z)\|_2}$. Hence, $\alpha_i - z\beta_i = t(\alpha_j - z\beta_j)$ for some $t \in \mathbb{T}$. Consequently, we have $z = \dfrac{\alpha_i - t\alpha_j}{\beta_i - t\beta_j}$. Then by (2.2) and (3.1), we have

$$\epsilon = \frac{|\alpha_i - z\beta_i|}{\|(1, z)\|_2} = \frac{|\alpha_i\beta_j - \alpha_j\beta_i|}{\|x_i - tx_j\|_2} \geq \mathrm{Gap}(x_i, x_j).$$

Conversely, if $\epsilon \geq \mathrm{Gap}(x_i, x_j)$ then we have $|\alpha_i - \lambda_\zeta\beta_i| = |\alpha_j - \lambda_\zeta\beta_j|$ and

$$\frac{|\alpha_i - \lambda_\zeta\beta_i|}{\|(1, \lambda_\zeta)\|_2} = \frac{|\alpha_i\beta_j - \alpha_j\beta_i|}{\|x_i + \zeta x_j\|_2} = \mathrm{Gap}(x_i, x_j) \leq \epsilon \tag{4.1}$$

showing that $\lambda_\zeta \in \Omega_\epsilon(\alpha_i, \beta_i) \cap \Omega_\epsilon(\alpha_j, \beta_j)$. Hence the result follows. The proof is similar when L is homogeneous. ▯

As an immediate consequence of Theorem 4.1, we have

$$\mathrm{d}(\mathrm{L}) \geq \min_{i \neq j} \mathrm{Gap}(x_i, x_j). \tag{4.2}$$

We show that the equality holds in (4.2) for all three norms in (2.1). The following result, which is a special case of [4, Theorem 5.1] when $M = \ell^2$, shows that it is fairly easy to construct a nearest pencil with a multiple eigenvalue. This however does not solve Wilkinson's problem as formulated in (1.1).

THEOREM 4.2. *Let* $x_j := (\beta_j, \alpha_j), j = 1 : n$. *Let* i *and* j *be such that* $\mathrm{Gap}(x_i, x_j) = \min_{k \neq l} \mathrm{Gap}(x_k, x_l)$. *Also, let* ζ, λ_ζ *be as in Theorem 4.1. Set*

$$U := [e_i, e_j] \quad \textit{and} \quad V := [\mathrm{sign}(\alpha_i - \lambda_\zeta\beta_i)e_i,\, \mathrm{sign}(\alpha_j - \lambda_\zeta\beta_j)e_j],$$

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where e_j is the j-th column of the identity matrix of size n. Define

$$[\Delta A, \Delta B] := \frac{\mathrm{Gap}(x_i, x_j)}{\|(1, \lambda_\zeta)\|_2}\,[-1, \overline{\lambda_\zeta}] \otimes UV^*$$

and consider the pencil $\Delta\mathrm{L}(z) := \Delta A - z\,\Delta B$. Then λ_ζ is a multiple eigenvalue of $\mathrm{L} + \Delta\mathrm{L}$ of geometric multiplicity 2 and $\|\Delta\mathrm{L}\|_M = \mathrm{d}(\mathrm{L}) = \min_{i\neq j}\mathrm{Gap}(x_i, x_j)$ for $M = 2, \ell^2$.

Proof. By (4.1) we have $|\alpha_i - \lambda_\zeta\beta_i| = |\alpha_j - \lambda_\zeta\beta_j| = \mathrm{Gap}(x_i, x_j)\,\|(1, \lambda_\zeta)\|_2$. It is easy to check that $(\mathrm{L}(\lambda_\zeta) + \Delta\mathrm{L}(\lambda_\zeta))V = 0$ and $U^*(\mathrm{L}(\lambda_\zeta) + \Delta\mathrm{L}(\lambda_\zeta)) = 0$. Hence λ_ζ is a multiple eigenvalue of $\mathrm{L} + \Delta\mathrm{L}$ of geometric multiplicity 2. Since $\|\Delta\mathrm{L}\|_M = \mathrm{Gap}(x_i, x_j) \geq \mathrm{d}(\mathrm{L})$ for $M = 2, \ell^2$, the desired result follows from (4.2). ∎

Similar construction holds for a homogeneous pencil $\Delta\mathrm{L}$. Note that the construction in Theorem 4.2 provides a pencil with a multiple eigenvalue which is closest to L with respect to $\|\cdot\|_2$ and $\|\cdot\|_{\ell^2}$ norms but not with respect to the Frobenius norm $\|\cdot\|_F$. This is due to the fact that the perturbation matrices ΔA and ΔB have rank 2. We now modify the construction in Theorem 4.2 so as to obtain a pencil based on rank-1 perturbation so that the resulting perturbed pencil is defective.

We need the following well known result [4] that characterizes multiple eigenvalues of a pencil. Let λ (resp., (λ, μ)) be an eigenvalue of $A - zB$ (resp., $cA - sB$). Then the following results hold:

$$\begin{array}{lll} (i)\ \lambda \text{ multiple} & \Longleftrightarrow & \exists\ x \text{ and } y \text{ such that } y^*Bx = 0, \\ (ii)\ (\lambda, \mu) \text{ multiple} & \Longleftrightarrow & \exists\ x \text{ and } y \text{ such that } (y^*Ax, y^*Bx) = (0, 0), \end{array} \tag{4.3}$$

where x and y denote right and left eigenvectors corresponding to λ (resp., (λ, μ)).

We now describe construction of a pencil $\Delta\mathrm{L}(\lambda)$ such that $\mathrm{L}(\lambda) + \Delta\mathrm{L}(\lambda)$ is defective and that $\|\Delta\mathrm{L}\|_M = \mathrm{d}(\mathrm{L})$ for $M = 2, F, \ell^2$.

