Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

ELA

COMBINATORIAL PROPERTIES OF GENERALIZED M-MATRICES*

MANIDEEPA SAHA† AND SRIPARNA BANDHOPADHAY‡

Abstract.

An $M_\vee$-matrix has the form $A = sI - B$ with $s \geq \rho(B)$ and B^k is entrywise nonnegative for all sufficiently large integers k. In this paper, the existence of a preferred basis for a singular $M_\vee$-matrix $A = sI - B$ with $\text{index}(B) \leq 1$ is proven. Some equivalent conditions for the equality of the height and level characteristics of A are studied. Well structured property of the reduced graph of A is discussed. Also possibility of the existence of preferred basis for another generalization of M-matrices, known as GM-matrices, is studied.

Key words. Preferred basis, Quasi-preferred basis, Height characteristic, Level characteristic.

AMS subject classifications. 15A48, 15A21, 15A18, 05C50.

1. Introduction. In this paper, we consider two types of generalizations of M-matrices, namely the class of GM-matrices [3] and $M_\vee$-matrices [10]. We show that the Preferred Basis Theorem and the Index Theorem for M-matrices are not true for GM-matrices of order greater than 2, whereas we prove the existence of a preferred basis for the subclass of $M_\vee$-matrices $A = sI - B$ with $\text{index}(B) \leq 1$, and we give a procedure to obtain a preferred basis from a quasi-preferred basis for the generalized null space for a certain subclass of $M_\vee$-matrices.

The existence of quasi-preferred bases for the class of $M_\vee$-matrices was shown by Naqvi and McDonald [9]. Rothblum, Schneider and Hershkowitz proved the existence of quasi-preferred and preferred bases for singular M-matrices ([11] and [6]).

In this paper, using similar techniques, we provide a constructive method to obtain a preferred basis from a given quasi-preferred basis for a subclass of singular $M_\vee$-matrices. Moreover, the procedure proves the existence of a preferred basis for this subclass of singular $M_\vee$-matrices.

In [9], it was proved that the height characteristic is always majorized by its level characteristic for a specific subclass of $M_\vee$-matrices. In this paper, we give some

*Received by the editors on October 15, 2014. Accepted for publication on June 8, 2015. Handling Editor: Michael Tsatsomeros.

†Department of Mathematics, National Institute of Technology Meghalaya, Shillong 793003, India (sahamanideepa@gmail.com).

‡Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati 781039, India (sriparna@iitg.ernet.in).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

necessary and sufficient conditions for the equality of these two characteristics. Later we describe the concept of well structured graphs and give a sufficient condition for the reduced graph of a subclass of $M_\vee$-matrices to be well structured.

The paper is organized as follows: we start with background and notation in Section 2. In Section 3, we consider the class of $M_\vee$-matrices, which consists of matrices of the form $A = sI - B$, where B is an eventually nonnegative matrix and $s \geq \rho(B)$. In particular, we give a procedure to obtain a preferred basis from a given quasi-preferred basis for M-matrices and for $M_\vee$-matrices with $\text{index}(B) \leq 1$, and summarize the entire procedure in Algorithm 1. We discuss height and level characteristics and give some necessary and sufficient conditions for their equality, and give a sufficient condition for the reduced graph of $M_\vee$-matrices to be well structured, introduced in [8]. In Section 4, we consider another generalization of M-matrices, known as GM-matrices, which are matrices of the form $A = sI - B$, where B and B^T possess the Perron-Frobenius property, and $s \geq \rho(B)$. We show that a quasi-preferred basis, and hence a preferred basis, may not exist for the generalized null space of these matrices of order more than two. It is shown that the Preferred Basis Theorem and the Index Theorem hold if the order is two.

2. Notation and preliminaries. This section contains basic notations and some preliminary results, mostly from [7]. We denote the set $\{1, 2, \ldots, n\}$ by $\langle n\rangle$. For a real $n \times m$ matrix $A = [a_{i,j}]$ we use the following terminology and notation.

- $A \geq 0$ (A is nonnegative) if $a_{i,j} \geq 0$, for all $i \in \langle n\rangle,\ j \in \langle m\rangle$.
- $A > 0$ (A is strictly positive) if $a_{i,j} > 0$, for all $i \in \langle n\rangle,\ j \in \langle m\rangle$.

If $n = m$, then we denote by

- $\sigma(A)$ the spectrum of A.
- $\rho(A) = \max_{\lambda\in\sigma(A)}\{|\lambda|\}$, the spectral radius of A.
- $N(A)$ the nullspace of A, and by $n(A)$ the nullity of A.
- $\text{index}_\lambda(A)$ the size of the largest Jordan block associated with the eigenvalue λ, and if A is singular we simply write $\text{index}_0(A)$ as $\text{index}(A)$.
- $E_\lambda(A)$, the generalized eigenspace of A corresponding to the eigenvalue λ, i.e., $N((\lambda I - A)^n)$. In case A is a singular matrix, we simply write $E(A)$ for $E_0(A)$.

Definition 2.1. For $n \geq 2$, an $n \times n$ matrix A is said to be *reducible* if there exists a permutation matrix Π such that

$$\Pi A \Pi^T = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}, \tag{2.1}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

where B and D are square, nonempty matrices. Otherwise A is called *irreducible*. If A is reducible and in the form (2.1), and if a diagonal block is reducible, then this block can be reduced further via permutation similarity. If this process is continued, then finally there exists a suitable permutation matrix Π such that A is in block triangular form

$$\Pi A \Pi^T = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ 0 & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{pp} \end{bmatrix}, \tag{2.2}$$

where each block $A_{i,i}$ is square and irreducible. This block triangular form is called a *Frobenius normal form* of A. An irreducible matrix consists of one block, is in Frobenius normal form.

If $A = [A_{i,j}]$ is an $n \times n$ matrix in Frobenius normal form with p block rows and columns, and when discussing matrix-vector multiplication with A or the structure of eigenvectors of A, we partition vectors b analogously in p vector components b_i conformably with A, and we define the *support* of b via $\text{supp}(b) = \{i \in \langle p \rangle : b_i \neq 0\}$.

For an $n \times n$ matrix A, the *directed graph* of A denoted by $\Gamma(A)$ is the directed graph with vertices $1, 2, \ldots, n$ in which (i, j) is an edge if and only if $a_{ij} \neq 0$. A *path* from vertex j to vertex m of *length* t is a sequence of t vertices $v_1, v_2, \ldots, v_t$ such that (v_l, v_{l+1}) is an edge in $\Gamma(A)$ for $l = 1, 2, \ldots, t-1$ where $v_1 = j$ and $v_t = m$. We say a vertex j has *access* to m, if $j = m$ or there is a path from j to m in $\Gamma(A)$, and in this case we write $j \to m$. We write $j \nrightarrow m$ if j does not have access to m. The *transitive closure* of $\Gamma(A)$, denoted by $\overline{\Gamma(A)}$, is the graph with the same vertex set as that of $\Gamma(A)$ and (i, j) is an edge in $\overline{\Gamma(A)}$ if i has access to j in $\Gamma(A)$. If j has access to m and m has access to j, we say j and m *communicate*. The communication relation is an equivalence relation on $\{1, 2, \ldots, n\}$ and an equivalence class α is called a *class* of A. For any two classes α and β of A, we say that α has access β in $\Gamma(A)$ if there are vertices $i \in \alpha$ and $j \in \beta$ such that i has access to j in $\Gamma(A)$.

The *reduced graph* of A, denoted by $R(A)$ is the graph with vertex set consisting of all the classes in A and (i, j) is an edge in $R(A)$ if and only if i has access to j in $\Gamma(A)$.

For any $\alpha, \beta \subset \{1, 2, \ldots, n\}$, $A_{\alpha\beta}$ denotes the submatrix of A whose rows are indexed by α and whose columns are indexed by β. If α is a class of A, then we say that α is a *basic class* if $\rho(A_{\alpha\alpha}) = \rho(A)$, a *singular class* if $A_{\alpha\alpha}$ is singular, an *initial class* if it is not accessed by any other class of A and a *final class* if it does not have access to any other class of A.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

ELA

A *chain* of classes is a collection of classes such that each class in the collection has access to or from every other class in the collection. A chain of classes with initial class J and final class K is called a chain from J to K. The *length of a chain* is the number of singular classes it contains. We say J has access to K in *n steps* if the length of the longest chain from J to K is n.

DEFINITION 2.2. For a set W of vertices in the vertex set $V(A)$ of $R(A)$ we introduce the following sets.

$$\begin{aligned}
\text{below}(W) &= \{i \in V(A) : \text{ there exists } j \in W \text{ such that } i \to j\};\\
\text{above}(W) &= \{i \in V(A) : \text{ there exists } j \in W \text{ such that } j \to i\};\\
\text{top}(W) &= \{i \in W : j \in W, i \to j \text{ imply } i = j\};\\
\text{bottom}(W) &= \{i \in W : j \in W, j \to i \text{ imply } i = j\}.
\end{aligned}$$

DEFINITION 2.3. Let A be an $n \times n$ singular matrix in Frobenius normal form (2.2). We say a vertex i in $R(A)$ is a *singular vertex* if the corresponding block A_{ii} in (2.2), is singular. Let $H(A)$ be the collection of all singular vertices in $R(A)$.

(i) We define the *singular graph* $S(A)$ associated with $R(A)$ as the graph with vertex set $H(A)$ and (i, j) is an edge if and only if $i = j$ or there is a path from i to j in $R(A)$.
(ii) The *level of a vertex i* in $R(A)$, denoted by level(i), is the maximal number of singular vertices on a path in $R(A)$ that terminates at i.
(iii) Let x be a block-vector with p blocks, partitioned according to the Frobenius normal form of A. The *level of x*, denoted by level(x), is defined to be $\max\{\text{level}(i) : i \in \text{supp}(x)\}$.
(iv) For a nonzero vector x in the generalized nullspace $E(A)$, we define the *height of x*, denoted by height(x), to be the smallest nonnegative integer k such that $A^k x = 0$.

The other essential objects in our analysis are appropriately chosen sets of basis vectors for the generalized eigenspace associated with the spectral radius.

DEFINITION 2.4. Let A be a square matrix in Frobenius normal form (2.2), and let $H(A) = \{\alpha_1, \dots, \alpha_q\}$, with $\alpha_1 < \cdots < \alpha_q$ be the set of singular vertices in $R(A)$.

