

A NOTE ON EIGENVALUES LOCATION FOR TRACE ZERO DOUBLY STOCHASTIC MATRICES*

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Abstract. Some results on the location of the eigenvalues of trace zero doubly stochastic matrices are provided. A result similar to that provided in [H. Perfect and L. Mirsky. Spectral properties of doubly-stochastic matrices. *Monatshefte für Mathematik*, 69(1):35–57, 1965.] for doubly stochastic matrices is given.

Key words. Doubly stochastic matrices, Eigenvalues.

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1. Introduction. A real square matrix is said to be *stochastic* if its elements are all nonnegative and all its rows sum to one. A real square matrix is said to be *doubly stochastic* if it is stochastic and all its columns sum to one. The classes of stochastic and doubly stochastic matrices have been the subject of numerous studies since the beginning of the last century, but very little is known about the spectral properties of these classes.

The regions Θ_n of the complex plane consisting of all the eigenvalues of all n -dimensional stochastic matrices were first characterized by Dmitriev and Dynkin [2] in 1946 for $n = 2, \dots, 5$, and by Karpelevich [4] for all $n \geq 2$. A useful reformulation of the Karpelevich result can be found in Ito [3] and diagrams of the regions are depicted in [1].

In 1965, Perfect and Mirsky [7] considered the analogous problem for doubly stochastic matrices, that is the characterization of the regions Ω_n of the complex plane consisting of all the eigenvalues of all n -dimensional doubly stochastic matrices. They proved that for $n \geq 2$,

$$\Pi_2 \cup \Pi_3 \cup \dots \cup \Pi_n \subseteq \Omega_n, \quad (1.1)$$

where Π_k denotes the convex hull of the k -th roots of unity. Moreover, they demonstrated that the sign of inclusion in (1.1) can be replaced by the sign of equality for

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$n = 3$, that is,

$$\Omega_3 = \Pi_2 \cup \Pi_3.$$

It has been then conjectured that

$$\Omega_n = \Pi_2 \cup \Pi_3 \cup \cdots \cup \Pi_n$$

for all n . Recently, Levick, Pereira and Kribs [5] have proved that the conjecture holds true for $n = 4$, that is,

$$\Omega_4 = \Pi_2 \cup \Pi_3 \cup \Pi_4$$

while Mashreghi and Rivard [6] in 2007 exhibited a counterexample for $n = 5$. In more detail, they considered the doubly stochastic matrix

$$D_5 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and showed that D_5 has two complex conjugate eigenvalues lying outside the region $\Pi_2 \cup \Pi_3 \cup \Pi_4 \cup \Pi_5$.

In [7], Perfect and Mirsky considered also the regions Ω_n^0 consisting of those complex numbers which can serve as characteristic roots of n -dimensional doubly stochastic matrices with zero trace. They proved that for $n \geq 2$,

$$\Pi_n^0 \subseteq \Omega_n^0, \quad (1.2)$$

where Π_n^0 denotes the union of the point 1 and the convex hull of the n -th roots of unity different from 1. Moreover, they demonstrated that the sign of inclusion in (1.2) can be replaced by the sign of equality for $n = 3$, that is,

$$\Omega_3^0 = \Pi_3^0.$$

Finally, the authors found the necessary and sufficient conditions for four real numbers to be the characteristic roots of a doubly stochastic 4×4 matrix with zero trace (see the corollary of Theorem 16 in [7]).

In this brief note, we propose some additional results on the location of the eigenvalues of trace zero doubly stochastic matrices.

2. Results. From the Corollary of Theorem 16 in [7], it follows that $\Omega_4^0 \cap \mathbb{R} = [-1, 1]$. To prove this in a different way, consider, for any $-1 \leq \lambda \leq 1$, the trace zero doubly stochastic matrix

$$\frac{1}{2} \begin{pmatrix} 0 & 1 + \lambda & 1 - \lambda & 0 \\ 1 + \lambda & 0 & 0 & 1 - \lambda \\ 1 - \lambda & 0 & 0 & 1 + \lambda \\ 0 & 1 - \lambda & 1 + \lambda & 0 \end{pmatrix}$$

that has λ as one of its characteristic roots.

The following theorem extends this result to the case of n -dimensional matrices:

THEOREM 2.1. *For $n \geq 4$, the region Ω_n^0 of the complex plane consisting of all the eigenvalues of all n -dimensional doubly stochastic matrices with zero trace is such that*

$$\Omega_n^0 \cap \mathbb{R} = [-1, 1].$$

Proof. The statement is obviously true for $n = 4$ by virtue of the Corollary of Theorem 16 in [7]. Consider then the 5×5 trace zero doubly stochastic matrix

$$D_5^0(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 - a & a & 0 \\ a & 1 - a & 0 & 0 & 0 \\ 1 - a & a & 0 & 0 & 0 \\ 0 & 0 & a & 1 - a & 0 \end{pmatrix}$$

with $0 \leq a \leq 1$. We will show that for any real number $-1 \leq \bar{\lambda} \leq 1$ there exists a value \bar{a} of the parameter a such that $0 \leq \bar{a} \leq 1$ and $\bar{\lambda} \in \sigma(D_5^0(\bar{a}))$. To this end note that the characteristic polynomial of $D_5^0(a)$ is

$$p_{D_5^0}(\lambda) = (\lambda - 1)(\lambda^4 + \lambda^3 + (2a - 2a^2)\lambda^2 + (4a - 4a^2 - 1)\lambda + (4a - 4a^2 - 1)),$$

and it has a root in $\bar{\lambda}$ for

$$a = \bar{a} := \frac{1}{2} + \frac{\bar{\lambda}}{2} \sqrt{\frac{2\bar{\lambda}^2 + 2\bar{\lambda} + 1}{\bar{\lambda}^2 + 2\bar{\lambda} + 2}} \quad \text{or} \quad a = \bar{a} := \frac{1}{2} - \frac{\bar{\lambda}}{2} \sqrt{\frac{2\bar{\lambda}^2 + 2\bar{\lambda} + 1}{\bar{\lambda}^2 + 2\bar{\lambda} + 2}}.$$