THEOREM 4.3. *Consider $\mathrm{L}(z) = \mathrm{diag}(\alpha_i) - z\mathrm{diag}(\beta_i)$ and set $x_j := (\beta_j, \alpha_j), j = 1 : n$. Then we have $\mathrm{d}(\mathrm{L}) = \min_{i\neq j}\mathrm{Gap}(x_i, x_j)$ for all three norms in (2.1). Let i and j be such that $\mathrm{Gap}(x_i, x_j) = \mathrm{d}(\mathrm{L})$. Also, let ζ, λ_ζ be as in Theorem 4.1. Set*

$$u := te_i + \sqrt{1 - t^2}\; e_j \quad \text{and} \quad v := (te_i - \sqrt{1 - t^2}\; \zeta e_j)\,\mathrm{sign}(\alpha_i - \lambda_\zeta\beta_i),$$

where $t := \dfrac{\sqrt{\|x_j\|_2^2 + |\langle x_i,\, x_j\rangle|}}{\|x_i + \zeta x_j\|_2}$ and e_j is the j-th column of the identity matrix of size n. Define

$$[\Delta A, \Delta B] := \frac{\mathrm{Gap}(x_i, x_j)}{\|(1, \lambda_\zeta)\|_2}\,[-1, \overline{\lambda_\zeta}] \otimes uv^*$$

and consider the pencil $\Delta\mathrm{L}(z) := \Delta A - z\,\Delta B$. Then λ_ζ is a defective eigenvalue of the pencil $\mathrm{L} + \Delta\mathrm{L}$ and $\|\Delta\mathrm{L}\|_M = \mathrm{d}(\mathrm{L}) = \mathrm{Gap}(x_i, x_j)$ for $M = 2, F, \ell^2$.

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Proof. First, we show that u and v, respectively, are normalized left and right eigenvectors of $\mathrm{L}+\Delta\mathrm{L}$ corresponding to the eigenvalue λ_ζ and that $u^*(B+\Delta B)v=0$.

Note that $\alpha_i-\lambda_\zeta\beta_i=-\zeta(\alpha_j-\lambda_\zeta\beta_j)$ and $\mathrm{sign}(\alpha_i-\lambda_\zeta\beta_i)=-\overline{\zeta}\ \mathrm{sign}(\alpha_j-\lambda_\zeta\beta_j)$. By Theorem 4.1, λ_ζ is the common boundary point of the components $\Omega_\epsilon(\alpha_i,\beta_i)$ and $\Omega_\epsilon(\alpha_j,\beta_j)$ for $\epsilon=\mathrm{Gap}(x_i,x_j)$. Hence, $\sigma_{\min}(\lambda_\zeta):=|\alpha_i-\lambda_\zeta\beta_i|=|\alpha_j-\lambda_\zeta\beta_j|=\mathrm{Gap}(x_i,x_j)\|(1,\lambda_\zeta)\|_2$ is the smallest singular value of $\mathrm{L}(\lambda_\zeta)$ and

$$\|(1,\lambda_\zeta)\|_2=\frac{\|x_i+\zeta x_j\|_2}{|\beta_i+\zeta\beta_j|}. \tag{4.4}$$

Note that e_k and $e_k\mathrm{sign}(\alpha_k-\lambda_\zeta\beta_k)$ are normalized left and right singular vectors of $\mathrm{L}(\lambda_\zeta)$ corresponding to $\sigma_{\min}(\lambda_\zeta)$ for $k=i,j$. Since

$$t^2=\frac{\|x_j\|_2^2+|\langle x_i,\,x_j\rangle|}{\|x_i+\zeta x_j\|_2^2}\quad\text{and}\quad 1-t^2=\frac{\|x_i\|_2^2+|\langle x_i,\,x_j\rangle|}{\|x_i+\zeta x_j\|_2^2}, \tag{4.5}$$

a little calculation shows that u and v are unit left and right singular vectors of $\mathrm{L}(\lambda_\zeta)$ corresponding to the smallest singular value $\sigma_{\min}(\lambda_\zeta)$. Also, by construction u and v are unit left and right eigenvectors of $\mathrm{L}+\Delta\mathrm{L}$ corresponding to the eigenvalue λ_ζ. Indeed, we have $\Delta\mathrm{L}(\lambda_\zeta)=-\sigma_{\min}(\lambda_\zeta)uv^*$ and $\mathrm{L}(\lambda_\zeta)v=\sigma_{\min}(\lambda_\zeta)u$, which give $(\mathrm{L}(\lambda_\zeta)+\Delta\mathrm{L}(\lambda_\zeta))v=\sigma_{\min}(\lambda_\zeta)u-\sigma_{\min}(\lambda_\zeta)u=0$. Similarly, $u^*(\mathrm{L}(\lambda_\zeta)+\Delta\mathrm{L}(\lambda_\zeta))=0$.

Now by (4.4) and (4.5), we have

$$\begin{aligned}
u^*(B+\Delta B)v &= u^*Bv+\frac{\overline{\lambda_\zeta}\,|\alpha_i-\lambda_\zeta\beta_i|}{\|(1,\lambda_\zeta)\|_2^2}=\omega\left(t^2\beta_i-(1-t^2)\zeta\beta_j+\frac{\overline{\lambda_\zeta}(\alpha_i-\lambda_\zeta\beta_i)}{\|(1,\lambda_\zeta)\|_2^2}\right)\\
&=\omega\left(t^2(\beta_i+\zeta\beta_j)-\frac{\|(1,\lambda_\zeta)\|_2^2(\beta_i+\zeta\beta_j)-(\beta_i+\overline{\lambda_\zeta}\alpha_i)}{\|(1,\lambda_\zeta)\|_2^2}\right)\\
&=\omega\left(t^2(\beta_i+\zeta\beta_j)-\left[(\beta_i+\zeta\beta_j)-\frac{\|x_i\|_2^2+|\langle x_i,\,x_j\rangle|}{(\overline{\beta_i+\zeta\beta_j})\|(1,\lambda_\zeta)\|_2^2}\right]\right)\\
&=\omega(\beta_i+\zeta\beta_j)\left[t^2-1+\frac{\|x_i\|_2^2+|\langle x_i,\,x_j\rangle|}{|\beta_i+\zeta\beta_j|^2\,\|(1,\lambda_\zeta)\|_2^2}\right]=0,
\end{aligned}$$

where $\omega:=\mathrm{sign}(\alpha_i-\lambda_\zeta\beta_i)$. Therefore, by (4.3), λ_ζ is a multiple eigenvalue of $\mathrm{L}+\Delta\mathrm{L}$. Since $\mathrm{rank}(\mathrm{L}(\lambda_\zeta)+\Delta\mathrm{L}(\lambda_\zeta))=n-1$, λ_ζ is a nonderogatory defective eigenvalue of $\mathrm{L}+\Delta\mathrm{L}$. By construction, we have $|||\Delta\mathrm{L}|||_M=\mathrm{Gap}(x_i,x_j)=\mathrm{d}(\mathrm{L})$ for $M=2,F,\ell^2$. Hence the proof follows. □