A set of vectors $x^1 = [x^1_j], \dots, x^q = [x^q_j] \geq 0$ is called a *quasi-preferred set* for A if

$$x^i_j > 0 \text{ if } j \to \alpha_i, \text{ and } x^i_j = 0 \text{ if } j \not\to \alpha_i$$

for all $i = 1, \dots, q$ and $j = 1, \dots, p$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

ELA

If in addition we have

$$-Ax^i = \sum_{k=1}^{q} c_{k,i} x^k,\ i = 1, \ldots, q,$$

where $c_{k,i}$ satisfy

$$c_{k,i} > 0 \text{ if } \alpha_k \to \alpha_i, i \neq k; \text{ and } c_{k,i} = 0 \text{ if } \alpha_k \nrightarrow \alpha_i \textit{ or } i = k$$

then the set of vectors $x^1, \ldots, x^q$ is said to be a *preferred set* for A. A (quasi-) preferred set that forms a basis for $E(A)$ is called a *(quasi-) preferred basis* for A.

Throughout this paper we will assume that the matrix A is in Frobenius normal form (see (2.2)), and we denote the (i, j)-block of the Frobenius normal form of A by A_{ij}. Every x with n entries will be assumed to be partitioned into p vector components x_i conformably with A.

Definition 2.5. An $n \times n$ matrix A is called an *M-matrix* if it can be written as $A = sI - B$, where $B \geq 0$ and $s \geq \rho(B)$. The following results are well-known.

Theorem 2.6. [7] *(Preferred Basis Theorem) If A is a singular M-matrix, then there exists a preferred basis for the generalized eigenspace $E(A)$ of A.*

Theorem 2.7. [7, 11] *(Index Theorem) If A is a singular M-matrix, then $\text{index}_{\rho(A)}(A)$ is equal to the length of the longest chain in $R(A)$.*

After having introduced the basic concepts, in the next section we consider one generalization of M-matrices, known as $M_\vee$-matrices.

3. Combinatorial structure of singular $M_\vee$-matrices.

3.1. Preferred basis for singular $M_\vee$-matrices. In this section, we first prove some results on the combinatorial properties of quasi-preferred bases of a subclass of $M_\vee$-matrices which will be used subsequently to give a constructive method for obtaining a preferred basis from a quasi-preferred basis.

Definition 3.1. Let $A \in \mathbb{R}^{n,n}$. For any two vertices i and j of $R(A)$, let $\text{hull}(i, j) := \text{above}(i) \bigcap \text{below}(j)$.

Definition 3.2. A square matrix A is called an *eventually nonnegative (positive)* matrix if there is a positive integer n_0 such that $A^k \geq 0$ ($A^k > 0$) for all $k \geq n_0$.

Definition 3.3. A square matrix A is called an *$M_\vee$-matrix* if it can be expressed as $A = sI - B$ with eventually nonnegative B and $s \geq \rho(B)$.

Throughout the remaining two sections we assume that a singular $M_\vee$-matrix A has the form $A = \rho I - B$, where B is an eventually nonnegative matrix with $\rho = \rho(B)$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

and A has q singular classes with $H(A) = \{\alpha_1, \dots, \alpha_q\}$ as the set of singular classes of A, where $\alpha_1 < \cdots < \alpha_q$.

The following results are well known.

THEOREM 3.4. [5] *Let A be a square matrix in block triangular form and let x be a vector. Then* $\mathrm{supp}(Ax) \subseteq \mathrm{below}(\mathrm{supp}(x))$.

THEOREM 3.5. [9] *Suppose that A is an eventually nonnegative matrix with* $\mathrm{index}(A) \leq 1$ *and* $D_A = \{d \mid \theta - \alpha = \frac{c}{d},$ *where* $re^{2\pi i\theta} \in \sigma(A), re^{2\pi i\alpha} \in \sigma(A),\ r > 0, c \in \mathbb{Z}^+, d \in \mathbb{Z} \setminus \{0\}, gcd(c,d) = 1\}$. *Let g be a prime number such that $g \notin D_A$ and $A^k \geq 0$ for all $k \geq g$. Then $\overline{R(A)} = \overline{R(A^g)}$.*

LEMMA 3.6. [9] *Let $A \in \mathbb{C}^{n,n}$ and $\lambda \in \sigma(A), \lambda \neq 0$. Then for all $k \notin D_A$ we have $N(\lambda I - A) = N(\lambda^k I - A^k)$ and the Jordan blocks of λ^k in $J(A^k)$ are obtained from the Jordan blocks of λ in $J(A)$ by replacing λ with λ^k.*

THEOREM 3.7. [9] *Let A be an eventually nonnegative matrix with* $\mathrm{index}(A) \leq 1$. *Then A has a quasi-preferred basis for $E_{\rho(A)}(A)$.*

REMARK 3.8. Let $A = \rho I - B$ be an $M_\vee$-matrix with $\mathrm{index}(B) \leq 1$. Then by Theorem 3.7, there exists a quasi-preferred basis $\mathcal{B} = \{x^1, x^2, \dots, x^q\}$ for $E_\rho(B)$, and hence, $\mathcal{B}$ is a basis for $E(A)$. Since $Ax^i \in E(A)$ for all $i \in \{1, \dots, q\}$, there always exists a matrix Z(coefficient matrix) such that $-AX = XZ$, where $X = [x^1\ x^2\ \cdots\ x^q]$.

LEMMA 3.9. *If i, j are vertices of $\Gamma(A)$, then there is a path of length k from i to j in $\Gamma(A)$ if and only if the (i,j)-entry of A^k is nonzero.*

Proof. If $(A^k)_{ij}$ denotes the (i,j)-entry of A^k, then

$$(A^k)_{ij} = \sum_{i_1}\sum_{i_2}\cdots\sum_{i_{k-1}} a_{ii_1}a_{i_1i_2}\cdots a_{i_{k-1}j}$$

and $(A^k)_{ij} \neq 0$ if and only if $a_{ii_1}a_{i_1i_2}\cdots a_{i_{k-1}j} \neq 0$ for some $i_1, i_2, \dots, i_{k-1}$, that is, if and only if there is a path of length k from i to j through $i_1, i_2, \dots, i_{k-1}$. ☐

LEMMA 3.10. *Let A be a singular matrix and let X be such that its columns form a quasi-preferred basis of $E(A)$. If Z is such that $AX = XZ$, then*

$$z_{ij} = 0 \text{ if } \alpha_i \not\twoheadrightarrow \alpha_j$$

In particular, Z is triangular with all its diagonal entries equal to 0.

Proof. Since $AX = XZ$ and $X = [x^1\ \cdots\ x^q]$, we have

$$Ax^j = \sum_{i=1}^{q} z_{ij}x^i \quad \text{for all } j = 1, \dots, q. \tag{3.1}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

ELA

Take any $\alpha_j \in H(A)$ and consider the set $Q = \{\alpha_i \in H(A)|\ z_{ij} \neq 0\}$. Since $\alpha_i \not\rightarrow \alpha_j$ implies $\alpha_i \notin \text{below}(\alpha_j)$, to prove (3.1) we have to essentially show $Q \subseteq \text{below}(\alpha_j)$. To show $Q \subseteq \text{below}(\alpha_j)$, it is enough to show, $\text{top}(Q) \subseteq \text{below}(\alpha_j)$.

Consider any $\alpha_k \in \text{top}(Q)$. If $\alpha_k \notin \text{below}(\alpha_j)$, then $\left[(A - z_{jj}I)x^j\right]_{\alpha_k} = 0$, since $\text{supp}\left((A - z_{jj}I)x^j\right) \subseteq \text{below}(\text{supp}(x^j))$. Then equation (3.1) gives $\sum_{\substack{i\in Q\\ i\neq j}} z_{ij}x^i_{\alpha_k} = 0$. But $\alpha_k \in \text{top}(Q)$ implies $z_{kj}x^k_{\alpha_k} = 0$ which is not possible, hence $\alpha_k \in \text{below}(\alpha_j)$. Thus, we have that $\text{top}(Q) \subseteq \text{below}(\alpha_j)$, and hence, $Q \subseteq \text{below}(\alpha_j)$.

Since $AX = XZ$ and $\{x^1, \dots, x^q\} \subseteq E(A)$, $A^nX = 0 = XZ^n$. As Z is triangular and X is of full column rank, all the diagonal entries of Z must be equal to 0. □

LEMMA 3.11. *Let A be a singular M-matrix and X be such that the columns of X form a quasi-preferred basis for $E(A)$. Let Z be a matrix satisfying the condition $-AX = XZ$. If α_i and α_j are two singular classes with* $\text{hull}(\alpha_i, \alpha_j) \bigcap H(A) = \{\alpha_i, \alpha_j\}$, *then $z_{ij} > 0$.*

Proof. Let there exist a pair of singular classes α_i, $\alpha_j \in H(A)$ such that $\text{hull}(\alpha_i, \alpha_j) \bigcap H(A) = \{\alpha_i, \alpha_j\}$ and $z_{ij} \leq 0$. Since $X = [x^1\ \cdots\ x^q]$, by Lemma 3.10,

$$\left(-Ax^j\right)_{\alpha_i} = x^i_{\alpha_i}z_{ij} + \cdots + x^{j-1}_{\alpha_i}z_{j-1,j}. \tag{3.2}$$

As $\{x^1, \dots, x^q\}$ is a quasi-preferred basis for A and $\text{hull}(\alpha_i, \alpha_j) \bigcap H(A) = \{\alpha_i, \alpha_j\}$, equation (3.2) gives $(-Ax^j)_{\alpha_i} = x^i_{\alpha_i}z_{ij} \leq 0$. Also since A is an M-matrix and $(Ax^j)_{\alpha_i} = A_{\alpha_i,\alpha_i}x^j_{\alpha_i} + \sum_{k=\alpha_i+1}^{\alpha_j} A_{\alpha_i,k}x^j_k$, it follows that $A_{\alpha_i,\alpha_i}x^j_{\alpha_i} \geq 0$. Since A_{α_i,α_i} is an irreducible singular M-matrix, $A_{\alpha_i,\alpha_i}x^j_{\alpha_i} \geq 0$ implies $A_{\alpha_i,\alpha_i}x^j_{\alpha_i} = 0$ [1, p.156]. Hence, it follows that $\sum_{k=\alpha_i+1}^{\alpha_j} A_{\alpha_i,k}x^j_k = 0$ and for any $k = \alpha_i + 1, \dots, \alpha_j$, if $A_{\alpha_i,k} < 0$ then $x^j_k = 0$. This contradicts $\alpha_i \rightarrow \alpha_j$, hence $z_{ij} > 0$. □