It is now easy to prove that, in both cases, $0 \leq \bar{a} \leq 1$. In fact, it suffices to show that

$$0 \leq 2\bar{\lambda}^2 + 2\bar{\lambda} + 1 \leq \bar{\lambda}^2 + 2\bar{\lambda} + 2.$$

The left inequality is obvious while the right one reduces to

$$2\bar{\lambda}^2 + 2\bar{\lambda} + 1 - (\bar{\lambda}^2 + 2\bar{\lambda} + 2) = (\bar{\lambda} + 1)(\bar{\lambda} - 1) \leq 0,$$



which obviously holds. Hence, the statement of the theorem holds true also for $n = 5$.

Consider now the case $n \geq 6$ and the $n \times n$ trace zero doubly stochastic matrix

$$D_n^0 = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & \\ \hline & & D_{n-2}^0 \end{array} \right),$$

where D_{n-2}^0 is a trace zero doubly stochastic matrix of dimension $n - 2$. The spectrum of the matrix D_n^0 is the union of the spectra of its blocks, that is

$$\sigma(D_n^0) = \{1, -1\} \cup \sigma(D_{n-2}^0)$$

so that

$$\Omega_{n-2}^0 \subseteq \Omega_n^0$$

and the theorem is then proved by induction on n . \square

On the basis of this result, for $n \geq 4$, relation (1.2) can be refined as

$$[-1, 1] \cup \Pi_n^0 \subseteq \Omega_n^0. \tag{2.1}$$

In what follows, we show that for $n \geq 5$, the sign of inclusion in (2.1) cannot be replaced by the sign of equality and propose a further refinement of relation (2.1). The case $n = 4$ remains open even though numerical calculation suggests that, in this case, the sign of equality should be used in (2.1).

THEOREM 2.2. *For $n \geq 4$, the region Ω_n^0 of the complex plane consisting of all the eigenvalues of all n -dimensional doubly stochastic matrices with zero trace is such that*

$$\Pi_2^0 \cup \Pi_3^0 \cup \dots \cup \Pi_{n-2}^0 \cup [-1, 1] \cup \Pi_n^0 \subseteq \Omega_n^0. \tag{2.2}$$

Proof. Let $n \geq 4$ and consider a trace zero doubly stochastic matrix of dimension n of the form

$$D_n^0 = \left(\begin{array}{c|c} D_k^0 & 0 \\ \hline 0 & D_{n-k}^0 \end{array} \right),$$

where $2 \leq k \leq n - 2$ and D_j^0 is a trace zero doubly stochastic matrix of dimension j . Since

$$\sigma(D_n^0) = \sigma(D_k^0) \cup \sigma(D_{n-k}^0),$$

it follows

$$\Pi_{n-k}^0 \subseteq \Omega_{n-k}^0 \subseteq \Omega_n^0$$

for any $2 \leq k \leq n - 2$. Hence, the theorem is proved. \square

Note that, for $n = 4$, relation (2.2) reduces to relation (2.1) while for $n \geq 5$, since $e^{\frac{2\pi i}{n-2}} \in \Pi_{n-2}^0$ but $e^{\frac{2\pi i}{n-2}} \notin \Pi_n^0$, relation (2.2) is an actual refinement of relation (2.1).

Moreover, even in relation (2.2), in general, the sign of inclusion cannot be replaced by the sign of equality. As an example, consider the case $n = 5$ and the following trace zero doubly stochastic matrix

$$D_5^0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1/5 & 0 & 0 & 4/5 & 0 \\ 4/5 & 1/5 & 0 & 0 & 0 \\ 0 & 4/5 & 1/5 & 0 & 0 \\ 0 & 0 & 4/5 & 1/5 & 0 \end{pmatrix}.$$

As Figure 2.1 makes clear, the complex conjugate eigenvalues of D_5^0 are not in the set $\Pi_3^0 \cup \Pi_5^0$ so that, at least in this case, the inclusion in relation (2.2) is strict.

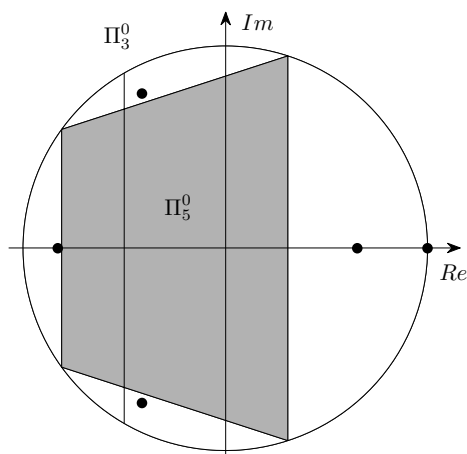


FIG. 2.1. Eigenvalues of the matrix D_5^0 and the regions Π_3^0 and Π_5^0 .



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