For completeness, we now describe homogeneous version of Theorem 4.3, that is, when L is considered as a homogeneous pencil.

THEOREM 4.4. *Consider* $\mathrm{L}(z)=c\,\mathrm{diag}(\alpha_i)-s\,\mathrm{diag}(\beta_i)$ *and set* $x_j:=(\beta_j,\alpha_j)$, $j=1:n$. *Let* i *and* j *be such that* $\mathrm{Gap}(x_i,x_j)=\mathrm{d}(\mathrm{L})$. *Also, let* ζ, c_ζ, s_ζ *be as in*

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Theorem 4.1 and t be as in Theorem 4.3. Set $u := te_i + \sqrt{1-t^2}\, e_j$ *and* $v := (te_i - \sqrt{1-t^2}\, \zeta e_j)\mathrm{sign}(c_\zeta \alpha_i - s_\zeta \beta_i)$. *Define*

$$[\Delta A, \Delta B] := \mathrm{Gap}(x_i, x_j)\, [-\overline{c_\zeta},\ \overline{s_\zeta}] \otimes uv^*$$

and consider the pencil $\Delta\mathrm{L}(c,s) := c\,\Delta A - s\,\Delta B$. *Then* (c_ζ, s_ζ) *is a defective eigenvalue of* $\mathrm{L} + \Delta\mathrm{L}$ *and* $\|\Delta\mathrm{L}\|_M = \mathrm{Gap}(x_i, x_j) = \mathrm{d}(\mathrm{L})$ *for* $M = 2, F, \ell^2$.

Proof. By Theorem 4.1, $(c_\zeta,\, s_\zeta) \in \mathbb{S}^1$ and is a common boundary point of the components $\Omega_\epsilon(\alpha_i, \beta_i)$ and $\Omega_\epsilon(\alpha_j, \beta_j)$ for $\epsilon = \mathrm{Gap}(x_i, x_j)$. Hence, $|c_\zeta \alpha_i - s_\zeta \beta_i| = |c_\zeta \alpha_j - s_\zeta \beta_j| = \mathrm{Gap}(x_i, x_j)$. By construction, u and v are unit left and right singular vectors of $\mathrm{L}(c_\zeta, s_\zeta)$ corresponding to the smallest singular value $\mathrm{Gap}(x_i, x_j)$. Also, by construction, we have $(\mathrm{L}(c_\zeta, s_\zeta) + \Delta\mathrm{L}(c_\zeta, s_\zeta))v = 0$ and $u^*(\mathrm{L}(c_\zeta, s_\zeta) + \Delta\mathrm{L}(c_\zeta, s_\zeta)) = 0$, that is, u and v are left and right eigenvectors of L corresponding to (c_ζ, s_ζ), respectively.

We now show that $u^*Av - \overline{c_\zeta}\mathrm{Gap}(x_i, x_j) = 0$ and $u^*Bv + \overline{s_\zeta}\mathrm{Gap}(x_i, x_j) = 0$. We have

$$\begin{aligned} u^*Av - \overline{c_\zeta}\mathrm{Gap}(x_i, x_j) &= \Big[t^2(\alpha_i + \zeta\alpha_j) - \zeta\alpha_j\Big]\mathrm{sign}(c_\zeta \alpha_i - s_\zeta \beta_i) - \overline{c_\zeta}|c_\zeta \alpha_i - s_\zeta \beta_i| \\ &= \mathrm{sign}(c_\zeta \alpha_i - s_\zeta \beta_i)\Big[[t^2(\alpha_i + \zeta\alpha_j) - \zeta\alpha_j] - \overline{c_\zeta}(c_\zeta \alpha_i - s_\zeta \beta_i)\Big]. \end{aligned}$$

Since $|c_\zeta|^2 + |s_\zeta|^2 = 1$, by (4.5) we have

$$\begin{aligned} [t^2(\alpha_i + \zeta\alpha_j) - \zeta\alpha_j] - \overline{c_\zeta}(c_\zeta \alpha_i - s_\zeta \beta_i) &= t^2(\alpha_i + \zeta\alpha_j) - (\alpha_i + \zeta\alpha_j) + s_\zeta(\overline{s_\zeta}\alpha_i + \overline{c_\zeta}\beta_i) \\ &= (\alpha_i + \zeta\alpha_j)\left[t^2 - 1 + \frac{\|x_i\|_2^2 + \overline{\zeta}\,\overline{\langle x_j, x_i\rangle}}{\|x_i + \zeta x_j\|_2^2}\right] = 0. \end{aligned}$$

This shows that $u^*(A + \Delta A)v = u^*Av - \overline{c_\zeta}\mathrm{Gap}(x_i, x_j) = 0$. Similarly, it can be shown that $u^*(B + \Delta B)v = u^*Bv + \overline{s_\zeta}\mathrm{Gap}(x_i, x_j) = 0$. Hence, by (4.3), (c_ζ, s_ζ) is a multiple eigenvalue of $\mathrm{L} + \Delta\mathrm{L}$. Since $\mathrm{rank}(\mathrm{L}(c_\zeta, s_\zeta) + \Delta\mathrm{L}(c_\zeta, s_\zeta)) = n - 1$, (c_ζ, s_ζ) is a defective eigenvalue. By construction, we have $\|\Delta\mathrm{L}\|_M = \mathrm{Gap}(x_i, x_j) = \mathrm{d}(\mathrm{L})$ for $M = 2, F, \ell^2$. Hence the proof follows. ∎