LEMMA 3.12. *Let $A = \rho I - B$ be an $M_\vee$-matrix with $\rho = \rho(B)$ and* $\text{index}_\rho(A) \leq 1$. *Let the matrix X be such that its columns form a quasi-preferred basis in $E(A)$ and let Z be a matrix satisfying the condition $-AX = XZ$. If α_i and α_j are two singular classes with* $\text{hull}(\alpha_i, \alpha_j) \bigcap H(A) = \{\alpha_i, \alpha_j\}$, *then $z_{ij} > 0$.*

Proof. Given $A = \rho I - B$, where B is an eventually nonnegative matrix with $\text{index}(B) \leq 1$ and $\rho = \rho(B)$. As D_B, defined in Theorem 3.5, is finite and B is eventually nonnegative matrix, so we can always choose a prime number g such that $g \notin D_B$ and $B^l \geq 0$ for all integer $l \geq g$. Since $-AX = XZ$, $B^kX = X\bar{Z}^k$ for any positive integer k, where $\bar{Z} = Z + \rho I$. Take $\tilde{B} = B^g$ and $\tilde{Z} = \bar{Z}^g$, then $\tilde{B} \geq 0$ and since by Theorem 3.5 the accessibility relations in B and $\tilde{B}$ are same, columns of X

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

will also be a quasi-preferred basis for $E(\rho^g I-\tilde{B})$. If α_i,α_j are singular classes of A with $\text{hull}(\alpha_i,\alpha_j)\bigcap H(A)=\{\alpha_i,\alpha_j\}$ then by Lemma 3.11, $\tilde{z}_{ij}>0$. Let $(\bar{Z}^k)_{ij}$ be the (i,j)-entry of $\bar{Z}^k$, for any k and we simply write $\bar{Z}_{ij}$ when $k=1$. We will use strong induction on l to show that $(\bar{Z}^l)_{ij}=l\rho^{l-1}z_{ij}$ for any integer $l\geq 2$, hence $\tilde{z}_{ij}>0$ will imply $z_{ij}>0$.

For $l=2$, $(\bar{Z}^2)_{ij}=2\rho\bar{Z}_{ij}+\sum\limits_{l=i+1}^{j-1}\bar{Z}_{il}\bar{Z}_{lj}=2\rho z_{ij}+\sum\limits_{l=i+1}^{j-1}z_{il}z_{lj}$.

Since $\text{hull}\{\alpha_i,\alpha_j\}\bigcap H(A)=\{\alpha_i,\alpha_j\}$, from Lemma 3.10 it follows that $z_{il}z_{lj}=0$ for all l, $i+1\leq l\leq j-1$. Thus, $(\bar{Z}^2)_{ij}=2\rho z_{ij}$. Let $(\bar{Z}^l)_{ij}=l\rho^{l-1}z_{ij}$ for all $l<k$ and $k>2$.
Now,

$$\begin{aligned}(\bar{Z}^k)_{ij}&=\bar{Z}_{ii}(\bar{Z}^{k-1})_{ij}+\sum_{l=i+1}^{j-1}\bar{Z}_{il}(\bar{Z}^{k-1})_{lj}+\bar{Z}_{ij}(\bar{Z}^{k-1})_{jj}\\&=\rho(k-1)\rho^{k-2}z_{ij}+\sum_{l=i+1}^{j-1}z_{il}(\bar{Z}^{k-1})_{lj}+z_{ij}\rho^{k-1}\\&=k\rho^{k-1}z_{ij}+\sum_{l=i+1}^{j-1}z_{il}(\bar{Z}^{k-1})_{lj}.\end{aligned}$$

From Lemma 3.9, if $z_{il}(\bar{Z}^{k-1})_{lj}\neq 0$ for some l, $i+1\leq l\leq j-1$ then there is a path from i to l in $\Gamma(Z)$ and from l to j in $\Gamma(\bar{Z})$. Hence, by Lemma 3.10, there is a path from i to j in $\Gamma(A)$ through at least 3 singular classes i,l and j of A, which contradicts the fact that $\text{hull}(i,j)\bigcap H(A)=\{i,j\}$. Thus, $\sum\limits_{l=i+1}^{j-1}z_{il}(\bar{Z}^{k-1})_{lj}=0$, or $(\bar{Z}^k)_{ij}=k\rho^{k-1}z_{ij}$. Hence, $\tilde{z}_{ij}=g\rho^{g-1}z_{ij}>0$, which implies $z_{ij}>0$ and the result follows. □

If B is an eventually nonnegative matrix with $\text{index}(B)>1$, then B need not have a quasi-preferred basis. However if $\text{index}(B)\leq 1$ it is known from [9] that B, and hence, $A=\rho I-B$ has a quasi-preferred basis. In this section, we give a procedure to obtain a preferred basis from a quasi-preferred basis for any $M_\vee$-matrix A, where $A=\rho I-B$ with $\text{index}(B)\leq 1$.

Procedure 3.13. ***Constructive method of obtaining a preferred basis from a quasi-preferred basis:***

Let $A=\rho I-B$ be an $M_\vee$-matrix with $\text{index}_\rho(A)\leq 1$ and let $X=[x^1\ x^2\ \cdots\ x^q]$ be an $n\times q$ matrix whose columns form a quasi-preferred basis for $E(A)$. Then by Remark 3.8, we can choose a matrix Z satisfying $-AX=XZ$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

We now construct a preferred basis (from the given quasi-preferred basis X) $\tilde{X}$ such that $-A\tilde{X} = \tilde{X}\tilde{Z}$ for some nonnegative matrix $\tilde{Z}$.

If the columns of X already give a preferred basis for $E(A)$, then we are done. If the columns of X form a quasi-preferred basis but not a preferred basis for $E(A)$, then there exist indices i_0 and j_0 such that $\alpha_{i_0} \to \alpha_{j_0}$ and $z_{i_0,j_0} \leq 0$. If $\mathcal{I} := \{j \in \langle q \rangle \mid z_{ij} < 0 \text{ for some } i\} \bigcup \{j \in \langle q \rangle \mid \alpha_i \to \alpha_j \text{ and } z_{ij} = 0 \text{ for some } i\}$, then $\mathcal{I} \neq \emptyset$ since $j_0 \in \mathcal{I}$. Let j be the least index in $\mathcal{I}$. Then the first $j-1$ columns of X forms a preferred set for $E(A)$. To find an $\tilde{x}^j$ such that if $\tilde{X}$ is the matrix obtained by replacing the jth column x_j of X by $\tilde{x}^j$, then the first j columns of $\tilde{X}$ will be a preferred set of $E(A)$. Finally we show that it can be done for every $j \geq 2$.

Let

$$\begin{aligned} Q &= \{i \in \langle j-1 \rangle \mid z_{ij} < 0\} \\ R &= \{i \in \langle j-1 \rangle \mid z_{ij} = 0, \alpha_i \to \alpha_j\} \\ S &= Q \bigcup R \\ \bar{Q} &= \langle j-1 \rangle \setminus Q \\ \bar{R} &= \langle j-1 \rangle \setminus R \\ \bar{S} &= \bar{Q} \bigcap \bar{R}. \end{aligned}$$

We claim that $S \neq \emptyset$. Since for all $i \in S$, $\alpha_i \to \alpha_j$, there exists an $l(i) \in H(A)$ such that $\alpha_{l(i)} \to \alpha_j$ and $\text{hull}(\alpha_i, \alpha_{l(i)}) \bigcap H(A) = \{\alpha_i, \alpha_{l(i)}\}$. Since for all $i \in S$, $z_{ij} \leq 0$ and from Lemma 3.12, $z_{i,l(i)} > 0$, so $l(i) < j$ for all $i \in S$.

Case I: $Q = \emptyset$. Let $\tilde{x}^j = x^j + \sum_{i \in R} x^{l(i)}$. Since $-Ax^{l(i)} = z_{i,l(i)}x^i + \sum_{\substack{k=1 \\ k \neq i}}^{l(i)-1} z_{k,l(i)}x^k$ and $-Ax^j = \sum_{i \in \bar{R}} z_{ij}x^i$, we have $-A\tilde{x}^j = \sum_{i \in R} z_{i,l(i)}x^i + \sum_{i \in R} \sum_{\substack{k=1 \\ k \neq i}}^{l(i)-1} z_{k,l(i)}x^k + \sum_{i \in \bar{R}} z_{ij}x^i$. Since the first $j-1$ columns of X formed a preferred set for $E(A)$ and $z_{i,l(i)} > 0$ for all $i \in R$, $\{x^1, \ldots, x^{j-1}, \tilde{x}^j\}$ forms a preferred set for $E(A)$.

Case II: $Q \neq \emptyset$. Let $\tilde{x}^j = x^j + \beta \sum_{i \in S} x^{l(i)}$. Then

$$-A\tilde{x}^j = \beta \sum_{i \in R} z_{i,l(i)}x^i + \sum_{i \in Q} \left(\beta z_{i,l(i)} + z_{ij}\right) x^i + \sum_{i \in S} \sum_{\substack{k=1 \\ k \neq i}}^{l(i)-1} \beta z_{k,l(i)}x^k + \sum_{i \in \bar{S}} z_{ij}x^i.$$

For $\beta > \max_{i \in Q} \left\{ \frac{-z_{ij}}{z_{i,l(i)}} \right\} > 0$, $\{x^1, \ldots, x^{j-1}, \tilde{x}^j\}$ forms a preferred set for $E(A)$. Hence, in both cases if we take $\tilde{X} = \begin{bmatrix} x^1 & \cdots & \tilde{x}^j & \cdots & x^q \end{bmatrix}$ and if $\bar{Z}$ is the matrix satisfying the condition $-A\tilde{X} = \tilde{X}\bar{Z}$, then the leading j columns of $\tilde{X}$ form a preferred set for $E(A)$. The above process is repeated with X replaced by $\tilde{X}$. Since at every stage at

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

least one more column is included in the preferred set, after at most $q - j$ steps we will get a preferred basis for $E(A)$. □

The following theorem is an immediate consequence of Procedure 3.13.

THEOREM 3.14. *If $A = \rho I - B$ is an $M_\vee$-matrix with* $\text{index}_\rho(A) \leq 1$, *then there is a preferred basis for $E(A)$.*

REMARK 3.15. The Procedure 3.13 can also be used to obtain a preferred basis from a given quasi-preferred basis for M-matrices.

We summarize the entire procedure below.

