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INVERTIBLE AND REGULAR COMPLETIONS OF OPERATOR MATRICES*

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Abstract. In this paper, for given operators $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$, the set of all $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ such that the operator matrix $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ is injective, invertible, left invertible and right invertible, is described. Answers to some open questions are given. Also, in the case when A and B are relatively regular operators, the set of all $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ such that M_C is regular is described. In addition, a necessary and a sufficient conditions are given for M_C to be regular with the inner inverse of a certain given form.

Key words. Invertibility, Operator matrix, Regularity, Inner inverse.

AMS subject classifications. 47A10, 47A53, 47A55.

1. Introduction. Let $\mathcal{X},\mathcal{Y}$ be Banach spaces and let $\mathcal{B}(\mathcal{X},\mathcal{Y})$ denote the set of all bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$. For simplicity, we also write $\mathcal{B}(\mathcal{X},\mathcal{X})$ as $\mathcal{B}(\mathcal{X})$. For a given $A \in \mathcal{B}(\mathcal{X},\mathcal{Y})$, the symbols $\mathcal{N}(A)$ and $\mathcal{R}(A)$ denote the null space and the range of A, respectively and define $\alpha(A) = \dim \mathcal{N}(A)$ and $\beta(A) = \operatorname{codim}\mathcal{R}(A)$. If $\mathcal{X} = \mathcal{M} \oplus \mathcal{N}$, by $P_{\mathcal{M},\mathcal{N}}$ we denote the projection onto $\mathcal{M}$ along $\mathcal{N}$.

An operator $A \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ is relatively regular (or simply, regular), if there exists an operator $X \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ satisfying $AXA = A$. In that case any such X is an inner inverse of A which we denote by $A^{(1)}$ or A^-. The set of all inner inverses of A is denoted by $A\{1\}$. If $X \in A\{1\}$ satisfies $XAX = X$, then X is a reflexive or $\{1,2\}$-inverse of A, denoted by $A^{(1,2)}$. The set of all reflexive inverses of A is denoted by $A\{1,2\}$.

It is well-known that A is relatively regular if and only if $\mathcal{R}(A)$ and $\mathcal{N}(A)$ are closed and complemented subspaces of $\mathcal{Y}$ and $\mathcal{X}$, respectively. Suppose that $A \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ is relatively regular and let $\mathcal{T} = \mathcal{R}(XA)$ and $\mathcal{S} = \mathcal{N}(AX)$, for some $X \in A\{1\}$. We have the following decomposition of spaces $\mathcal{X} = \mathcal{T} \oplus \mathcal{N}(A)$ and $\mathcal{Y} =$

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$\mathcal{R}(A) \oplus \mathcal{S}$, and consequently the matrix form of A:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{T} \\ \mathcal{N}(A) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ S \end{pmatrix},$$

where $A_1 \in \mathcal{B}(\mathcal{T}, \mathcal{R}(A))$ is invertible. Evidently, X must be of the form

$$X = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & N \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix} \to \begin{pmatrix} \mathcal{T} \\ \mathcal{N}(A) \end{pmatrix}$$

for some $N \in \mathcal{B}(\mathcal{S}, \mathcal{N}(A))$. Using the above decomposition of $\mathcal{X}$, by $P_{\mathcal{N}(A)}$ we denote the projection on $\mathcal{N}(A)$.

Among the various types of spectra of an operator $A \in \mathcal{B}(\mathcal{X})$ that can be found in literature, we will consider the following:

the spectrum: $\sigma(A) = \{\lambda \in \mathbb{C} : \ A - \lambda I \text{ is not invertible}\}$,

the point spectrum: $\sigma_p(A) = \{\lambda \in \mathbb{C} : \ A - \lambda I \text{ is not injective}\}$,

the left spectrum: $\sigma_l(A) = \{\lambda \in \mathbb{C} : \ A - \lambda I \text{ is not left invertible}\}$,

the right spectrum: $\sigma_r(A) = \{\lambda \in \mathbb{C} : \ A - \lambda I \text{ is not right invertible}\}$.

If $\sigma_*(A)$ is any of the spectra defined above, by $\rho_*(A)$ we will denote the set $\mathbb{C} \setminus \sigma_*(A)$. In the case when A is right (left) invertible, an arbitrary fixed right (left) inverse will be denoted by $(A)_r^{-1}$ ($(A)_l^{-1}$).

If for Banach spaces $\mathcal{X}$, $\mathcal{Y}$, there exists a left invertible operators $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, then we say that $\mathcal{X}$ can be embedded in $\mathcal{Y}$, which is denoted by $\mathcal{X} \preceq \mathcal{Y}$.

In this paper, we will consider some types of invertibility and regularity of upper-triangular operator matrices which have already been subject of research before (see [5]-[14] and for some related problems [3, 8, 9]). In fact, the study of 2×2 upper-triangular operator matrices arises from the following fact: If $A \in \mathcal{B}(\mathcal{X})$ and $\mathcal{S}$ is a closed, complemented A-invariant subspace of $\mathcal{X}$, then A has the following representation:

$$A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : \begin{pmatrix} \mathcal{S} \\ \mathcal{P} \end{pmatrix} \to \begin{pmatrix} \mathcal{S} \\ \mathcal{P} \end{pmatrix},$$

where $\mathcal{X} = \mathcal{S} \oplus \mathcal{P}$.

In this paper, we will consider the injectivity, invertibility, left (right) invertibility and regularity of an operator matrix

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

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where $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ are given. There are few papers, which present necessary and sufficient conditions for the existence of $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ such that M_C is injective, invertible, left (right) invertible and regular (see [5, **?**, 7, 10, 6]). However, in our paper we approach the problem using a technique different than those employed in these papers and, in addition, for given operators $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$, we also completely describe the set of all $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ such that M_C is invertible, right invertible, left invertible or regular, respectively, and answer Question 3 below which has never been done before.

2. Left and right invertibility of M_C. The invertibility of the operator matrix M_C for given operators $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ was considered for the first time in [5] in the case when $\mathcal{X}$ and $\mathcal{Y}$ are separable Hilbert spaces. Since then, many papers have considered various types of spectra of M_C for given operators $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ and in them the following three questions repeatedly arise:

QUESTION 1. Is there an operator $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ such that M_C is invertible (right invertible, left invertible, regular, ...)?

QUESTION 2. $\bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma_*(M_C) =?$, where σ_* is any type of spectrum such as the point, continuous, residual, defect, left, right, essential, Weyl spectrum etc.