Often in practice it is necessary to perturb the coefficient matrices of a pencil relative to some weights. For example, $\mathrm{P}(z) := A - zB$ may be perturbed to $A + \Delta A - z(B + \Delta B)$ such that $\|\Delta A\|_M \le w_A\epsilon$ and $\|\Delta B\|_M \le w_B\epsilon$, where w_A and w_B are nonnegative real numbers called weights. However, as shown in [4], weighted perturbations require no special machinery and are easily incorporated by considering weighted norm of matrix pencils. Indeed, consider the weight vector $w := (w_A, w_B)$ with the convention that $w^{-1} := (w_A^{-1},\, w_B^{-1})$ and $w_A^{-1} = 0$ (resp., $w_B^{-1} = 0$) if $w_A = 0$ (resp., $w_B = 0$), and define the weighted scalar product on $\mathbb{C}^2$ by

$$\langle x,\, y\rangle_w := w_A^2 x_1 \overline{y}_1 + w_B^2 x_2 \overline{y}_2 \quad \text{for } x, y \in \mathbb{C}^2.$$

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Defining the norm/seminorm $\|x\|_{w,2} := \sqrt{\langle x,\, x\rangle_w}$, we have $|\langle x,\, y\rangle_w| \leq \|x\|_{w,2}\, \|y\|_{w,2}$. We define the weighted gap by

$$\mathrm{Gap}_w(x,y) := \min_{\zeta\in\mathbb{T}} \frac{|x_2y_1 - x_1y_2|}{\|x-\zeta y\|_{w,2}} = \frac{|x_2y_1 - x_1y_2|}{(\|x\|_{w,2}^2 + \|y\|_{w,2}^2 + 2|\langle x,\, y\rangle_w|)^{1/2}},$$

where the minimum is attained at $\zeta_{\min} := -\mathrm{sign}(\langle y,\, x\rangle_w)$. The weighted norm/seminorm

$$|\!|\!|\mathrm{P}|\!|\!| := \|[w_A^{-1}A,\, w_B^{-1}B]\|_2,\ |\!|\!|\mathrm{P}|\!|\!| := \|[w_A^{-1}A,\, w_B^{-1}B]\|_F,\ |\!|\!|\mathrm{P}|\!|\!| := \|[w_A^{-1}\|A\|_2,\, w_B^{-1}\|B\|_2]\|_2 \tag{4.6}$$

gives $\Lambda_\epsilon(\mathrm{L}) = \cup_{j=1}^n \Lambda_\epsilon(\alpha_j, \beta_j)$, where $\Lambda_\epsilon(\alpha_j,\beta_j) := \{(c,s) \in \mathbb{S}_w^1 : |\alpha_j c - \beta_j s| \leq \epsilon\}$ and $\mathbb{S}_w^1 := \{(c,s) \in \mathbb{C}^2 : \|(c,s)\|_{w,2} = 1\}$. See [4] for more on weighted pseudospectra. Consequently, we have the following result whose proof is easy to check.

THEOREM 4.5. *Consider* $\mathrm{L}(z) = c\,\mathrm{diag}(\alpha_i) - s\,\mathrm{diag}(\beta_i)$ *and set* $x_j := (\beta_j, \alpha_j),\ j = 1:n$. *Then*

$$\mathrm{d(L)} = \min_{i\neq j} \mathrm{Gap}_w(x_i, x_j)$$

for all three norms in (4.6). Let i *and* j *be such that* $\mathrm{Gap}_w(x_i,x_j) = \mathrm{d(L)}$. *Set* $\zeta := \mathrm{sign}(\langle x_j,\, x_i\rangle_w)$ *and define*

$$c_\zeta := \frac{\beta_i + \zeta\beta_j}{\|x_i + \zeta x_j\|_{w,2}}, \quad s_\zeta := \frac{\alpha_i + \zeta\alpha_j}{\|x_i + \zeta x_j\|_{w,2}} \quad \text{and} \quad t := \frac{\sqrt{\|x_j\|_{w,2}^2 + |\langle x_i,\, x_j\rangle_w|}}{\|x_i + \zeta x_j\|_{w,2}}.$$

Also set $u := te_i + \sqrt{1-t^2}\; e_j$ *and* $v := (te_i - \sqrt{1-t^2}\;\zeta e_j)\mathrm{sign}(c_\zeta\alpha_i - s_\zeta\beta_i)$. *Define*

$$[\Delta A, \Delta B] := \mathrm{Gap}_w(x_i,x_j)\,[-\overline{c_\zeta} w_A^2,\ \overline{s_\zeta} w_B^2] \otimes uv^*$$

and consider the pencil $\Delta\mathrm{L}(c,s) := c\,\Delta A - s\,\Delta B$. *Then* (c_ζ, s_ζ) *is a defective eigenvalue of* $\mathrm{L} + \Delta\mathrm{L}$ *and* $|\!|\!|\Delta\mathrm{L}|\!|\!|_M = \mathrm{Gap}_w(x_i,x_j) = \mathrm{d(L)}$ *for all three norms in (4.6).*

5. Bounds for Wilkinson's distance.

Let $\mathrm{L}(z) := A - zB$ be an n-by-n pencil with n distinct eigenvalues. We now derive computable upper and lower bounds for d(L) and show that d(L) is almost inversely proportional to the condition number of the most sensitive eigenvalue of L. First, note that $\mathrm{d(L)} = \mathrm{d}(U\mathrm{L}V)$ for unitary matrices U and V.