Algorithm 1 Given $A \in \mathbb{R}^{n,n}, X \in \mathbb{R}^{n,q}$

$H(A) = \{\alpha_1, \ldots, \alpha_q\}$ basis classes of A
$Z = X^+AX$ (X^+ is the pseudo inverse of X)
$\mathcal{I} = \{j \in \langle q\rangle \mid z_{ij} < 0 \text{ for some } i\} \bigcup \{j \in \langle q\rangle \mid z_{ij} = 0, \alpha_i \to \alpha_j, \text{ for some } i\}$
while $I \neq \emptyset$ **do**
  $j = \min I$
  $Q = \{i \in \langle j-1\rangle | z_{ij} < 0\} = \{i_1, \ldots, i_m\}$
  $R = \{i \in \langle j-1\rangle | z_{ij} = 0, \alpha_i \to \alpha_j\} = \{i_{m+1}, \ldots, i_t\}$
  if $Q = \emptyset$ **then**
    for $k = m+1 \colon t$ **do**
      $l(k) \longleftarrow \text{hull}\{\alpha_{i_k}, \alpha_{l(k)}\} \bigcap H(A) = \{\alpha_{i_k}, \alpha_{l(k)}\}$ and $\alpha_{l(k)} \to \alpha_j$
    end for
    for $r = 1 \colon n$ **do**
      $$X_{rj} \longleftarrow X_{rj} + \sum_{k=m+1}^{t} X_{rl(k)}$$
    end for
  else
    for $k = 1 \colon t$ **do**
      $l(k) \longleftarrow \text{hull}\{\alpha_{i_k}, \alpha_{l(k)}\} \bigcap H(A) = \{\alpha_{i_k}, \alpha_{l(k)}\}$ and $\alpha_{l(k)} \to \alpha_j$
    end for
    Choose $\beta > \max_{1 \leq k \leq m} \left\{ \frac{-z_{i_k j}}{z_{i_k l(k)}} \right\}$
    for $r = 1 \colon n$ **do**
      $$X_{rj} \longleftarrow X_{rj} + \beta \sum_{k=1}^{t} X_{rl(k)}$$
    end for
  end if
  $Z = X^+AX$
  $\mathcal{I} = \{j \in \langle q\rangle \mid z_{ij} < 0 \text{ for some } i\} \bigcup \{j \in \langle q\rangle \mid z_{ij} = 0, \alpha_i \to \alpha_j, \text{ for some } i\}$
end while

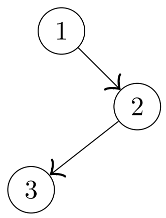
Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

We illustrate Procedure 3.13 with the help of the following example.

Example 3.16. Let

$$B = \begin{bmatrix} 2 & 2 & 4 & -1 & 0 & 0 \\ 2 & 2 & -1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 6 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 2 & 1 \end{bmatrix}.$$

Then $B^k \geq 0$ for all $k \geq 7$ with $\rho(B)$=4. Consider the $M_\vee$ matrix $A = 4I - B$ so that $E(A) = N(A^3)$ and $\text{index}_4(A) = 1$. The reduced graph of A is given by,



Consider the quasi-preferred basis for $E(A)$ given by,

$$x^1 = [2\ \ 2\ \ 0\ \ 0\ \ 0\ \ 0]^T$$
$$x^2 = [271\ \ 241\ \ 36\ \ 12\ \ 0\ \ 0]^T$$
$$x^3 = [3.0625\ \ 1\ \ 2.8\ \ 1\ \ 1.5\ \ 1]^T.$$

Take $X = [x^1\ x^2\ x^3]$. Then $-AX = XZ$ implies that

$$Z = \begin{bmatrix} 0 & 36 & -0.35 \\ 0 & 0 & 0.25 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the set $\mathcal{I} = \{j \in \langle 4 \rangle \mid z_{ij} < 0 \text{ for some } i\} \bigcup \{j \in \langle 4 \rangle \mid \{z_{ij} = 0, \alpha_i \to \alpha_j, \text{ for some } i\}\} = \{3\} \bigcup \emptyset$. So 3 is the least index in $\mathcal{I}$. Now consider the set $Q = \{i \mid z_{i3} < 0\} = \{1\}$. Again we have $\text{hull}(1,2) \bigcap H(A) = \{1,2\}$. Define the vector $x^3_{new} = x^3 + x^2$ so that

$$-Ax^3_{new} = 35.65x^1 + 0.25x^2 + 4x^3_{new}.$$

Then,

$$-A[x^1\ x^2\ x^3_{new}] = [x^1\ x^2\ x^3_{new}] \begin{bmatrix} 0 & 36 & 35.65 \\ 0 & 0 & 0.25 \\ 0 & 0 & 0 \end{bmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

Thus, we have the preferred basis $\{x^1, x^2, x^3_{new}\}$ for $E(A)$ such that if $X_{final} = [x^1\ x^2\ x^3_{new}]$, then

$$-AX_{final} = X_{final}\begin{bmatrix} 0 & 36 & 35.65 \\ 0 & 0 & 0.25 \\ 0 & 0 & 0 \end{bmatrix} := X_{final}Z_{final}.$$

3.2. Height and level characteristics of $M_\vee$-matrices and well structured graph. Most of following results were obtained by Schneider and Hershkowitz in [7, 8, 4], for the class of singular M-matrices. We try to give independent proofs of each of the results and extend it for the class of $M_\vee$-matrices. This section essentially deals with two different types of characteristics, namely height characteristic and level characteristic and we give some necessary and sufficient conditions for their equality. Later we give a sufficient condition for the reduced graph of an $M_\vee$-matrix to be well structured.

3.2.1. Height and level characteristics of $M_\vee$-matrices. We begin this section with some definitions, most of them are taken from [8].

Definition 3.17. [7, 8] Let $t = \text{index}(A)$. For $i \in \langle t\rangle$, let $\eta_i(A) = n(A^i) - n(A^{i-1})$. The sequence $(\eta_1(A), \ldots, \eta_t(A))$ is called the *height (*or *Weyr) characteristic* of A, and is denoted by $\eta(A)$. Normally we write η_i for $\eta_i(A)$ where no confusion should result.

Definition 3.18. [8] Let A be a singular matrix and let $\text{index}(A) = t$.

(i) Let S be a collection of vectors in $E(A)$, and let $\eta_k(S)$ be the number of vectors in S of height k. The *height signature* $\eta(S)$ of S is defined as the t-tuple $(\eta_1(S), \ldots, \eta_t(S))$.
(ii) A basis $\mathcal{B}$ for $E(A)$ is said to be a *height basis* for $E(A)$ if $\eta(\mathcal{B}) = \eta(A)$.

Definition 3.19. [8] Let A be a singular matrix.

(i) The *Segré* characteristic $j(A)$ of A is defined to be the nonincreasing sequence of sizes of the Jordan blocks of A associated with the eigenvalue 0.
(ii) A sequence $(x^1, \ldots, x^s)$ of vectors in $E(A)$ is said to be a *Jordan chain* for A if $Ax^i = x^{i-1},\ i \in \{2, \ldots, s\}$, and $Ax^1 = 0$. The vector x^s is called the *top* of the chain $(x^1, \ldots, x^s)$.
(iii) A basis for $E(A)$ that consists of disjoint Jordan chains for A is said to be a *Jordan basis* for $E(A)$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

ELA

Remark 3.20. It is known that $E(A)$ always has a Jordan basis.

Remark 3.21. Observe that every Jordan basis for A is a height basis, but clearly a height basis need not be a Jordan basis.

Definition 3.22. [4] Let $a = (a_1, \ldots, a_r)$ be a nonincreasing sequence of positive integers. Consider the diagram formed by r columns of stars such that the jth column has a_j stars. The sequence $a^\star$ *dual to* a is defined to be the sequence of row lengths of the diagram, reordered in a nonincreasing order.

It is well known that the height characteristic and the *Segré* characteristic are dual sequences (see [13]).

Definition 3.23. [8] The cardinality of the jth level of $S(A)$ is denoted by $\lambda_j(A)$. If $S(A)$ has m levels, then the sequence $(\lambda_1(A), \ldots, \lambda_m(A))$ is called the *level characteristic* of A, and is denoted by $\lambda(A)$. Normally we write λ_i for $\lambda_i(A)$ where no confusion should result.

Convention 3.24. We will always assume that the level characteristic and the height characteristic of A to be $(\lambda_1, \ldots, \lambda_m)$ and $(\eta_1, \ldots, \eta_t)$, respectively.

Remark 3.25. [9] If $A = \rho I - B$ is an $M_\vee$-matrix with $\text{index}_\rho(A) \leq 1$, then m and t in Convention 3.24 are equal.

Definition 3.26. [8] Let A be a square matrix.

- (i) Let S be a collection of vectors in $E(A)$, and let $\lambda_k(S)$ be the number of vectors in S of level k. We define the *level signature* $\lambda(S)$ of S as the m-tuple $(\lambda_1(S), \ldots, \lambda_m(S))$.
- (ii) A basis $\mathcal{B}$ for $E(A)$ is said to be a *level basis* for $E(A)$ if $\lambda(\mathcal{B}) = \lambda(A)$.
- (iii) A basis $\mathcal{B}$ for $E(A)$ is said to be a *height-level basis* for $E(A)$ if $\mathcal{B}$ is both height and level basis.

Definition 3.27. [7] Let A be an $n \times n$ singular matrix and let $\mathcal{B} = \{x^1, \ldots, x^q\}$ be a basis for $E(A)$. Denote $X = \begin{bmatrix} x^1 \cdots x^q \end{bmatrix} \in \mathbb{R}^{n,q}$. Then there exists a unique matrix $C \in \mathbb{R}^{q,q}$ such that $AX = XC$. This matrix is called the *induced matrix for* A *by* $\mathcal{B}$, and is denoted by $C(A, \mathcal{B})$.

Definition 3.28. [7] Let $\mathcal{P}$ be the set of p-tuples of nonnegative integers. $\mathcal{P}$ is partially ordered in the following way: If $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_p)$ are in $\mathcal{P}$, then we define $a \preccurlyeq b$ if

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

ELA

$$\begin{cases} \displaystyle\sum_{i=1}^{k} a_i \le \sum_{i=1}^{k} b_i, \quad k \in \langle p-1\rangle, \\ \displaystyle\sum_{i=1}^{p} a_i = \sum_{i=1}^{p} b_i. \end{cases}$$

If $a \preccurlyeq b$, then a is said to be majorized by b. If $a \preccurlyeq b$ and $a \neq b$, then it is written as $a \prec b$.

Remark 3.29. Let $\mathcal{B}$ be a basis of $E(A)$. If $\eta(\mathcal{B}) = (\eta_1(\mathcal{B}), \dots, \eta_t(\mathcal{B}))$ is the height signature of $\mathcal{B}$, then for any $k \in \langle t\rangle$, $\mathcal{B}$ has $\eta_1(\mathcal{B}) + \cdots + \eta_k(\mathcal{B})$ elements of height at most k, and hence, $\eta_1(\mathcal{B}) + \cdots + \eta_k(\mathcal{B}) \le \eta_1 + \cdots + \eta_k$, so $\eta(\mathcal{B}) \preccurlyeq \eta(A)$.