QUESTION 3. For given operators $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$, is there an operator $C' \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ such that

$$\sigma_*(M_{C'}) = \bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma_*(M_C),$$

where again σ_* is any type of spectrum?

The set $\bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma(M_C)$ is completely described in [5] but Question 3 is still not answered. The results from [5] are generalized in [7] in the case of Banach spaces, but Question 3 remains open. Also, in [5] and [7], necessary and sufficient conditions for the existence of $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ such that M_C is invertible are presented, but neither is the set of all such operators $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ described nor does the proof offer a clue as to what such operators could look like. In the main result of this section, we give necessary and sufficient conditions for the invertibility of M_C, using a method which is much simpler than the one used in [5] and completely different than the one used in [7]. Our method allows us to completely describe the set of all $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ for which M_C is invertible for given operators $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$, in the case when $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. Also, in some cases we partially answer Question 3.

We note that for a given $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$, the set of all $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ such that M_C is injective, invertible, left invertible and right invertible, will be denoted by $S_p(A,B)$, $S(A,B)$, $S_l(A,B)$, $S_r(A,B)$, respectively.

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Before proving the main result, given $A \in \mathcal{B}(\mathcal{X})$, $B \in \mathcal{B}(\mathcal{Y})$ and $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ we consider necessary and sufficient conditions for injectivity of the operator M_C. In connection with this problem, let us point out that the point spectrum is considered in [14] in the case of separable Hilbert spaces, where the set $\bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})}\sigma_p(M_C)$ is described by

$$\bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})}\sigma_p(M_C) = \sigma_p(A) \cup \{\lambda \in \mathbb{C} : \alpha(B-\lambda I) > \beta(A-\lambda I)\}$$

without considering the question of existence of an operator $C' \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ such that $\sigma_p(M_{C'}) = \bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})}\sigma_p(M_C)$.

The proof of the following result will allow us to describe the set of all $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ for which M_C is injective and also to partially answer Question 3 in the case of the point spectrum.

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ be given operators. The operator matrix M_C is injective for $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ if and only if A is injective and $C|_{\mathcal{N}(B)}$ is injective with range disjoint with $\mathcal{R}(A)$.*

Proof. ($\Rightarrow$) Suppose that M_C is injective for some $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$. Then for $x \in \mathcal{N}(A)$, we get that $\begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{N}(M_C)$ which implies that $x = 0$. Hence, $\mathcal{N}(A) = \{0\}$. Taking $y \in \mathcal{N}(B)$, we get that $M_C \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} Cy \\ 0 \end{pmatrix}$, so $\mathcal{N}(C) \cap \mathcal{N}(B) = \{0\}$. Now, under the assumptions $\mathcal{N}(A) = \{0\}$ and $\mathcal{N}(C) \cap \mathcal{N}(B) = \{0\}$, it is evident that the following holds:

$$\mathcal{N}(M_C) = \{0\} \Leftrightarrow C(\mathcal{N}(B)) \cap \mathcal{R}(A) = \{0\}. \tag{2.1}$$

($\Leftarrow$) If $\mathcal{N}(M_C) \neq \{0\}$, then there are three possibilities:

1. $\begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{N}(M_C),\ x \neq 0,$

2. $\begin{pmatrix} 0 \\ y \end{pmatrix} \in \mathcal{N}(M_C),\ y \neq 0,$

3. $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{N}(M_C), x \neq 0, y \neq 0.$

The first case holds if and only if $\mathcal{N}(A) \neq \{0\}$. The second, if and only if $\mathcal{N}(C) \cap \mathcal{N}(B) \neq \{0\}$. If neither of the assumptions 1. and 2. is satisfied, by (2.1), we get that the third one is satisfied if and only if $C(\mathcal{N}(B)) \cap \mathcal{R}(A) \neq \{0\}$. ∎

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Evidently, in the case when $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces:

$$\bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma_p(M_C)=\sigma_p(A)\cup\Big\{\lambda\in\mathbb{C}:\nexists C\in\mathcal{B}(\mathcal{Y},\mathcal{X}),\ C:\mathcal{N}(B-\lambda I)\overset{1-1}{\longrightarrow}\mathcal{X},$$

$$C(\mathcal{N}(B-\lambda I))\cap\mathcal{R}(A-\lambda I)=\{0\}\Big\}.$$

As a corollary, using Lemma 2.1 from [11], we get the result from [14] in the case when $\mathcal{X}$ and $\mathcal{Y}$ are separable Hilbert spaces:

Corollary 2.2. [14] *For given operators $A\in\mathcal{B}(\mathcal{X})$ and $B\in\mathcal{B}(\mathcal{Y})$, we have*

$$\bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma_p(M_C)=\sigma_p(A)\cup\{\lambda\in\mathbb{C}:\alpha(B-\lambda I)>\beta(A-\lambda I)\}.$$

Remark 2.3. We can partially answer Question 3 in the case of the point spectrum.

Denote by P_p the complement of the set $\bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})}\sigma_p(M_C)$ in $\mathbb{C}$. Thus, P_p consists of all those $\lambda\in\rho_p(A)$ for which there is a $C\in\mathcal{B}(\mathcal{Y},\mathcal{X})$ such that $\mathcal{N}(C)\cap\mathcal{N}(B-\lambda I)=\{0\}$ and $C(\mathcal{N}(B-\lambda I))\cap\mathcal{R}(A-\lambda I)=\{0\}$.

In Question 3, we actually seek a $C'\in\mathcal{B}(\mathcal{Y},\mathcal{X})$ such that $\mathcal{N}(C')\cap\mathcal{N}(B-\lambda I)=\{0\}$ and $C'(\mathcal{N}(B-\lambda I))\cap\mathcal{R}(A-\lambda I)=\{0\}$ for each $\lambda\in P_p$. If for each $\lambda\in R_0=P_p\cap\sigma_p(B)$ we fix a $C_\lambda\in\mathcal{B}(\mathcal{Y},\mathcal{X})$ with $\mathcal{N}(C_\lambda)\cap\mathcal{N}(B-\lambda I)=\{0\}$ and $C_\lambda(\mathcal{N}(B-\lambda I))\cap\mathcal{R}(A-\lambda I)=\{0\}$, then any common extension of the restrictions $C_\lambda|\mathcal{N}(B-\lambda I)$ for $\lambda\in P_p$ would be as required. Now, we point out in the sequel two particular cases in which such extensions exist.