THEOREM 5.1. *Let* $\mathrm{L}(z) := A - zB$ *be an* n*-by-*n *pencil having* n *distinct eigenvalues. Let* X *and* Y *be nonsingular matrices such that* $Y^*\mathrm{L}(z)X = \mathrm{diag}(\alpha_i) - z\,\mathrm{diag}(\beta_i)$. *Then obviously* $(\beta_j, \alpha_j),\ j = 1:n$, *are eigenvalues of* $\mathrm{L}(c,s) = cA - sB$ *and we have*

$$\min_{i\neq j} \frac{\mathrm{Gap}(\alpha_i,\beta_i;\alpha_j,\beta_j)}{\|Y\|_2\|X\|_2} \leq \mathrm{d(L)} \leq \min_{i\neq j} \mathrm{Gap}(\alpha_i,\beta_i;\alpha_j,\beta_j) \tag{5.1}$$

for all three norms in (2.1).

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Proof. Let U and V be unitary matrices such that $U^*\mathrm{L}(z)V = T_A - zT_B$, where T_A and T_B are upper triangular matrices with diagonal entries α_i and β_i, respectively, for $i = 1 : n$. Set $\mathrm{P}(z) := \mathrm{diag}(\alpha_i) - z\mathrm{diag}(\beta_i)$. Then by Theorem 4.3, there is a diagonal pencil $\Delta\mathrm{P}(z)$ such that $\mathrm{P}(z) + \Delta\mathrm{P}(z)$ has a multiple eigenvalue and $|||\Delta\mathrm{P}|||_M = \mathrm{d}(\mathrm{P}) = \min_{i\neq j} \mathrm{Gap}(\alpha_i, \beta_i; \alpha_j, \beta_j)$ for $M = 2, F, \ell^2$. Consequently, $T_A - zT_B + \Delta\mathrm{P}(z)$ has a multiple eigenvalue. Hence we have $\mathrm{d}(\mathrm{L}) \leq |||\Delta\mathrm{P}|||_M = \mathrm{d}(\mathrm{P})$. Since $Y^*\mathrm{L}(z)X = \mathrm{P}(z)$ and $\|XAY\|_F \leq \|X\|_2\|Y\|_2\|A\|_F$, by (1.1) we have $\mathrm{d}(\mathrm{P}) \leq \|Y^*\|_2\|X\|_2\, \mathrm{d}(\mathrm{L})$ for $M = 2, F, \ell^2$. Hence the bounds follow. □

It is well known [21, 22, 23] that in the case of a matrix, the distance to the nearest defective matrix is directly related to the ill-conditioning of its eigenvalues. More precisely, for $A \in \mathbb{C}^{n\times n}$ with distinct eigenvalues $\lambda_j, j = 1 : n$, we have [6, 22, 23]

$$\mathrm{d}(A) \leq \min_j \frac{\|A\|_2}{\sqrt{\mathrm{cond}(\lambda_j)^2 - 1}}, \tag{5.2}$$

where $\mathrm{cond}(\lambda_j)$ is the condition number [21] of the eigenvalue λ_j and $\mathrm{d}(A)$ is the Wilkinson's distance from A to the nearest defective matrix [7]. Generically the upper bound (5.2) provides a sharp estimate of $\mathrm{d}(A)$ - this is specially true for matrices with ill-conditioned eigenvalues; see [6].

An upper bound similar to (5.2) holds for matrix pencils as well and can be derived easily. Let X, Y and L be as in Theorem 5.1. Then (β_j, α_j) is an eigenvalue of $\mathrm{L}(c, s) = cA - sB$ and, $y_j := Ye_j$ and $x_j := Xe_j$, respectively, are corresponding left and right eigenvectors, that is, $y_j^*\mathrm{L}(\beta_j, \alpha_j) = 0$ and $\mathrm{L}(\beta_j, \alpha_j)x_j = 0$, for $j = 1 : n$. The eigenvalue (β_j, α_j) and, the left and the right eigenvectors y_j and x_j, respectively, are normalized in the sense that $\alpha_j = y_j^*Ax_j$ and $\beta_j = y_j^*Bx_j$ for $j = 1 : n$. Then the condition number of (β_j, α_j) with respect to the norms in (2.1) is given by [19]

$$\mathrm{cond}(\beta_j, \alpha_j) = \frac{\|x_j\|_2\|y_j\|_2}{\sqrt{|\alpha_j|^2 + |\beta_j|^2}}. \tag{5.3}$$

Similarly, when $\beta_j \neq 0$, the condition number of the eigenvalue $\lambda_j := \alpha_j/\beta_j$ of L is given by [1, 19]

$$\mathrm{cond}(\lambda_j) = \frac{\sqrt{1 + |\lambda_j|^2}\, \|x_j\|_2\|y_j\|_2}{|y_j^*Bx_j|}. \tag{5.4}$$

THEOREM 5.2. *Let* L, X *and* Y *be as in Theorem 5.1. Consider the left and right eigenvectors* $y_j := Ye_j$ *and* $x_j := Xe_j$*, respectively, corresponding to the eigenvalue* $(\beta_j, \alpha_j), j = 1 : n$*. Suppose that* $\|x_j\|_2\|y_j\|_2 > 1$ *for some* j*. Then for the homogeneous pencil* $\mathrm{L}(c, s) = cA - sB$*, for all three norms in (2.1), we have*

$$\mathrm{d}(\mathrm{L}) \leq \min_j \frac{|||\mathrm{L}|||}{\sqrt{(|\alpha_j|^2 + |\beta_j|^2)\,\mathrm{cond}(\beta_j, \alpha_j)^2 - 1}} = \min_j \frac{|||\mathrm{L}|||}{\sqrt{\|x_j\|_2^2\|y_j\|_2^2 - 1}}. \tag{5.5}$$

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Furthermore, when $y_j^* B x_j \neq 0$ *for* $j = 1 : n$*, considering* L *as a nonhomogeneous pencil, that is,* $\mathrm{L}(z) = A - zB$*, and the eigenvalues* $\lambda_j := \alpha_j/\beta_j, j = 1 : n$*, we have*

$$\mathrm{d}(\mathrm{L}) \leq \min_j \frac{\|\!|\mathrm{L}|\!\|}{\sqrt{\frac{|y_j^* B x_j|^2 \mathrm{cond}(\lambda_j)^2}{1+|\lambda_j|^2} - 1}} = \min_j \frac{\|\!|\mathrm{L}|\!\|}{\sqrt{\|x_j\|_2^2 \|y_j\|_2^2 - 1}}. \tag{5.6}$$