By a similar argument, $\lambda(\mathcal{B}) \preccurlyeq \lambda(A)$ for any basis $\mathcal{B}$ of $E(A)$.

Lemma 3.30. [7] *Given A, let y be a linear combination of the n-component vectors $x^1, \dots, x^r$. Then* $\text{level}(y) \le \max\{\text{level}(x^i) : i \in \langle r\rangle\}$.

Lemma 3.31. [7] *Given A, let y be a linear combination of the n-component vectors $x^1, \dots, x^r$. Then* $\text{height}(y) \le \max\{\text{height}(x^i) : i \in \langle r\rangle\}$.

Lemma 3.32. *If $\mathcal{B}$ is a preferred basis of an $M_\vee$-matrix $A = \rho I - B$ with* $\text{index}_\rho(A) \le 1$, *then* $\text{level}(A^k x) \le \text{level}(x) - k$ *for all $x \in \mathcal{B}$ and $k \ge 1$.*

Proof. Let $\mathcal{B} = \{x^1, \dots, x^q\}$. Since

$$(-1)^k A^k x^i = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_k} c_{i_1 i_2} \cdots c_{i_k i} x^{i_1}, \qquad i_1 \neq \cdots \neq i_k \neq i,$$

and $c_{i_1 i_2} \cdots c_{i_k i} > 0$ for some $i_1 \neq \cdots \neq i_k \neq i$ if and only if there is a chain of length k from i_1 to i, so it follows that $\text{level}(x^{i_1}) \le \text{level}(x^i) - k$, for all i_1. Hence, by Lemma 3.30, the result follows. ◻

Corollary 3.33. *For any preferred basis $\mathcal{B}$ of A,* $\text{height}(x) \le \text{level}(x)$, *for all $x \in \mathcal{B}$.*

Proof. The proof follows by Lemma 3.32. ◻

Lemma 3.34. *Let A be any $M_\vee$-matrix with* $\text{index}_\rho(A) \le 1$ *and $x \in E(A)$. Then* $\text{height}(x) \le \text{level}(x)$.

Proof. If $\mathcal{B} = \{x^1, \dots, x^q\}$ be a preferred basis for A, then $x = \displaystyle\sum_{i=1}^{q} c_i x^i$ for some c_i's. Let $Q = \{i \mid c_i \neq 0\}$. Then clearly, $l = \text{level}(x) = \max\{\text{level}(x^i) \mid i \in \text{top}(Q)\}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

ELA

From Corollary 3.33, it follows that for all $i \in Q$, $\text{height}(x^i) \leq \text{level}(x^i) \leq l$, or $A^l x^i = 0$. So it follows that $A^l x = 0$ and therefore, $\text{height}(x) \leq l = \text{level}(x)$. ☐

REMARK 3.35. From Lemma 3.34 we can easily conclude that if A is any $M_\vee$-matrix with $\text{index}_\rho(A) \leq 1$ and $\mathcal{B}$ is any basis for $E(A)$ then $\lambda(\mathcal{B}) \preceq \eta(\mathcal{B})$.

REMARK 3.36. If A is any $M_\vee$- matrix with $\text{index}_\rho(A) \leq 1$ then from Lemma 3.30 then the set $\Lambda_k(A)$ consisting of all vectors in $E(A)$, with level less than or equal to k form a vector space and in view of Lemma 3.34, $\Lambda_k(A) \subseteq N(A^k)$, hence $\lambda(A) \preccurlyeq \eta(A)$.

LEMMA 3.37. *Let $A = \rho I - B$ be an $M_\vee$-matrix with* $\text{index}_\rho(A) \leq 1$. *Then for any nonnegative vector x in $E(A)$,* $\text{height}(x) = \text{level}(x)$.

Proof. It suffices to show that $\text{level}(x) \leq \text{height}(x)$. Let $\{x^1, \ldots, x^q\}$ be a preferred basis for $E(A)$. Then $x = \displaystyle\sum_{i=1}^{q} c_i x^i$ for some scalars c_i, and $l = \text{level}(x) = \max\{\text{level}(x^i) \mid i \in \text{top}(Q)\}$, where Q is as defined in Lemma 3.34. Clearly since x is nonnegative, for any $i \in \text{top}(Q)$, $c_i > 0$. In view of the above argument it is enough to show $A^{l-1}x \neq 0$.

If $-Ax^i = \displaystyle\sum_{k=1}^{q} c_{ki} x^k$ where the c_{ki}'s are as in the definition of a preferred basis, then

$$(-1)^{l-1} A^{l-1} x = (-1)^{l-1} \left(\sum_{i \in Q} c_i A^{l-1} x^i \right).$$

From Lemma 3.34, $\text{height}(x) \leq \text{level}(x)$, and hence, it follows that

$$\begin{aligned}(-1)^{l-1} \left(\sum_{i \in Q} c_i A^{l-1} x^i \right) &= (-1)^{l-1} \sum_{\substack{i \in Q \\ \text{level}(x^i) = l}} c_i A^{l-1} x^i \\ &= \sum_{i_1} \sum_{i_2} \cdots \sum_{i_l - 1} \sum_{\substack{i \in Q \\ \text{level}(x^i) = l}} c_{i_1 i_2} \cdots c_{i_{l-1} i} c_i x^{i_1}.\end{aligned}$$

Since for every $i \in Q$ with $\text{level}(x^i) = l$, $c_i > 0$ and there is a sequence of distinct indices $i_i, i_2, \ldots, i_{l-1}$ such that $c_{i_1 i_2} \cdots c_{i_{l-1} i} > 0$, so it follows that $A^{l-1}x \neq 0$. ☐

REMARK 3.38. From Lemma 3.37 it is clear that for any nonnegative level basis of $E(A)$ and in particular for a preferred basis $\mathcal{B}$ of $E(A)$, $\eta(\mathcal{B}) = \lambda(\mathcal{B}) = \lambda(A)$.

REMARK 3.39. If $\mathcal{B}$ is a nonnegative height basis, then $\eta(\mathcal{B}) = \lambda(\mathcal{B}) = \eta(A)$ and, this together with Remark 3.38 and Remark 3.29 imply that $\eta(\mathcal{B}) = \lambda(\mathcal{B}) = \eta(A) =$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

$\lambda(A)$. Hence, $\mathcal{B}$ is also a level basis.

In [9], it was shown that the level characteristic of an eventually nonnegative matrix B with $\text{index}(B) \leq 1$ is majorized by the height characteristic which implies that the level characteristic of an $M_\vee$-matrix $A = \rho I - B$ with $\text{index}_\rho(A) \leq 1$, is majorized by the height characteristic. Motivated by the necessary and sufficient conditions obtained by Schneider and Hershkowitz in [7] for the equality of these two characteristics for singular M-matrices, we independently try to obtain similar conditions for the equality of these two characteristics for the class of $M_\vee$-matrices.

THEOREM 3.40. *Let A be an $M_\vee$-matrix with* $\text{index}_\rho(A) \leq 1$. *Then the following are equivalent:*

(i) $\eta(A) = \lambda(A)$.
(ii) *For all $x \in E(A)$,* $\text{height}(x) = \text{level}(x)$.
(iii) *For every basis $\mathcal{B}$ of $E(A)$, we have* $\text{height}(x) = \text{level}(x)$ *for all $x \in \mathcal{B}$.*
(iv) *For some height basis $\mathcal{B}$ of $E(A)$, we have* $\text{height}(x) = \text{level}(x)$ *for all $x \in \mathcal{B}$.*
(v) *Every height basis for A is a level basis for A.*
(vi) *Every level basis for A is a height basis for A.*
(vii) *Some preferred basis for A is a height basis for A.*
(viii) *There exists a nonnegative height-level basis for A.*
(ix) *There is a nonnegative height basis for A.*
(x) *For all $j \in \langle t \rangle$, there exists a nonnegative basis for $N(A^j)$.*
(xi) *For every level basis $\mathcal{B}$ for A with induced matrix $C = C(A, \mathcal{B})$, the block $C_{k-1,k}$ has full column rank for all $k \in \langle t \rangle$.*
(xii) *There exists a level basis $\mathcal{B}$ for A with induced matrix $C = C(A, \mathcal{B})$, such that for all $k \in \langle t \rangle$ the block $C_{k-1,k}$ has full column rank.*

Proof.

(i) $\Rightarrow$ (ii) : Condition (i) implies that for any k, $\dim(\Lambda_k(A)) = \lambda_1 + \cdots + \lambda_k = \eta_1 + \cdots + \eta_k = \dim(N(A^k))$. So from Remark 3.36, it follows that $\Lambda_k(A) = N(A^k)$ and hence (ii) follows.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) : Obvious.

(iv) $\Rightarrow$ (v) : By assumption we have a height basis $\mathcal{B}$ such that for each $x \in \mathcal{B}$, $\text{height}(x) = \text{level}(x)$, hence it follows that $\eta(A) = \eta(\mathcal{B}) = \lambda(\mathcal{B})$. Since $\lambda(\mathcal{B}) \preccurlyeq \lambda(A)$ from Remark 3.29, and $\eta(A) \succcurlyeq \lambda(A)$, it follows that $\eta(A) = \lambda(A)$ and hence (i) and (iii) hold.
If $\mathcal{B}'$ is any height basis, then (iii) and (i) imply $\lambda(\mathcal{B}') = \eta(\mathcal{B}') = \eta(A) = \lambda(A)$. Thus, $\mathcal{B}'$ is a level basis.

(v) $\Rightarrow$ (vi) : Consider a Jordan basis $\mathcal{B}$ for $E(A)$ derived from the set $T = \{y^1, \ldots, y^{\bar{t}}\}$ and let $\max\{\text{height}(y^k) \mid k \in \langle \bar{t} \rangle\} = l$. Since A is an $M_\vee$-matrix with $\text{index}_\rho(A) \leq 1$, $\text{index}(A)$ is equal to the length of the longest chain in A

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

and it follows that $\max\{\text{level}(y_k) \mid k \in \langle \bar{t} \rangle\} = l$.

Since $\mathcal{B}$ is a height basis, $\eta(\mathcal{B}) = \eta(A)$. Thus, by assumption, $\lambda(\mathcal{B}) = \lambda(A)$. Also for any basis $\mathcal{B}'$, $\lambda(\mathcal{B}') \preccurlyeq \eta(\mathcal{B}') \preccurlyeq \eta(A)$ and $\lambda(\mathcal{B}') = \lambda(A) \preccurlyeq \eta(A)$ if it is a level basis, then to show that every level basis is a height basis, it is enough to show $\eta(A) = \lambda(A)$ or $\eta(\mathcal{B}) = \lambda(\mathcal{B})$.