(1) The first case is when the set R_0 is finite. Obviously, if $R_0=\emptyset$ then any $C'\in\mathcal{B}(\mathcal{Y},\mathcal{X})$ is a common extension. If $R_0=\{\lambda_1,\dots,\lambda_k\}$, then we can take $C'=\sum_{i=1}^{k}C_{\lambda_i}\pi_i\pi$, where $\pi_i:\bigoplus_{j=1}^{k}\mathcal{N}(B-\lambda_jI)\to\mathcal{N}(B-\lambda_iI)$, for $1\le i\le k$, are projections corresponding to the given decomposition of the space $\mathcal{P}=\bigoplus_{j=1}^{k}\mathcal{N}(B-\lambda_jI)$ and $\pi:Y\to\mathcal{P}$ is an arbitrary (bounded) projection onto $\mathcal{P}$. The reason there is such a projection is because $\mathcal{P}$ can easily be seen to be closed by induction on k.

Note that R_0 is finite in particular if A is a compact operator(R_0 is finite also, when A is a Riesz or polynomial Riesz operator). Indeed, then $\sigma(A)\subseteq\sigma_p(A)\cup\{0\}$ as is well known. So if $\lambda\in P_p\setminus\{0\}$ then $A-\lambda I$ is an invertible operator. Hence, $\mathcal{R}(A-\lambda I)=\mathcal{X}$, and consequently, $\mathcal{N}(B-\lambda I)=\{0\}$. This means that $R_0\subseteq\{0\}$.

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(2) When $\mathcal{Y}$ is a separable Hilbert space and B is a normal operator, as is well known, all the eigenvalues of B can be arranged in a sequence $(\lambda_n : n \in \mathbb{N})$; and the eigenspaces $\mathcal{N}(B-\lambda_i I)$ and $\mathcal{N}(B-\lambda_j I)$ are mutually orthogonal for $i \neq j$. Let $\pi_n : \mathcal{Y} \to \mathcal{N}(B-\lambda_n I)$ be the orthogonal projection onto the corresponding eigenspace for $n \in \mathbb{N}$. Set $D_n = \dfrac{1}{\|C_{\lambda_n}\| n^2} C_{\lambda_n}$ for $n \in \mathbb{N}$. Clearly $\mathcal{N}(D_n) \cap \mathcal{N}(B-\lambda I) = \{0\}$ and $D_n(\mathcal{N}(B-\lambda I)) \cap \mathcal{R}(A-\lambda I) = \{0\}$. From $\|D_n \pi_n\| \leq \|D_n\| \leq \dfrac{1}{n^2}$ it follows that the series $\sum_{n=1}^{\infty} D_n \pi_n$ converges in the Banach space $\mathcal{B}(\mathcal{Y},\mathcal{X})$ to its sum which we denote by C'. Then C' is a common extension of the restrictions $D_n|\mathcal{N}(B-\lambda_n I)$. For if we take $x \in \mathcal{N}(B-\lambda_n)$ then $\pi_m(x) = 0$ for all $m \neq n$ so $C'(x) = D_n\pi_n(x) = D_n(x)$.

In the following theorem, we give necessary and sufficient conditions for the invertibility of M_C, in the case when $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. The method used in the proof of the following result allows us to completely describe the set of all $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$, for which M_C is invertible. Notice once again, that that for a given $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$, the set of all $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ such that M_C is invertible will be denoted by $S(A,B)$.

THEOREM 2.4. *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ be given operators. The operator matrix M_C is invertible for some $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ if and only if*

(i) *A is left invertible,*
(ii) *B is right invertible,*
(iii) *$\mathcal{N}(B) \cong \mathcal{X}/R(A)$.*

If conditions $(i)-(iii)$ are satisfied, the set of all $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ such that M_C is invertible is given by

$$S(A,B) = \Big\{ C \in \mathcal{B}(\mathcal{Y},\mathcal{X}) : C = \begin{pmatrix} C_1 & 0 \\ 0 & C_4 \end{pmatrix} : \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix}, \\ C_4 \textit{ is invertible}, \mathcal{X} = \mathcal{R}(A) \oplus \mathcal{S} \textit{ and } \mathcal{Y} = \mathcal{P} \oplus \mathcal{N}(B) \Big\}. \tag{2.2}$$

Proof. $(\Rightarrow)$ Suppose that M_C is invertible for some $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$. From

$$M_C = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} I & C \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \tag{2.3}$$

we have that B is right invertible and A is left invertible. Hence, $\mathcal{R}(A)$ and $\mathcal{N}(B)$ are complemented subspaces of $\mathcal{X}$ and $\mathcal{Y}$, respectively.

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From invertibility of M_C, we get that there exists an operator

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix}$$

such that

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{2.4}$$

and

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{2.5}$$

By (2.5), it follows that $X = (A)_l^{-1}$ and by $XC + YB = 0$, we get that

$$(A)_l^{-1} C (B)_r^{-1} B = (A)_l^{-1} C, \tag{2.6}$$

for any right inverse $(B)_r^{-1}$ of B. Analogous, by (2.4), it follows that $W = (B)_r^{-1}$ and by $AY + CW = 0$, we get that

$$A(A)_l^{-1} C (B)_r^{-1} = C (B)_r^{-1}, \tag{2.7}$$

for any left inverse $(A)_l^{-1}$ of A. Now, by (2.6) and (2.7), we get that

$$C(B)_r^{-1} B = A(A)_l^{-1} C, \tag{2.8}$$

for some $(A)_l^{-1}$ and $(B)_r^{-1}$. Now, we can take corresponding decompositions of $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{S}$ and $\mathcal{Y} = \mathcal{P} \oplus \mathcal{N}(B)$ such that

$$A(A)_l^{-1} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix}$$

and

$$(B)_r^{-1} B = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix}.$$

Let C be given by

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} : \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix}.$$

Using (2.7), we can check that $C_2 = 0$ and $C_3 = 0$. Now, by (2.4), it follows that $A(A)_l^{-1} + CZ = I$, which using the above decomposition of C gives that C_4 is right invertible. Analogous, by (2.5) and the equation $ZC + (B)_r^{-1} B = I$, we get that C_4

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is left invertible. Now, from the invertibility of C_4, we have that $\mathcal{N}(B) \cong X/R(A)$. Furthermore, we proved that if M_C is invertible, then there exist some decompositions of $\mathcal{X}$ and $\mathcal{Y}$ such that C is given by

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_4 \end{pmatrix} : \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix}, \quad \text{where } C_4 \text{ is invertible.}$$