Proof. The bounds (5.5) and (5.6) can be derived in almost the same way as the bound (5.2); see [6, 22, 23] for a proof of (5.2). Indeed, considering the eigenvalue (β_1, α_1) and without loss of generality assuming $A = \begin{bmatrix} \alpha_1 & a_1 \\ 0 & A_1 \end{bmatrix}$ and $B = \begin{bmatrix} \beta_1 & b_1 \\ 0 & B_1 \end{bmatrix}$, we have $x_1 = e_1$ and $y_1^* = [1, -(\beta_1 a_1 - \alpha_1 b_1)(\beta_1 A_1 - \alpha_1 B_1)^{-1}]$. This shows that

$$\begin{aligned} \|y_1\|_2^2 &\leq 1 + (|\alpha_1|^2 + |\beta_1|^2)\|[a_1, b_1]\|_2^2 \|(\beta_1 A_1 - \alpha_1 B_1)^{-1}\|_2^2 \\ &\leq 1 + (|\alpha_1|^2 + |\beta_1|^2)\, \|\!|\mathrm{L}|\!\|^2 \|(\beta_1 A_1 - \alpha_1 B_1)^{-1}\|_2^2. \end{aligned}$$

By (5.3) $\sqrt{(|\alpha_1|^2 + |\beta_1|^2)\mathrm{cond}(\beta_1, \alpha_1)^2 - 1} \leq \sqrt{|\alpha_1|^2 + |\beta_1|^2}\, \|\!|\mathrm{L}|\!\| \|(\beta_1 A_1 - \alpha_1 B_1)^{-1}\|_2$ which gives

$$\frac{\sigma_{\min}((\beta_1 A_1 - \alpha_1 B_1)}{\sqrt{|\alpha_1|^2 + |\beta_1|^2}} \leq \frac{\|\!|\mathrm{L}|\!\|}{\sqrt{(|\alpha_1|^2 + |\beta_1|^2)\mathrm{cond}(\beta_1, \alpha_1)^2 - 1}} = \frac{\|\!|\mathrm{L}|\!\|}{\sqrt{\|x_1\|_2^2 \|y_1\|_2^2 - 1}}.$$

Consequently, considering $\mathrm{L}_1(c, s) = cA_1 - sB_1$, by (2.2) and (2.3), we have

$$\eta(\beta_1, \alpha_1, \mathrm{L}_1) = \frac{\sigma_{\min}((\beta_1 A_1 - \alpha_1 B_1)}{\sqrt{|\alpha_1|^2 + |\beta_1|^2}}$$

and a pencil $\Delta\mathrm{L}_1(c, s) = c\Delta A_1 - s\Delta B_1$ such that $\|\!|\Delta\mathrm{L}_1|\!\| = \eta(\beta_1, \alpha_1, \mathrm{L}_1)$ and $(\beta_1, \alpha_1) \in \Lambda(\mathrm{L}_1 + \Delta\mathrm{L}_1)$. Hence, considering $\Delta A = \mathrm{diag}(0, \Delta A_1)$ and $\Delta B = \mathrm{diag}(0, \Delta B_1)$ and the pencil $\Delta\mathrm{L}(c, s) = c\Delta A - s\Delta B$ it follows that (β_1, α_1) is a multiple eigenvalue of $\mathrm{L} + \Delta\mathrm{L}$ and that $\|\!|\Delta\mathrm{L}|\!\| = \eta(\beta_1, \alpha_1, \mathrm{L}_1)$. This shows that

$$\mathrm{d}(\mathrm{L}) \leq \frac{\|\!|\mathrm{L}|\!\|}{\sqrt{(|\alpha_1|^2 + |\beta_1|^2)\mathrm{cond}(\beta_1, \alpha_1)^2 - 1}} = \frac{\|\!|\mathrm{L}|\!\|}{\sqrt{\|x_1\|_2^2 \|y_1\|_2^2 - 1}}.$$

Hence the bound (5.5) follows.

Next, when $\beta_j = y_j^* B x_j \neq 0$, by (5.3) and (5.4), we have

$$\mathrm{cond}(\beta_j, \alpha_j) = \frac{\|x_j\|_2 \|y_j\|_2}{|\beta_j|\sqrt{1 + |\lambda_j|^2}} = \frac{\mathrm{cond}(\lambda_j)}{1 + |\lambda_j|^2}.$$

Hence $(|\alpha_j|^2 + |\beta_j|^2)\mathrm{cond}(\beta_j, \alpha_j)^2 = |\beta_j|^2 \mathrm{cond}(\lambda_j)^2/(1 + |\lambda_j|^2)$ gives the bound (5.6). ∎

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We mention that the bounds (5.2) and (5.6) are closely related. This can be seen by considering A as the pencil $\mathrm{L}(z) = A - zI$. Indeed, let $X^{-1}AX = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and set $Y^* := X^{-1}$. Then $Y^*AX = \mathrm{diag}(\lambda_i)$ and $Y^*X = I$. Thus, the condition number $\mathrm{cond}(\lambda_j, A)$ of λ_j as an eigenvalue of A is given by [21] $\mathrm{cond}(\lambda_j, A) = \|x_j\|_2 \|y_j\|_2$. On the other hand, by (5.4), the condition number $\mathrm{cond}(\lambda_j)$ of λ_j as an eigenvalue of the pencil $\mathrm{L}(z) = A - zI$ is given by $\mathrm{cond}(\lambda_j) = \sqrt{1 + |\lambda_j|^2}\,\mathrm{cond}(\lambda_j, A)$. Hence, considering $|||\mathrm{L}||| = \|[A, I]\|_2$, by (5.6), we have

$$\mathrm{d}(\mathrm{L}) \leq \min_j \frac{|||\mathrm{L}|||}{\sqrt{\|x_j\|_2^2 \|y_j\|_2^2 - 1}} = \min_j \frac{\|[A, I]\|_2}{\sqrt{\mathrm{cond}(\lambda_j, A)^2 - 1}}. \tag{5.7}$$