For any y^i for which $\text{height}(y^i) = \text{level}(y^i)$, $\text{height}(A^k y^i) = \text{height}(y^i) - k = \text{level}(y^i) - k \geq \text{level}(A^k y^i)$. It follows that $\text{height}(A^k y^i) = \text{level}(A^k y^i)$ for any $k \leq \text{height}(y^i)$. Then for y^i with $\text{height}(y^i) = l$, $\text{height}(A^k y^i) = \text{level}(A^k y^i)$ for any $k \leq l$. From the above argument it follows that if $\lambda(\mathcal{B}) \neq \eta(\mathcal{B})$, then there exists a $y^i \in T$ with $\text{height}(y^i) < l$ such that $\text{height}(y^i) < \text{level}(y^i)$. Let $\text{height}(y^i) = s$ and $\text{level}(y^i) = p$. Consider any $y^j \in T$ with $\text{height}(y^j) = l = \text{level}(y^j)$. Then there exists an r such that $\text{height}(A^r y^j) = \text{height}(y^i) = s$. Consider the element $z = y^i + A^r y^j$, and the new basis $\bar{\mathcal{B}}$ obtained from $\mathcal{B}$ by replacing $A^r y^j$ with z. Since $\mathcal{B}$ is a height basis, the new basis $\bar{\mathcal{B}}$ so constructed will also be a height basis and since $\text{level}(A^r y^j) = \text{height}(A^r y^j) = s$, so $\text{level}(z) = p > s$. Hence, $\lambda(\bar{\mathcal{B}}) \prec \lambda(\mathcal{B}) = \lambda(A)$ which contradicts the assumption that every height basis is a level basis. So for any $y^i \in T$, $\text{height}(y^i) = \text{level}(y^i)$ which implies $\eta(\mathcal{B}) = \lambda(\mathcal{B})$.

(vi) $\Rightarrow$ (vii) & (vii) $\Rightarrow$ (viii) : Follow from the fact that every preferred basis is a level basis.

(viii) $\Rightarrow$ (ix) : Obvious.

(ix) $\Rightarrow$ (x) : Let $\mathcal{B}$ be a nonnegative height basis for A. Then $\eta(\mathcal{B}) = \eta(A) = (\eta_1, \ldots, \eta_p)$. Thus, for any $j \in \langle p \rangle$, there are $\eta_1 + \cdots + \eta_j$ elements in $\mathcal{B}$ of height at most j and since $\dim(N(A^j)) = \eta_1 + \cdots + \eta_j$, these elements will form a nonnegative basis for $N(A^j)$.

(x) $\Rightarrow$ (xi) : Suppose that for each $j \in \langle p \rangle$, there exists a nonnegative basis for $N(A^j)$. Let $\mathcal{B}$ be a level basis for A with the induced matrix $C = C(A, \mathcal{B})$. To show that for all k, $C_{k-1,k}$ has full column rank.

Suppose that there is a k such that $C_{k-1,k}$ does not have full column rank and we assume that k is the least of such indices. We have,

$$A[X^{(1)} \cdots X^{(t)}] = [X^{(1)} \cdots X^{(t)}] \begin{bmatrix} 0 & C_{12} & C_{13} & \cdots & C_{1t} \\ & 0 & C_{23} & \cdots & C_{2t} \\ & & 0 & \ddots & \vdots \\ & & & \ddots & C_{t-1,t} \\ & & & & 0 \end{bmatrix} \tag{3.3}$$

with $X^{(i)} = [x_1^i \cdots x_{\lambda_i}^i]$ in which the columns give the elements of $\mathcal{B}$ having level i.

Since $C_{k-1,k} = [C_1^{k-1,k} \cdots C_{\lambda_k}^{k-1,k}]$ does not have full column rank, so there is a column say $C_j^{k-1,k}$ in $C_{k-1,k}$ which is a linear combination of its preceding

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

columns. Since every column of $C_{k-1,k}$ is a nonzero vector, there exists scalars $\beta_1,\ldots,\beta_{j-1}$, not all zeros, such that $C_j^{k-1,k} = \sum_{r=1}^{j-1}\beta_r C_r^{k-1,k}$. Then from equation (3.3), for $r = 1,\ldots,j-1$ we have,

$$Ax_r^k = X^{(1)}C_r^{1,k} + X^{(2)}C_r^{2,k} + \cdots + X^{(k-1)}C_r^{k-1,k},$$

and

$$Ax_j^k = X^{(1)}C_j^{1,k} + X^{(2)}C_j^{2,k} + \cdots + X^{(k-1)}\left(\sum_{r=1}^{j-1}\beta_r C_r^{k-1,k}\right).$$

If $z = x_j^k - \sum_{r=1}^{j-1}\beta_r x_r^k$, then it follows that $\text{height}(z) \le k-1$. Let $\text{height}(z) = h$. Then by assumption $N(A^h)$ has a nonnegative basis, say, $\{y^1,\ldots,y^m\}$ and let $z = d_1y^1 + \cdots + d_my^m$ for some scalars d_i. If $\text{level}(z) = l$, then since $\text{level}(y^i) = \text{height}(y^i) \le h$, so $l \le h < k$. Construct a new basis $\tilde{\mathcal{B}}$ from $\mathcal{B}$ by replacing x_j^k with z in $\mathcal{B}$. Then $\lambda_i(\tilde{\mathcal{B}}) = \lambda_i$ for all $i \notin \{l,k\}$; $\lambda_l(\tilde{\mathcal{B}}) = \lambda_l + 1$; $\lambda_k(\tilde{\mathcal{B}}) = \lambda_k - 1$. Hence, it follows that $\lambda(\tilde{\mathcal{B}}) \succ \lambda(A)$, which is a contradiction. Thus, (xii) holds.

(xi) $\Rightarrow$ (xii) : Obvious.

(xii) $\Rightarrow$ (i) : Let there exist a level basis $\mathcal{B}$ for A with the induced matrix $C = C(A,\mathcal{B})$ such that for all $k \in \langle t\rangle$ the block $C_{k-1,k}$ has full column rank. We show that $\lambda(A) = \eta(A)$.

From equation (3.3), we have $A^{k-1}X^{(k)} = X^{(1)}C_{12}C_{23}\cdots C_{k-1,k}$. Since $C_{j-1,j}$'s are of full column rank, $\text{height}(x_i^k) = k$ for all $i \in \langle\lambda_k\rangle$. Hence, we have $\text{height}(x) = \text{level}(x)$ for all $x \in \mathcal{B}$ and, $\eta(\mathcal{B}) = \lambda(\mathcal{B}) = \lambda(A)$.

If $\eta(A) \succ \lambda(A)$ then there exists a k for which $\lambda_k > \eta_k$. Since $A^{k-1}X^{(k)} = X^{(1)}C_{12}C_{23}\cdots C_{k-1,k}$ and each of the matrices $X^{(1)}, C_{12}, C_{23},\ldots,C_{k-1,k}$ is of full column rank, $\text{rank}(A^{k-1}X^{(k)}) = \lambda_k(\mathcal{B}) = \lambda_k$, which is equal to the number of columns in $X^{(k)}$. Hence, no linear combination of the columns in $X^{(k)}$ can belong to $N(A^{k-1})$. Also since $A^kX^{(k)}$ is the $\mathbf{0}$ matrix, $\eta_k = n(A^k) - n(A^{k-1}) \ge \lambda_k$, which is a contradiction. Hence, it follows that $\eta(A) = \lambda(A)$. □

Theorem 3.41. *Let A be an $M_\vee$-matrix with* $\text{index}_\rho(A) \le 1$. *Then* $\eta(A) = \lambda(A)$ *if and only if there exists a nonnegative Jordan basis for* $-A$.

Proof. Since every nonnegative Jordan basis for $-A$ is a nonnegative height basis for A, the 'if' part follows from Theorem 3.40(ix).

The 'only if' part can be obtained by proceeding as in Theorem 6.10 of [7]. □

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

ELA

We next consider two extreme cases: (i) Each path in $R(A)$ has at most one singular vertex, (ii) all singular vertices lie on a single path.

Theorem 3.42. [12] *Let A be an M-matrix. Then the following are equivalent:*

(i) *The Segré characteristic of A is $(1, 1, \ldots, 1)$.*
(ii) *The level characteristic of A is (t).*

Theorem 3.43. [12] *Let A be an M-matrix. Then the following are equivalent:*

(i) *The Segré characteristic of A is (t).*
(ii) *The level characteristic of A is $(1, 1, \ldots, 1)$.*

Theorems 3.42 and 3.43 are also true for an $M_\vee$-matrix A with $\text{index}_\rho(A) \leq 1$, due to Theorem 3.5 and Lemma 3.6.

Remark 3.44. Let A be an $M_\vee$-matrix with $\text{index}_\rho(A) \leq 1$. Then in Theorem 3.42 since 0 is a simple eigenvalue of every singular block in the Frobenius normal form of A, t is the algebraic multiplicity of 0. Also, the number of 1's in the *Segré* characteristic in Theorem 3.42 is t. Therefore, Theorem 3.42 states that in the extreme case (i) we have that $\lambda(A) = j(A)^\star = \eta(A)$.
Similarly for the other extreme case (ii), considered in Theorem 3.43, $\lambda(A) = j(A)^\star = \eta(A)$.

The following examples show that the results in Theorem 3.42 and Theorem 3.43 need not be true for an $M_\vee$-matrix A having $\text{index}_\rho(A) > 1$.

Example 3.45. Consider the $M_\vee$-matrix

$$A = 4I - B = 4I - \left[\begin{array}{cc|cc|cc} 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ \hline 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 2 & 2 \end{array}\right].$$

Clearly A has $\text{index}_4(A) = 2 > 1$; $t = 2$ and A is in Frobenius normal form having irreducible diagonal blocks A_{11},A_{22}, A_{33} so that the singular vertices in $R(A)$ are 1 and 3. *Segré characteristic* is $(1, 1)$ since it has two Jordan blocks of size 1 corresponding to the eigenvalue 0, whereas *level characteristic* is $(1, 1)$.

Example 3.46. Consider the $M_\vee$ matrix A given by,

$$A = 4I - B = 4I - \begin{bmatrix} 2 & 2 & 0.5 & 0.5 \\ 2 & 2 & 0.5 & 0.5 \\ 1 & -1 & 2 & 2 \\ -1 & 1 & 2 & 2 \end{bmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

Then A has $\text{index}_4(A) = 2 > 1$. Since A has only one Jordan block corresponding to 0 of size 2, and it has only one irreducible block, the matrix itself, so *Segré characteristic* of A is $j(A) = (2)$ and, *level characteristic* is $\lambda(A) = (1)$.