($\Leftarrow$) From (i) and (ii), it follows that there exist closed subspaces $\mathcal{S}$ and $\mathcal{P}$ such that $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{S}$ and $\mathcal{Y} = \mathcal{P} \oplus \mathcal{N}(B)$. Using these decompositions, we have that $A = \begin{pmatrix} A_1 \\ 0 \end{pmatrix} : \mathcal{X} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix}$ and $B = \begin{pmatrix} B_1 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix} \to \mathcal{Y}$, where A_1 and B_1 are invertible. By the condition (iii), there exists an invertible operator $C_4 : \mathcal{N}(B) \to \mathcal{S}$. Let $C_1 : \mathcal{P} \to \mathcal{R}(A)$ be arbitrary. It is easy to check that

$$M_C = \begin{pmatrix} A_1 & C_1 & 0 \\ 0 & 0 & C_4 \\ 0 & B_1 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \\ \mathcal{Y} \end{pmatrix}$$

is invertible with the inverse given by $\begin{pmatrix} A_1^{-1} & 0 & -A_1^{-1}C_1B_1^{-1} \\ 0 & 0 & B_1^{-1} \\ 0 & C_4^{-1} & 0 \end{pmatrix}$.

Now, it is clear that (2.2) holds. ☐

REMARK 2.5. 1. We can check that if the conditions $(i) - (iii)$ of Theorem 2.4 hold and if we take arbitrary but fixed decompositions of $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{S}$ and $\mathcal{Y} = \mathcal{P} \oplus \mathcal{N}(B)$, then

$$S(A,B) = \Big\{ C \in \mathcal{B}(\mathcal{Y},\mathcal{X}) : C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} : \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix}, \quad C_4 \text{ is invertible} \Big\}. \tag{2.9}$$

To prove that, let denote by $S_1(A,B)$ the right side of (2.9). For $C \in S_1(A,B)$, it is easy to check that M_C is invertible and in that case the inverse is given by

$$(M_C)^{-1} = \begin{pmatrix} A_1^{-1} & -A_1^{-1}C_2C_4^{-1} & -A_1^{-1}(C_1 - C_2C_4^{-1}C_3)B_1^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & B_1^{-1} \\ 0 & (C_4)_r^{-1} & -C_4^{-1}C_3B_1^{-1} \end{pmatrix}.$$

Hence, $S_1(A,B) \subseteq S(A,B)$. If $C \in S(A,B)$, then using the given decompositions of $\mathcal{X}$ and $\mathcal{Y}$, we get that $A = \begin{pmatrix} A_1 \\ 0 \end{pmatrix} : \mathcal{X} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix}$ and $B = (B_1 \quad 0) :$

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$\begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix} \to \mathcal{Y}$, where A_1 and B_1 are invertible. In that case, any right inverse of B and any left inverse of A are given by $(B)_r^{-1} = \begin{pmatrix} B_1^{-1} \\ T \end{pmatrix} : \mathcal{Y} \to \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix}$ and $(A)_l^{-1} = \begin{pmatrix} A_1^{-1} & R \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix} \to \mathcal{X}$, for some operators T and R. Using (2.4) and (2.5), we get that

$$\begin{aligned} &Z = (I - (B)_r^{-1}B)Z(I - A(A)_l^{-1}), \\ &A(A)_l^{-1} + CZ = I, \\ &ZC + (B)_r^{-1}B = I. \end{aligned} \tag{2.10}$$

Now if we suppose that

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} : \begin{pmatrix} P \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix},$$

by (2.10), we get that C_4 is invertible, which means that $C \in S_1(A, B)$. Hence, (2.9) holds.

2. As in [7], we get that

$$\bigcap_{C \in \mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma(M_C) = \sigma_l(A) \cup \sigma_r(B) \cup \{\lambda \in \mathbb{C} : \mathcal{N}(B - \lambda I) \cong \mathcal{X}/\mathcal{R}(A - \lambda I)\}$$

but using (2.2), we can partially answer Question 3. For arbitrary $C' \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$, it is evident that $\bigcap_{C\in\mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma(M_C) \subseteq \sigma(M_{C'})$ while the reverse inclusion holds if and only if for every $\lambda \in P = \rho_l(A) \cap \rho_r(B) \cap \{\lambda \in \mathbb{C} : \mathcal{N}(B - \lambda I) \cong \mathcal{Y}/\mathcal{R}(A - \lambda I)\}$ the operator C' belongs to $S(A - \lambda I, B - \lambda I)$. In other words, there exists $C' \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that

$$\sigma(M_{C'}) = \bigcap_{C \in \mathcal{B}(\mathcal{Y},\mathcal{X})} \sigma(M_C) \tag{2.11}$$

if and only if

$$\bigcap_{\lambda \in P} S(A - \lambda I, B - \lambda I) \neq \emptyset.$$

In the case of separable Hilbert spaces, $P = \rho_l(A) \cap \rho_r(B) \cap \{\lambda \in \mathbb{C} : \alpha(B - \lambda I) = \beta(A - \lambda I)\}$. In particular if A is a compact (Riesz and polynomial Riesz) operator on a Banach space, then $P = \rho(A) \cap \rho(B)$, so it is evident, that C' can be arbitrary operator from $\mathcal{B}(\mathcal{Y}, \mathcal{X})$. Analogous as in Remark 2.3 (2), we can find an operator C' which satisfies (2.11), in the case when B is normal and $\mathcal{Y}$ is separable Hilbert space.

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In the following theorem, we consider the right invertibility of M_C.

THEOREM 2.6. *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ be given operators. The operator matrix M_C is right invertible for $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ if and only if*

(i) B is right invertible,
(ii) $\begin{pmatrix} A & CP_{\mathcal{N}(B)} \end{pmatrix}$ is right invertible.

Proof. There is a $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that the operator M_C is right invertible if and only if there exist appropriate operators X, Y, Z, W, T such that

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} = \begin{pmatrix} I & T \\ 0 & I \end{pmatrix},$$

i.e., if and only if the following system is solvable:

$$\begin{aligned} AX + CZ &= I \\ BZ &= 0 \\ BW &= I. \end{aligned} \tag{2.12}$$

Evidently, (2.12) is solvable if and only if B is right invertible and the equation $AX + CP_{\mathcal{N}(B)}Q = I$ is solvable for some $Q \in \mathcal{B}(\mathcal{Y})$ (the last condition being equivalent to the right invertibility of $\begin{pmatrix} A & CP_{\mathcal{N}(B)} \end{pmatrix}$). □

REMARK 2.7. From Theorem 2.6, we can conclude that the right invertibility of M_C is equivalent to the right invertibility of B and the solvability of the equation

$$AX + CP_{\mathcal{N}(B)}Q = I \tag{2.13}$$

for some $Q \in \mathcal{B}(\mathcal{Y})$.