On the other hand, by restricting perturbations to A in (1.1) and thereby leaving I unperturbed, we have $\mathrm{d}(\mathrm{L}) \leq \mathrm{d}(A)$. Consequently, for the pencil $\mathrm{L}(z) = A - zI$, by (5.7), we have

$$\mathrm{d}(\mathrm{L}) \leq \mathrm{d}(A) \leq \min_j \frac{\|A\|_2}{\sqrt{\mathrm{cond}(\lambda_j, A)^2 - 1}} \leq \min_j \frac{\|[A, I]\|_2}{\sqrt{\mathrm{cond}(\lambda_j, A)^2 - 1}}. \tag{5.8}$$

We now illustrate these bounds by considering a few numerical examples. For the rest of the paper, we denote by LB and GP, respectively, the lower and the upper bounds in Theorem 5.1. Also, we denote the upper bound in (5.5) by UB.

Example 5.3. First, consider the diagonal pencils given by

$$\mathrm{L}(z) = \begin{bmatrix} 1+2i & 0 \\ 0 & 2+i \end{bmatrix} - z \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \mathrm{P}(z) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - z \begin{bmatrix} -0.3-0.2i & 0 \\ 0 & 0.2+0.5i \end{bmatrix}.$$

By (3.1), we have GP $= 0.5628$. Hence, by Theorem 4.3, we have $\mathrm{d}(\mathrm{L}) = \mathrm{GP} = 0.5628$ and $\lambda_\zeta = 1.1564 + 0.9877i$. These values are also confirmed by the contour plot of $\Lambda_\epsilon(\mathrm{L})$. Indeed, the left plot in Figure 5.1 shows the contour plot of $\Lambda_\epsilon(\mathrm{L})$ and the coalescence of pseudospectral components at λ_ζ for $\epsilon = 0.5628$.

The eigenvalues of P are $\lambda_1 := -4.6154 + 3.0769i$ and $\lambda_2 := 2.0690 - 5.1724i$. Again by (3.1) we have GP $= 0.4115$. Thus, by Theorem 4.3 we have $\mathrm{d}(\mathrm{P}) = \mathrm{GP} = 0.4115$ and $\lambda_\zeta = -5.0868 - 14.6297i$. Again, these values are confirmed by the contour plot of $\Lambda_\epsilon(\mathrm{P})$. Indeed, the right plot in Figure 5.1 shows the contour plot of $\Lambda_\epsilon(\mathrm{P})$ and the coalescence of pseudospectral components at λ_ζ for $\epsilon = 0.4115$.

The contour plot of $\Lambda_\epsilon(\mathrm{P})$ in Figure 5.1 needs to be interpreted properly as it contains ∞. For sufficiently small ϵ, the components of $\Lambda_\epsilon(\mathrm{P})$ are bounded regions in the complex plane containing the eigenvalues λ_1 and λ_2 in their interiors. As ϵ grows gradually to 0.3606, the component containing λ_2 remains bounded but the component containing λ_1 becomes unbounded and contains ∞. When ϵ is further increased to 0.4115, the two components coalesce at λ_ζ. Indeed, for $\epsilon = 0.4115$, $\Lambda_\epsilon(\mathrm{P})$

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is multiply connected and consists of the entire complex plane except for the region enclosed by two almost elliptical disks (almost crescent shaped region) as shown in Figure 5.1. The region enclosed by the inner elliptical disk is the component containing λ_2 and the region exterior to the outer elliptical disk is the component containing λ_1. The complex number λ_ς is the common boundary point of the two elliptical disks and is the point of coalescence of the two components.

Note that ∞ enters into the component of $\Lambda_\epsilon(\mathrm{P})$ containing λ_1 before it coalesces with the component containing λ_2 at λ_ς for $\epsilon = 0.4115$.

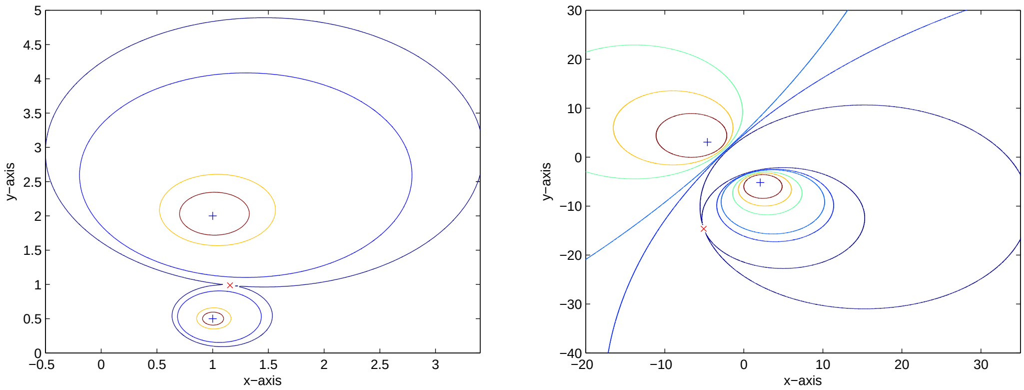

Fig. 5.1. *The left and the right figures show contour plots of $\Lambda_\epsilon(\mathrm{L})$ and $\Lambda_\epsilon(\mathrm{P})$, respectively, and the coalescence of pseudospectral components. The eigenvalues are indicated by $+$ and the point of coalescence λ_ς by $\times$.*

The upper and lower bounds in Theorem 5.1 are expected to be tight for a pencil L with well-conditioned eigenvalues. In such a case, GP is expected to provide a better estimate of d(L) than UB. On the other hand, if the eigenvalues of L are ill-conditioned then the upper bound UB in (5.5) is expected to provide a better estimate of d(L) than GP. We illustrate this fact by considering a few examples.