3.2.2. Hall condition and well structured graphs. In this section, we show with the help of the Hall Marriage condition that the reduced graph of a singular $M_\vee$-matrix with $\text{index}_\rho(A) \leq 1$ is a well structured graph.

We first state Hall's theorem essentially as it is found in [2].

THEOREM 3.47. [2] *Let $E_1, \ldots, E_h$ be subsets of a given set E. Then the following are equivalent:*

(i) *We have,*

$$\left|\bigcup_{i\in\alpha} E_i\right| \geq |\alpha|, \qquad \textit{for all} \;\; \alpha \subseteq \langle h\rangle. \tag{3.4}$$

(ii) *There exist distinct elements $e_1, \ldots, e_h$ of E such that $e_i \in E_i$, $i \in \langle h\rangle$.*

Condition (3.4) is often referred to as the Hall Marriage condition.

DEFINITION 3.48. [8] Let S be an acyclic graph. A chain $(i_1, \ldots, i_m)$ is called an *anchored chain* if the level of i_k is k, $k \in \langle m\rangle$.

DEFINITION 3.49. [8] Let S be an acyclic graph.

(i) A set κ of chains in S is said to be a *chain decomposition* of S if each vertex of S belongs to exactly one chain in κ.
(ii) A chain decomposition κ of S is said to be an *anchored chain decomposition* of S if every chain in κ is anchored.
(iii) S is said to be *well structured* if there exists an anchored chain decomposition of S.

The following result is due to [8].

THEOREM 3.50. [8] *Let S be an acyclic graph with levels $L_1, \ldots, L_t$. Then the following are equivalent:*

(i) *The sets $E_i = \text{below}(i) \bigcap L_k$, $i \in L_{k+1}$, satisfy the Hall Marriage Condition for all $k \in \langle t-1\rangle$.*
(ii) *S is well structured.*

In the next theorem we show that the reduced graph of certain $M_\vee$-matrices is well structured.

THEOREM 3.51. *Let A be an $M_\vee$-matrix with* $\text{index}_\rho(A) \leq 1$. *If* $\eta(A) = \lambda(A)$,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

then the reduced graph $R(A)$ is well structured.

Proof. Let $\{\alpha_1, \ldots, \alpha_q\}$ be the set of all singular vertices of A ordered according to levels. $L_1, \ldots, L_t$ mentioned in Theorem 3.50, are the levels of $R(A)$, i.e., L_i is the collection of singular vertices of level i and, t is the length of the longest chain in $R(A)$. It suffices to show that $E_i = \text{below}(\alpha_i) \bigcap L_k$, where $\alpha_i \in L_{k+1}$ satisfies condition (i) of Theorem 3.50, for all $k \in \langle t-1 \rangle$.

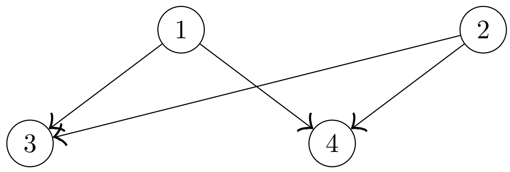
Suppose the E_i's as defined above do not satisfy the Hall marriage condition for all $k \in \langle t-1 \rangle$. Then there exists a k_0 and an $\alpha \subseteq \langle \lambda_{k_0+1} \rangle$ such that $|\bigcup_{i \in \alpha} E_i| < |\alpha|$. Without loss of generality let $\alpha = \{1, 2, \ldots, r\}$.

Consider a preferred basis $\mathcal{B}$. Since $\eta(A) = \lambda(A)$, $\mathcal{B}$ is also a height basis. If X is the matrix such that the columns of which give the elements of $\mathcal{B}$ and C is the corresponding induced matrix, then since $\eta(A) = \lambda(A)$, so $C_{k,k+1}$'s are of full column rank, for all $k \in \langle t-1 \rangle$. Since $\mathcal{B}$ is a preferred basis, $C_{ij} \neq 0$ if and only if $\alpha_i \to \alpha_j$. Hence, $|\bigcup_{i=1}^{r} E_i| < r$ implies that in the submatrix of C_{k_0,k_0+1} of order $\lambda_{k_0} \times r$, formed by taking only the first r columns of C_{k_0,k_0+1}, there are less than r nonzero rows, which contradicts the fact that the r columns are linearly independent. $\square$

REMARK 3.52. Note that $\eta(A) = \lambda(A)$ is a sufficient condition for the reduced graph $R(A)$ to be well structured, but is not a necessary condition. For example, consider the M-matrix $A = I - B$ where,

$$B = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The reduced graph $R(A)$ of A is given by,



Then $\{(1,3), (2,4)\}$ is an anchored chain decomposition for A, and hence, $R(A)$ is well structured. But note that $\lambda(A) = (2,2)$ whereas $\eta(A) = (3,1)$.

4. Combinatorial structure of GM-matrices. In this section, we consider another generalization of the class of M-matrices known as GM-matrices. We extend some results on the combinatorial spectral properties of M-matrices to this class. In particular, it is shown that the *Preferred Basis Theorem* and the *Index Theorem* do not hold for the class of GM-matrices of order $n \geq 3$, whereas the theorems are true

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

ELA

for $n < 3$.

DEFINITION 4.1. A matrix $A \in \mathbb{R}^{n,n}$ is said to have the *Perron-Frobenius property* if the spectral radius is an eigenvalue that has an entry-wise nonnegative eigenvector. WPF_n denotes the collection of all $n \times n$ matrices A, for which both A and A^T possess the Perron-Frobenius property.

DEFINITION 4.2. A matrix $A \in \mathbb{R}^{n,n}$ is said to be a GZ-matrix if it can be expressed in the form $A = sI - B$, where s is a positive scalar and $B \in WPF_n$. Moreover, if $A = sI - B$ is a GZ-matrix such that $\rho(B) \leq s$, then A is called a GM-matrix.

Through out this section we write a (singular) GM-matrix A in the form $A = \rho I - B$, where $B \in WPFn$ and $\rho = \rho(B)$.

EXAMPLE 4.3. Consider the matrix

$$A = 2I - B = 2I - \begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

The eigenvalues of B are $2, 2, -2$. As $[1\ 0\ 0]^T$ and $[0\ 1\ 1]^T$ are nonnegative left and right eigenvectors of B corresponding to 2 respectively, so A is a GM-matrix.

We will show that the size of the largest Jordan block associated with 0, in the Jordan form of a GM-matrix of order 2, is combinatorially determined, but the *Index Theorem* need not be true if the size of the matrix exceeds 2.

LEMMA 4.4. *For any $A \in \mathbb{R}^{2,2}$ with the spectral radius $\rho(A) \in \sigma(A)$, the length of the longest chain of A is always less than or equal to* $\text{index}_{\rho(A)}(A)$.

Proof. If $\text{index}_{\rho(A)}(A) = 2$, then the result is obviously true. If suppose

$$\text{index}_{\rho(A)}(A) = 1 \text{ and length of the longest chain} = 2.$$

So there are exactly two basic classes $\{1\}$ and $\{2\}$ such that either $\{1\} \to \{2\}$ or $\{2\} \to \{1\}$, and hence, either A or A^T is of the form $\begin{bmatrix} \rho(A) & * \\ 0 & \rho(A) \end{bmatrix}$, where $*$ is nonzero. In each of the cases $\text{index}_{\rho(A)}(A) = 2$, a contradiction to our assumption. Hence, the result follows. ☐

The following example shows that Lemma 4.4 does not hold if the order of the matrix exceeds 2.

EXAMPLE 4.5. Consider the GM-matrix A in Example 4.3. Note that $[0, 1, 1]^T$ and $[2, 0, 1]^T$ are two linearly independent eigenvectors of A corresponding to the eigenvalue 0 so that $\text{index}(A) = 1$. But the maximal level of a vertex in $\Gamma(A)$ is 2.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

ELA

We now give an example of a 2×2 matrix that satisfies the hypothesis of Lemma 4.4 and for which $\text{index}_{\rho(A)}(A) >$ length of the longest chain in $\Gamma(A)$. Hence, even for 2×2 matrices, the condition $\rho(A) \in \sigma(A)$ is not sufficient for their equality.

EXAMPLE 4.6. Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$. Then $\rho(A) = 1 \in \sigma(A)$ and $\text{index}_{\rho(A)}(A) = 2$ whereas the length of the longest chain in $\Gamma(A)$ is 1.

In the next lemma we give a subclass of 2×2 matrices for which $\text{index}_{\rho(A)}(A) =$ length of the longest chain in $\Gamma(A)$.

LEMMA 4.7. *If $A = (a_{ij}) \in \mathbb{R}^{2,2}$ is in $WPF2$ but not a nonnegative matrix, then the following statements are equivalent:*

(i) $\text{index}_{\rho(A)}(A) = 2$.
(ii) *$a_{ij} < 0$ for some $i \neq j$.*
(iii) *A is in triangular form with diagonal entries equal to $\rho(A)$.*

Proof. Since $A \in WPF2$, so there exists nonnegative vectors $x = [x_j]$ and $y = [y_j]$ such that,

$$Ax = \rho(A)x \text{ and } y^T A = \rho(A)y^T$$

which give,

$$(a_{11} - \rho(A))x_1 + a_{12}x_2 = 0 \tag{4.1}$$

$$a_{21}x_1 + (a_{22} - \rho(A))x_2 = 0 \tag{4.2}$$

$$(a_{11} - \rho(A))y_1 + a_{21}y_2 = 0 \tag{4.3}$$

$$a_{12}y_1 + (a_{22} - \rho(A))y_2 = 0. \tag{4.4}$$

(ii) $\Rightarrow$ (iii): Assume that $a_{12} < 0$. We claim that x_2 cannot be positive. If $x_2 > 0$, then from equation (4.1) we must have $x_1 > 0$ and,

$$a_{11} > \rho(A).$$

Since $a_{11} + a_{22} = \lambda + \rho(A)$ where λ is the other eigenvalue of A, so $a_{22} - \rho(A) < 0$. Thus, equation (4.4) implies $a_{12} \geq 0$, which is a contradiction. So $x_2 = 0$. From conditions (4.1) and (4.2) we get, $a_{11} = \rho(A)$ and $a_{21} = 0$, and hence, A is an upper

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

ELA

triangular matrix. Similarly, one can show that $y_1 = 0$ and $a_{22} = \rho(A)$. Thus, $\lambda = \rho(A)$.

(iii) $\Rightarrow$ (i) : Trivial.