If we impose the additional condition of the regularity of $A \in \mathcal{B}(\mathcal{X})$, we have the following result:

THEOREM 2.8. *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ be given operators. If A is regular, then the operator matrix M_C is right invertible for some $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ if and only if*

(i) B is right invertible,
(ii) $\mathcal{X}/\mathcal{R}(A) \preceq \mathcal{N}(B)$.

If conditions $(i) - (ii)$ are satisfied, the set of all $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_C is right invertible is described by

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$$S_r(A,B)=\Big\{C\in\mathcal{B}(\mathcal{Y},\mathcal{X}):C=\begin{pmatrix}C_1 & C_2\\ C_3 & C_4\end{pmatrix}:\begin{pmatrix}\mathcal{P}\\ \mathcal{N}(B)\end{pmatrix}\to\begin{pmatrix}\mathcal{R}(A)\\ \mathcal{S}\end{pmatrix},\quad C_4 \text{ is right invertible}\Big\}, \tag{2.14}$$

where $\mathcal{X}=\mathcal{R}(A)\oplus\mathcal{S}$ *and* $\mathcal{Y}=\mathcal{P}\oplus\mathcal{N}(B)$.

Proof. ($\Rightarrow$) From Remark 2.7, we get that the right invertibility of M_C is equivalent to the right invertibility of $B\in\mathcal{B}(\mathcal{Y})$ and the solvability of the equation (2.13). Let X and Y be any solutions of (2.13). Since A is regular, we have that

$$A=\begin{pmatrix}A_1 & 0\\ 0 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{T}\\ \mathcal{N}(A)\end{pmatrix}\to\begin{pmatrix}\mathcal{R}(A)\\ \mathcal{S}\end{pmatrix}, \tag{2.15}$$

where $A_1\in\mathcal{B}(\mathcal{T},\mathcal{R}(A))$ is invertible and $\mathcal{X}=\mathcal{T}\oplus\mathcal{N}(A)=\mathcal{R}(A)\oplus\mathcal{S}$. Also,

$$B=\begin{pmatrix}B_1 & 0\end{pmatrix}:\begin{pmatrix}\mathcal{M}\\ \mathcal{N}(B)\end{pmatrix}\to\mathcal{Y}, \tag{2.16}$$

where $B_1\in\mathcal{B}(\mathcal{M},\mathcal{Y})$ is invertible and $\mathcal{Y}=\mathcal{M}\oplus\mathcal{N}(B)$. Suppose that

$$X=\begin{pmatrix}X_1 & X_2\\ X_3 & X_4\end{pmatrix}:\begin{pmatrix}\mathcal{R}(A)\\ \mathcal{S}\end{pmatrix}\to\begin{pmatrix}\mathcal{T}\\ \mathcal{N}(A)\end{pmatrix},$$

$$Y=\begin{pmatrix}Y_1 & Y_2\\ Y_3 & Y_4\end{pmatrix}:\begin{pmatrix}\mathcal{R}(A)\\ \mathcal{S}\end{pmatrix}\to\begin{pmatrix}\mathcal{M}\\ \mathcal{N}(B)\end{pmatrix}$$

and

$$C=\begin{pmatrix}C_1 & C_2\\ C_3 & C_4\end{pmatrix}:\begin{pmatrix}\mathcal{M}\\ \mathcal{N}(B)\end{pmatrix}\to\begin{pmatrix}\mathcal{R}(A)\\ \mathcal{S}\end{pmatrix}.$$

We have that (2.13) is equivalent to

$$\begin{pmatrix}A_1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}X_1 & X_2\\ X_3 & X_4\end{pmatrix}+\begin{pmatrix}C_1 & C_2\\ C_3 & C_4\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}Y_1 & Y_2\\ Y_3 & Y_4\end{pmatrix}=\begin{pmatrix}I & 0\\ 0 & I\end{pmatrix},$$

i.e.,

$$A_1X_1+C_2Y_3=I,\quad A_1X_2+C_2Y_4=0,\quad C_4Y_3=0,\quad C_4Y_4=I. \tag{2.17}$$

Evidently, (2.13) is solvable if and only if C_4 is right invertible. Hence, $\mathcal{X}/\mathcal{R}(A)\preceq\mathcal{N}(B)$.

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($\Leftarrow$) If (i) and (ii) are satisfied, we can take $C = \begin{pmatrix} 0 & 0 \\ 0 & C_4 \end{pmatrix} : \begin{pmatrix} \mathcal{M} \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix}$, where C_4 is right invertible. It is evident that the right inverse of M_C is given by

$$\begin{pmatrix} A_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_1^{-1} \\ 0 & (C_4)_r^{-1} & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \\ \mathcal{Y} \end{pmatrix} \to \begin{pmatrix} \mathcal{T} \\ \mathcal{N}(A) \\ \mathcal{M} \\ \mathcal{N}(B) \end{pmatrix}.$$

To prove that the set of all $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ such that M_C is right invertible is described by (2.14), it is enough to note that if $C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} : \begin{pmatrix} \mathcal{M} \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix}$ for decompositions $\mathcal{X} = \mathcal{R}(A) \oplus \mathcal{S}$ and $\mathcal{Y} = \mathcal{M} \oplus \mathcal{N}(B)$, then we have that A and B are given by (2.15) and (2.16), respectively, and in that case, a right inverse of M_C is given by

$$(M_C)_r^{-1} = \begin{pmatrix} A_1^{-1} & -A_1^{-1}C_2(C_4)_r^{-1} & -A_1^{-1}(C_1B_1^{-1} - C_2(C_4)_r^{-1}C_3B_1^{-1}) \\ 0 & 0 & 0 \\ 0 & 0 & B_1^{-1} \\ 0 & (C_4)_r^{-1} & -(C_4)_r^{-1}C_3B_1^{-1} \end{pmatrix}. \quad \Box \tag{2.18}$$

REMARK 2.9. The condition (ii) of Theorem 2.6, in the case when $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces is equivalent to the condition $\mathcal{R}(A) + \mathcal{R}(CP_{\mathcal{N}(B)}) = \mathcal{X}$. In the case, when $\mathcal{X}$ and $\mathcal{Y}$ are separable Hilbert spaces, the condition (ii) of Theorem 2.6 is equivalent to $\alpha(B) \le \beta(A)$.