Example 5.4. The numerical results given below are correct to the digits shown and have been obtained by using MATLAB. First, we consider two pencils

$$\mathrm{L}(z) = \begin{bmatrix} 1 & 3 & 2 \\ 5 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix} - z \begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad \mathrm{P}(z) = \begin{bmatrix} 7 & -2 & 0 \\ -2 & 6 & -2 \\ 0 & -2 & 5 \end{bmatrix} - z \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

with well-conditioned eigenvalues (condition numbers < 14). The MATLAB `qz` com-

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mand returns the following unitarily equivalent upper triangular pencils

$$\mathrm{L}(z) \simeq \begin{bmatrix} 2.4495 & -0.8693 & 1.8012 \\ 0 & 4.2769 & -4.5142 \\ 0 & 0 & 3.0546 \end{bmatrix} - z \begin{bmatrix} 1.2247 & -3.0594 & 0.3740 \\ 0 & 2.5570 & -4.2420 \\ 0 & 0 & 5.4283 \end{bmatrix}$$

$$\mathrm{P}(z) \simeq \begin{bmatrix} 3.5857 & -2.8060 & 0.5455 \\ 0 & 6.5455 & 3.8060 \\ 0 & 0 & 6.9024 \end{bmatrix} - z \begin{bmatrix} 0.6425 & -0.7687 & 0.1818 \\ 0 & 2.1818 & 1.4353 \\ 0 & 0 & 2.8536 \end{bmatrix}.$$

The bounds on d(L) and d(P) are given in Table 5.1. The upper bound UB does not exist for the pencil P as the condition on eigenvectors in Theorem 5.2 is not satisfied.

Table 5.1
Bounds on d(L) *and* d(P).

Pencil	LB	d(Pencil)	GP	UB
L	0.0063	0.0076	0.1329	0.8681
P	0.1307	0.2297	0.2520	-

Next, we consider bounds for randomly generated pencils. We generate n-by-n pencil $\mathrm{L}(z) = A - zB$ using the MATLAB commands `A = rand(n)` and `B = rand(n)`. Table 5.2 shows the results for various values of n. Observe that LB and GP provide good estimates of d(L) and that the upper bound GP is a better estimate of d(L) than UB because randomly generated pencils are not expected to have highly ill-conditioned eigenvalues.

Table 5.2
Bounds on d(L) *for randomly generated pencil* L *of size* n.

n	LB	GP	UB
50	4.8385e-4	6.2000e-2	1.7599
100	7.3282e-6	1.0200e-2	2.4590e-1
150	2.1514e-4	7.6500e-2	1.5212
200	1.9335e-4	4.4000e-2	4.0096
250	5.5859e-5	3.7400e-2	1.7628

Finally, we illustrate that for a matrix pencil L with ill-conditioned eigenvalues the upper bound UB usually provides a better estimate of d(L) than GP. For this purpose, in view of (5.8), we consider $\mathrm{L}(z) = A - zI$ and choose A having ill-conditioned eigenvalues. Then using MATLAB command `qz` we obtain unitariliy equivalent upper triangular pencil. We choose A to be the Wilkinson matrix W which is a 20-by-20 bi-diagonal matrix whose diagonal entries are $20, 19, \ldots, 1$ and the supper-diagonals are 20. The matrix W is known to have highly ill-conditioned eigenvalues and it is

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shown by Wilkinson that $\mathrm{d}(W) \simeq 10^{-14}$, see [21, pp. 90–92]. It is shown in [7] that $\mathrm{d}(W) = 6.13 \times 10^{-14}$. In view of (5.8), we have $\mathrm{d}(\mathrm{L}) \leq \mathrm{d}(W) \leq \mathrm{UB}$.

Further, we consider Frank matrix which is also known to have ill-conditioned eigenvalues, see [21, pp. 90–92]. We denote by F_n the Frank matrix of size n, which is generated by the MATLAB command `Gallery('frank', n)`. The ill-conditioning of the eigenvalues of F_n increases rapidly with n and $\mathrm{d}(F_n) = \mathcal{O}(10^{-15})$ for $n = 15$, see [7, 21]. The bounds LB and UB in Table 5.3 confirm these results. The values of $\mathrm{d}(A)$ in Table 5.3 for $A = W$ and $A = F_n$ are taken from [7].

TABLE 5.3
Bounds on d(L) *for pencils* L *with ill-conditioned eigenvalues.*

A	LB	$\mathrm{d}(A)$	UB	GP
W	5.2290e-16	6.13e-14	7.2577e-012	2.5600e-02
F_6	3.6083e-04	5.56e-04	2.5940e-01	5.3900e-02
F_{10}	1.7389e-08	3.93e-08	1.3415e-04	1.4300e-02
F_{12}	5.9423e-11	1.85e-10	1.2314e-06	9.2000e-03
F_{15}	7.9951e-15	$\mathcal{O}(10^{-15})$	4.9733e-10	5.5000e-03

Conclusion. Given a regular pencil $\mathrm{L}(\lambda)$ with distinct eigenvalues, we have described construction (Theorems 4.3 and 4.4) of a defective pencil $\mathrm{L}(\lambda) + \Delta\mathrm{L}(\lambda)$ which is closest to $\mathrm{L}(\lambda)$ when $\mathrm{L}(\lambda)$ is unitarily diagonalizable. We have shown that the construction of $\Delta\mathrm{L}(\lambda)$ requires only the eigenvalues and eigenvectors of $\mathrm{L}(\lambda)$. Thus, we have shown that the infimum in (1.1) is attained for nongeneric pencils. For the general case when $\mathrm{L}(\lambda)$ is regular with distinct eigenvalues, we have derived computable upper and lower bounds (Theorems 5.1, 5.2) for d(L). The bound in Theorem 5.2 shows that d(L) is almost inversely proportional to the condition number of the most sensitive eigenvalue of $\mathrm{L}(\lambda)$.

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