(i) $\Rightarrow$ (ii): Since $\text{index}_{\rho(A)}(A) = 2$ and eigenvalues of A are

$$\frac{(a_{11} + a_{22}) \pm \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}}{2},$$

so $(a_{11} - a_{22})^2 + 4a_{12}a_{21} = 0$. Thus, either a_{12} and a_{21} are both nonzero and of opposite sign, in which case (ii) holds, or at least one of them must be zero. If any one of $a_{12},\ a_{21}$ is zero, then $a_{11} = a_{22} = \rho(A)$. Since A is not a nonnegative matrix, so at least one of $a_{12},\ a_{21}$ must be negative which implies that (ii) holds. □

Corollary 4.8. *Suppose $A \in \mathbb{R}^{2,2}$ is in $WPF2$. Then* $\text{index}_{\rho(A)}(A) = 2$ *if and only if either A or A^T is of the form* $\begin{bmatrix} \rho(A) & * \\ 0 & \rho(A) \end{bmatrix}$, *where $*$ is nonzero.*

Corollary 4.9. *If $A \in WPF2$, then* $\text{index}_{\rho(A)}(A) =$ *length of the longest chain in* $\Gamma(A)$.

Proof. If $A \geq 0$, then the result is known to be true. Suppose A is not a nonnegative matrix. If $\text{index}_{\rho(A)}(A) = 1$, then either A is a diagonal matrix or has two distinct eigenvalues, but in both the cases length of the longest chain in $\Gamma(A)$ is 1. If $\text{index}_{\rho(A)}(A) = 2$, then the result follows from Lemma 4.7. □

The Index Theorem for GM-matrices of order 2 is an immediate consequence of Corollary 4.9.

Theorem 4.10. *If $A = \rho I - B$ is a singular GM-matrix of order 2, then* $\text{index}(A)$ *is equal to the length of the longest chain in $R(A)$.*

We next show that there is a nonnegative basis for the generalized nullspace $E(A)$ and, the positive entries of which are combinatorially determined.

Lemma 4.11. *Suppose $A \in WPF2$ and let $\alpha_1 \cdots \alpha_M (M = 1$ or $2)$ be the basic classes for A. Then there always exists a nonnegative basis $\{x^1, \ldots, x^M\}$ for $E_{\rho(A)}(A)$ such that $x^i_j > 0$ if and only if j has access to the ith basic class α_i.*

Proof. The result is known to be true if A is a nonnegative matrix. Hence, assume that A has at least one negative entry. We consider two cases:

Case I: Suppose that $\text{index}_{\rho(A)}(A) = 1$. Then by Corollary 4.9, length of the longest chain in $\Gamma(A)$ is 1. If A has two basic classes, then A is a nonnegative diagonal matrix with diagonal entries equal to $\rho(A)$ in which case the result follows. Suppose A has only one basic class. By Lemma 4.7 both a_{12} and a_{21} are nonnegative.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

ELA

Suppose one of a_{12}, a_{21} is 0, say $a_{12} = 0$. Then A has two different diagonal entries, $\rho(A)$ and say, λ. If $a_{11} = \rho(A)$, then $x^1 = [1, \frac{a_{21}}{\rho(A)-\lambda}]^T$ will be the required vector, and if $a_{22} = \rho(A)$, then $x^1 = [0, 1]^T$ will be the required vector.

Suppose that both a_{12} and a_{21} are positive. Then the only basic class of A will be $\{1, 2\}$. Since $A \in WPF2$, so there is a nonnegative vector $x^1 = [x_1^1, x_2^1]^T \neq 0$ such that $Ax^1 = \rho(A)x^1$ which implies

$$(a_{11} - \rho(A))x_1^1 + a_{12}x_2^1 = 0$$

$$a_{21}x_1^1 + (a_{22} - \rho(A))x_2^1 = 0,$$

and hence $x_j^1 > 0 \ \ \forall j = 1, 2$.

Case II: Suppose that $\text{index}_{\rho(A)}(A) = 2$. If A is a nonnegative matrix, then the result follows from Theorem 2.6. If A is not a nonnegative matrix, then by Lemma 4.7, A has two basic classes $\{1\}, \{2\}$ such that either $2 \to 1$ or $1 \to 2$. If $2 \to 1$, then the required generalized eigenvectors are $x^1 = [1, 1]^T$ and $x^2 = [0, 1]^T$ that satisfy $x_j^i > 0$ if and only if j has access to the ith basic class, for $i, j \in \{1, 2\}$. □

We now prove the Preferred Basis Theorem for GM-matrices of order 2.

THEOREM 4.12. *If A is a singular GM-matrix of order 2, then there exists a preferred basis for $E(A)$.*

Proof. The result is known to be true if A is an M-matrix, hence let $A \in \mathbb{R}^{2,2}$ be a GM-matrix which is not an M-matrix. The existence of a quasi-preferred basis for $E(A)$ is an immediate consequence of Lemma 4.11. We now show that every quasi-preferred basis of $E(A)$ is a preferred basis.

Let the columns of X form a quasi-preferred basis for $E(A)$. If $\text{index}(A) = 1$, then $AX = 0$, and hence, the columns of X form a preferred basis for A. Suppose $\text{index}(A) = 2$. Then $X = [x^1 \ x^2]$, where x^2 is a positive vector and $x_1^1 > 0$. Thus, by Lemma 4.7, exactly one of a_{12} or a_{21} must be zero. Assume that $a_{12} \neq 0$. Then $a_{21} = 0 = a_{11} = a_{22}$. Then $AX = X \begin{bmatrix} 0 & \frac{a_{12}x_2^2}{x_1^1} \\ 0 & 0 \end{bmatrix}$. Thus, the columns of X form a preferred basis for $E(A)$. □

The following examples show that the conclusions of Theorem 4.10 and Theorem 4.12 do not hold for $WPFn$ matrices if $n > 2$.

EXAMPLE 4.13. Let,

$$A = 3I - B = 3I - \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

Clearly, $B \in WPF3$, and hence, A is a GM-matrix of order 3. Note that index$(A) = 2$ whereas the maximal level of a vertex in $\Gamma(A)$ is 1.

EXAMPLE 4.14. Consider the matrix

$$A = 2I - B = 2I - \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Clearly A is a singular GM-matrix with the singular classes $\{1, 2\}$ and $\{3\}$. Suppose that there is a preferred basis $\{x^1, x^2\}$ for $E(A)$ such that $x_i^j > 0$ if and only if i has access to the jth singular class. So by assumption $x_i^1 > 0$, for all $i = 1, 2$. But $Ax^1 = 0$ implies that $x_1^1 + x_2^1 = 0$, which cannot happen. Thus, Theorem 4.12 is not true for $n = 3$.

REMARK 4.15. By taking $\tilde{A} = \text{diag}(A, B) \in \mathbb{R}^{n,n}$, where A is as in Example 4.14 or in Example 4.13 and any matrix B having $\rho(B) < \rho(A)$, we can conclude that Theorems 4.10 and 4.12 do not hold for $n > 3$.

5. Conclusion. We have considered two types of generalizations of M-matrices, namely, the GM-matrices and the $M_\vee$-matrices. Initially we considered a generalization of M-matrices, known as $M_\vee$-matrices and we proved the existence of preferred basis for a subclass of these matrices. In particular, we gave a method to obtain a preferred basis for singular M-matrices and singular $M_\vee$-matrices, from a quasi-preferred basis. We next considered different types of characteristics, known as height, level and *Segré* characteristics and tried to understand their mutual relationship. Based on results obtained for singular M-matrices in [7], we stated and proved some equivalent conditions for the equality of the height characteristic and the level characteristic for a subclass of singular $M_\vee$-matrices. We also have given a sufficient condition for the reduced graph of this subclass of $M_\vee$-matrices to be well structured.

Finally, we showed the existence of a preferred basis for singular GM-matrices of order 2 and we have also demonstrated with the help of an example, the fact that a quasi-preferred (and hence, a preferred) basis need not exist if the order of the matrix exceeds 2.

Acknowledgment. The authors would like to express their gratitude to an anonymous referee for valuable suggestions and helpful comments.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 550-576, September 2015

ELA

REFERENCES

[1] A. Berman and R. Plemmons. *Nonnegative matrices in the Mathematical Sciences.* SIAM, Philadelphia, PA, 1994.

[2] R.A. Brualdi and H.J. Ryser. *Combinatorial Matrix Theory.* Encyclopedia of Mathematics and its Applications, Vol. 39, Cambridge University Press, Cambridge, 1991.

[3] A. Elhashash and D.B. Szyld. Generalizations of M-matrices which may not have a nonnegative inverse. *Linear Algebra Appl.*, 429(10):2435–2450, 2008.

[4] D. Hershkowitz. The combinatorial structure of generalized eigenspaces – from nonnegative matrices to general matrices. *Linear Algebra Appl.*, 302/303:173–191, 1999. Special issue dedicated to Hans Schneider (Madison, WI, 1998).

[5] D. Hershkowitz, U. Rothblum, and H. Schneider. The combinatorial structure of the generalized nullspace of a block triangular matrix. *Linear Algebra Appl.*, 116:9–26, 1989.

[6] D. Hershkowitz and H. Schneider. On the generalized nullspace of M-matrices and Z-matrices. *Linear Algebra Appl.*, 106:5–23, 1988.

[7] D. Hershkowitz, and H. Schneider. Height bases, level bases and the equality of the height and level characteristics of an M-matrix. *Linear Multilinear Algebra*, 25(2):149–171, 1989.

[8] D. Hershkowitz and H. Schneider. Combinatorial bases, derived Jordan sets and the equality of the height and level characteristics of an M-matrix. *Linear Multilinear Algebra*, 29(1):21–42, 1991.

[9] S. Carnochan Naqvi and J.J. McDonald. The combinatorial structure of eventually nonnegative matrices. *Electron. J. Linear Algebra*, 9:255–269, 2002.

[10] D.D. Olesky, M.J. Tsatsomeros, and P.van den Driessche. $M_\vee$-matrices: a generalization of M-matrices based on eventually nonnegative matrices. *Electron. J. Linear Algebra*, 18:339–351, 2009.

[11] U. Rothblum. Algebraic eigenspace of nonnegative matrices. *Linear Algebra Appl.*, 12:281–292, 1975.

[12] H. Schneider. The elementary divisors associated with 0 of nonsingular M-matrices. *Proc. Edinburgh Math. Soc. (Ser. 2)*, 10:108–122, 1956.

[13] H. Schneider. The influence of the marked reduced graph of a nonnegative matrix on the Jordan form and on related properties: A survey. *Linear Algebra Appl.*, 84:161–189, 1986.