Similarly, we can consider the left invertibility of M_C:

THEOREM 2.10. *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ be given operators. The operator matrix M_C is left invertible for $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ if and only if*

(i) *A is left invertible,*
(ii) $\begin{pmatrix} (I - P_{\mathcal{R}(A)})C \\ B \end{pmatrix}$ *is left invertible.*

With the additional assumption of the regularity of B, we have the following:

THEOREM 2.11. *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ be given operators such that B is regular. The operator matrix M_C is left invertible for some $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ if and only if*

(i) *A is left invertible,*
(ii) $\mathcal{N}(B) \preceq \mathcal{X}/\mathcal{R}(A)$.

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If the conditions $(i)-(ii)$ are satisfied, the set of all $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ such that M_C is right invertible is described by

$$S_l(A,B)=\left\{C\in\mathcal{B}(\mathcal{Y},\mathcal{X}):\ C=\begin{pmatrix} C_1 & C_2 \\ C_3 & C_4\end{pmatrix}:\begin{pmatrix}\mathcal{P}\\ \mathcal{N}(B)\end{pmatrix}\to\begin{pmatrix}\mathcal{R}(A)\\ \mathcal{S}\end{pmatrix},\right.$$
$$\left. C_4 \text{ is left invertible}\right\}.$$

where $\mathcal{X}=\mathcal{R}(A)\oplus\mathcal{S}$ and $\mathcal{Y}=\mathcal{P}\oplus\mathcal{N}(B)$.

In the case when $\mathcal{X}$ and $\mathcal{Y}$ are separable Hilbert spaces, the condition (ii) of Theorem 2.11 is equivalent to $\alpha(B)\leq\beta(A)$.

Remark 2.12. Analogous as in the case of the point spectrum, when $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, we can answer Question 3 in the case of right (left) spectrum and find such operator C' in the following cases: when A is compact or Riesz or polynomial Riesz operator; when B is a normal operator; when the set $P_r=\rho_l(B)\cap\{\lambda\in\mathbb{C}: 0<\beta(A-\lambda I)\leq\alpha(B-\lambda I)\}$ ($P_l=\rho_l(A)\cap\{\lambda\in\mathbb{C}:0<\alpha(B-\lambda I)<\beta(A-\lambda I)\}$) is finite.

3. Regularity of M_C. In this section, we will consider the regularity of an operator matrix

$$M_C=\begin{pmatrix} A & C\\ 0 & B\end{pmatrix},$$

where $A\in\mathcal{B}(\mathcal{X})$ and $B\in\mathcal{B}(\mathcal{Y})$ are given. We will completely describe the set of all $C\in\mathcal{B}(\mathcal{Y},\mathcal{X})$ such that M_C is regular. Furthermore, we will give a necessary and sufficient condition under which M_C is regular and has an inner inverse of the form $\begin{pmatrix} A^{(1,2)} & X\\ Y & B^{(1,2)}\end{pmatrix}$, where $A^{(1,2)}\in A\{1,2\}$, $B^{(1,2)}\in B\{1,2\}$ and one of the operators $X\in\mathcal{B}(\mathcal{Y},\mathcal{X})$ and $Y\in\mathcal{B}(\mathcal{X},\mathcal{Y})$ is zero.

In the next theorem, we present a necessary and sufficient condition under which M_C is regular for a given $A\in\mathcal{B}(\mathcal{X})$ and $B\in\mathcal{B}(\mathcal{Y})$. Using its proof, we can completely describe the set $M_C\{1\}$.

Theorem 3.1. *Let $A\in\mathcal{B}(\mathcal{X})$ and $B\in\mathcal{B}(\mathcal{Y})$ be relatively regular. The operator matrix M_C is relatively regular if and only if $(I-AA^{(1)})C(I-B^{(1)}B)$ is regular, for some $A^{(1)}\in A\{1\}$ and $B^{(1)}\in B\{1\}$.*

Proof. Since $A\in\mathcal{B}(\mathcal{X})$ and $B\in\mathcal{B}(\mathcal{Y})$ are relatively regular, it follows that $\mathcal{X}$ and $\mathcal{Y}$ have the following decompositions

$$\mathcal{X}=\mathcal{T}\oplus\mathcal{N}(A)=\mathcal{R}(A)\oplus\mathcal{S},\quad Y=\mathcal{P}\oplus\mathcal{N}(B)=\mathcal{R}(B)\oplus\mathcal{Q},$$

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and consequently, that A and B are given by

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{T} \\ \mathcal{N}(A) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix},$$

and

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(B) \\ \mathcal{Q} \end{pmatrix},$$

where $A_1 : \mathcal{T} \to \mathcal{R}(A)$ and $B_1 : \mathcal{P} \to \mathcal{R}(B)$ are invertible. There exist $A^{(1)} \in A\{1\}$ and $B^{(1)} \in B\{1\}$ such that $AA^{(1)} = P_{\mathcal{R}(A),\mathcal{S}}$, $A^{(1)}A = P_{\mathcal{T},\mathcal{N}(A)}$, $BB^{(1)} = P_{\mathcal{R}(B),\mathcal{Q}}$ and $B^{(1)}B = P_{\mathcal{P},\mathcal{N}(B)}$. Let C be given by

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} : \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix} \to \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix} \tag{3.1}$$

and suppose that M_C is relatively regular. In that case, there exists an operator $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix}$ such that

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \tag{3.2}$$

which is equivalent to the following system of the equations:

$$\begin{aligned} &AXA + CZA = A, \\ &BZC + BWB = B, \\ &BZA = 0, \\ &AXC + CZC + AYB + CWB = C. \end{aligned} \tag{3.3}$$

Suppose that X, Y, Z and W are given by

$$\begin{aligned} X &= \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix} \to \begin{pmatrix} \mathcal{T} \\ \mathcal{N}(A) \end{pmatrix}, \\ Y &= \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(B) \\ \mathcal{Q} \end{pmatrix} \to \begin{pmatrix} \mathcal{T} \\ \mathcal{N}(A) \end{pmatrix}, \\ Z &= \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix} \to \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix}, \\ W &= \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(B) \\ \mathcal{Q} \end{pmatrix} \to \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix}. \end{aligned} \tag{3.4}$$

We get that the third equation of (3.3), $BZA = 0$, is equivalent with $B_1 Z_1 A_1 = 0$, i.e.,

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$Z_1 = 0$. Now, using that $Z_1 = 0$, the first equation of (3.3) is equivalent with

$$\begin{aligned} &X_1 = A_1^{-1}(I - C_2Z_3), \\ &C_4Z_3 = 0, \end{aligned} \tag{3.5}$$

while the second one holds if and only if

$$\begin{aligned} &W_1 = (I - Z_2C_3)B_1^{-1}, \\ &Z_2C_4 = 0. \end{aligned} \tag{3.6}$$

Now, the last equation of (3.3) is equivalent with

$$\begin{aligned} &A_1X_2C_3 + C_2Z_4C_3 + A_1Y_1B_1 + C_2W_3B_1 + C_1 = 0, \\ &A_1X_2C_4 + C_2Z_4C_4 = 0, \\ &C_4Z_4C_3 + C_4W_3B_1 = 0 \\ &C_4Z_4C_4 = C_4. \end{aligned} \tag{3.7}$$

From the last equation of (3.7), we conclude that C_4 must be relatively regular. So, we get that the regularity of $(I - AA^{(1)})C(I - B^{(1)}B)$ is a necessary condition for the regularity of M_C. On the other hand, if C_4 is relatively regular, taking $Z_4 = C_4^{(1)}$, $Z_1 = 0$, $Z_2 = 0$, $Z_3 = 0$, $X_1 = A_1^{-1}(I - C_2Z_3)$, $X_2 = -A_1^{-1}C_2C_4^{(1)}$, $Y_1 = -A_1^{-1}(C_1 + A_1X_2C_3 + C_2Z_4C_3 + C_2W_3B_1)B_1^{-1}$, $W_1 = (I - Z_2C_3)B_1^{-1}$, $W_3 = -C_4^{(1)}C_3B_1^{-1}$, where $X_3, X_4, Y_2, Y_3, Y_4, Z_2, Z_3, W_2, W_4$ are arbitrary, using (3.5), (3.6) and (3.7) we can easily check that (3.2) holds. Thus, M_C is relatively regular. ∎

Remark 3.2. (1) Using the previous theorem, we can completely describe the set $M_C\{1\}$ in the case when M_C is regular. So, $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix} : \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix} \to \begin{pmatrix} \mathcal{X} \\ \mathcal{Y} \end{pmatrix} \in M_C\{1\}$ if and only if X, Y, Z, W are given by

$$\begin{aligned} &X = \begin{pmatrix} A_1^{-1}(I - C_2Z_3) & -A_1^{-1}C_2C_4^{(1)} + M(I - C_4C_4^{(1)}) \\ X_3 & X_4 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix} \to \begin{pmatrix} \mathcal{T} \\ \mathcal{N}(A) \end{pmatrix}, \\ &Y = \begin{pmatrix} -A_1^{-1}NB_1^{-1} & Y_2 \\ Y_3 & Y_4 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(B) \\ \mathcal{Q} \end{pmatrix} \to \begin{pmatrix} \mathcal{T} \\ \mathcal{N}(A) \end{pmatrix}, \\ &Z = \begin{pmatrix} 0 & Z_2 \\ Z_3 & C_4^{(1)} \end{pmatrix} : \begin{pmatrix} \mathcal{R}(A) \\ \mathcal{S} \end{pmatrix} \to \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix}, \\ &W = \begin{pmatrix} (I - Z_2C_3)B_1^{-1} & W_2 \\ -C_4^{(1)}C_3B_1^{-1} + (I - C_4^{(1)}C_4)S & W_4 \end{pmatrix} : \begin{pmatrix} \mathcal{R}(B) \\ \mathcal{Q} \end{pmatrix} \to \begin{pmatrix} \mathcal{P} \\ \mathcal{N}(B) \end{pmatrix}, \end{aligned} \tag{3.8}$$

where $Z_2 = E(I - C_4C_4^{(1)})$, $Z_3 = (I - C_4^{(1)}C_4)F$, $N = C_1 + A_1X_2C_3 + C_2Z_4C_3 +$

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$C_2W_3B_1$ and the others operators which appear are arbitrary.

(2) Evidently, Theorem 3.1 is a generalization of Theorem 2.1 [6].

In the following theorem, we present a necessary and sufficient condition under which M_C is regular and has an inner inverse of the form $\begin{pmatrix} A^{(1,2)} & X \\ Y & B^{(1,2)} \end{pmatrix}$, where $A^{(1,2)} \in A\{1,2\}$, $B^{(1,2)} \in B\{1,2\}$ and one of the operators $X \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ and $Y \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ is zero with the assumption that the operator C has the matrix representation from the proof of Theorem 3.1.

THEOREM 3.3. *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ be relatively regular. For $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$, the operator matrix M_C is relatively regular with an inner inverse of the form $\begin{pmatrix} A^{(1,2)} & X \\ Y & B^{(1,2)} \end{pmatrix}$ where $A^{(1,2)} \in A\{1,2\}$, $B^{(1,2)} \in B\{1,2\}$ and one of the operators $X \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ and $Y \in \mathcal{B}(\mathcal{X},\mathcal{Y})$ is zero, if and only if C satisfies one of the following conditions:*

(1) $C(\mathcal{N}(B)) \subseteq \mathcal{R}(A)$,
(2) C_1 *is regular,* $C_1C_1^-C_2 = 0$, $C_3C_1^-C_1 = 0$, $C_3C_1^-C_2 = -C_4$,

where $C_1 = (I - AA^{(1,2)})C(I - B^{(1,2)}B)$, $C_2 = (I - AA^{(1,2)})CB^{(1,2)}B$, $C_3 = AA^{(1,2)}C(I - B^{(1,2)}B)$, $C_4 = AA^{(1,2)}CB^{(1,2)}B$ *and* $C_1^- \in C_1\{1\}$.

Proof. The fact that for some $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ the operator matrix M_C is relatively regular with an inner inverse of the form $\begin{pmatrix} A^{(1,2)} & X \\ Y & B^{(1,2)} \end{pmatrix}$, for $X \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ and $Y \in \mathcal{B}(\mathcal{X},\mathcal{Y})$, is equivalent with the fact that the following system of the equations is solvable:

$$\begin{aligned} &CYA = 0, \\ &AA^{(1,2)}C + CYC + AXB + CB^{(1,2)}B = C, \\ &BYA = 0, \\ &BYC = 0. \end{aligned} \tag{3.9}$$

Since we have an additional condition that one of the operators X and Y is equal to zero, we will consider two cases.

First, we will consider the case when $Y = 0$. Then the system (3.9) is equivalent to:

$$AA^{(1,2)}C + AXB + CB^{(1,2)}B = C. \tag{3.10}$$

Hence, for a given $C \in \mathcal{B}(\mathcal{Y},\mathcal{X})$ the operator M_C is regular and has an inner inverse

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of the form $\begin{pmatrix} A^{(1,2)} & X \\ 0 & B^{(1,2)} \end{pmatrix}$, if and only if the following equation is solvable:

$$AXB = C - AA^{(1,2)}C - CB^{(1,2)}B. \tag{3.11}$$

This in turn holds if and only if $(I - AA^{(1,2)})C(I - B^{(1,2)}B) = 0$, which is equivalent with the fact that $C(\mathcal{N}(B)) \subseteq \mathcal{R}(A)$.

In the case when $X = 0$, the system (3.9) is equivalent to:

$$\begin{aligned} &CYA = 0, \\ &AA^{(1,2)}C + CYC + CB^{(1,2)}B = C, \\ &BYA = 0, \\ &BYC = 0, \end{aligned} \tag{3.12}$$

Now, it is clear that M_C will be relatively regular with an inner inverse of the form $\begin{pmatrix} A^{(1,2)} & 0 \\ Y & B^{(1,2)} \end{pmatrix}$, for each $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ which satisfies the conditions for the solvability of the system (3.12).

Suppose that such C exists. Then C and Y can be represented by

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B) \\ \mathcal{R}(B^{(1,2)}) \end{pmatrix} \to \begin{pmatrix} \mathcal{N}(A^{(1,2)}) \\ \mathcal{R}(A) \end{pmatrix}, \tag{3.13}$$

$$Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A^{(1,2)}) \\ \mathcal{R}(A) \end{pmatrix} \to \begin{pmatrix} \mathcal{N}(B) \\ \mathcal{R}(B^{(1,2)}) \end{pmatrix}, \tag{3.14}$$

for some operators C_i and Y_i, $i = \overline{1,4}$. Since the equation $BYA = 0$ is equivalent with the equation $B^{(1,2)}BYAA^{(1,2)} = 0$ and since

$$AA^{(1,2)} = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} : \begin{pmatrix} \mathcal{N}(A^{(1,2)}) \\ \mathcal{R}(A) \end{pmatrix} \to \begin{pmatrix} \mathcal{N}(A^{(1,2)}) \\ \mathcal{R}(A) \end{pmatrix},$$

$$B^{(1,2)}B = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} : \begin{pmatrix} \mathcal{N}(B) \\ \mathcal{R}(B^{(1,2)}) \end{pmatrix} \to \begin{pmatrix} \mathcal{N}(B) \\ \mathcal{R}(B^{(1,2)}) \end{pmatrix},$$

we conclude that $Y_4 = 0$. Now, $CYA = 0$ is equivalent with $C_1Y_2 = 0$ and $C_3Y_2 = 0$, while $BYC = 0$ is equivalent with $Y_3C_1 = 0$ and $Y_3C_2 = 0$. Now, we are looking for the conditions under which the second equation of the system (3.12) is satisfied. The same computations as above show that the second equation of the system (3.12) is satisfied for some operators C and Y given by (3.13) if and only if operators C_i and Y_i satisfy the following system:

$$C_1Y_1C_1 = C_1,\ C_1Y_1C_2 = 0,\ C_3Y_1C_1 = 0,\ C_3Y_1C_2 = -C_4. \tag{3.15}$$

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From the first equation of the system (3.15), it follows that C_1 must be relatively regular and that Y_1 is an inner inverse of C_1. Now, by (3.13) we get that (3.15) is equivalent to (ii). $\square$

Remark 3.4. (1) Theorem 3.3 is a special case of Theorem 3.1, so each of the conditions (1) and (2) of Theorem 3.3 implies the regularity of $(I - AA^{(1)})C(I - B^{(1)}B)$, for some $A^{(1)} \in A\{1\}$ and $B^{(1)} \in B\{1\}$. That this is true for the condition (2) is evident. For the condition (1) this follows from the fact that it is equivalent with the fact that $(I - AA^{(1)})C(I - B^{(1)}B) = 0$.

(2) Obviously, (3.12) can be written as

$$\begin{pmatrix} B \\ C \end{pmatrix} Y \begin{pmatrix} A & C \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & C - AA^{(1,2)}C - CB^{(1,2)}B \end{pmatrix}. \tag{3.16}$$

Now, it is clear that M_C will be relatively regular with an inner inverse of the form $\begin{pmatrix} A^{(1,2)} & 0 \\ Y & B^{(1,2)} \end{pmatrix}$, for each $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ which satisfies the conditions for the solvability of the equation (3.16). Hence, (ii) is equivalent to

$$\begin{pmatrix} B \\ C \end{pmatrix} \begin{pmatrix} B \\ C \end{pmatrix}^{(1)} \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix} \begin{pmatrix} A & C \end{pmatrix}^{(1)} \begin{pmatrix} A & C \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix},$$

where $T = C - AA^{(1,2)}C - CB^{(1,2)}B$. Using (3.16), we can describe all such Y.

(3) It is interesting to note that by Theorem 3.3, we have that for any relatively regular operator $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ which satisfies $C(I - P_{\mathcal{N}(B)}) = 0$ and $P_{\mathcal{R}(A)}C = 0$, for some projections $P_{\mathcal{N}(B)}$ and $P_{\mathcal{R}(A)}$, the matrix operator M_C is relatively regular with an inner inverse of the form $\begin{pmatrix} A^{(1,2)} & 0 \\ Y & B^{(1,2)} \end{pmatrix}$.

In the special case when $\mathcal{X}$ and $\mathcal{Y}$ are separable Hilbert spaces, we state the following result concerning the regularity and the form of the Moore-Penrose inverse of M_C.

Theorem 3.5. *Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ be relatively regular. For $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$, the operator matrix M_C is relatively regular with the Moore-Penrose of the form $\begin{pmatrix} A^\dagger & X \\ Y & B^\dagger \end{pmatrix}$, where one of the operators $X \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and $Y \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ is zero if and only if C satisfies one of the following conditions:*

(1) $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$, $\mathcal{R}(C) \subseteq \mathcal{R}(A)$,
(2) C is relatively regular, $\mathcal{R}(C) \subseteq \mathcal{N}(A^*)$, $\mathcal{R}(B^*) \subseteq \mathcal{N}(C)$.

Furthermore, in the case when $X = 0$, the operator Y must be equal to $C^\dagger$.

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Proof. First, if one of the conditions (1) – (2) is satisfied then by Theorem 3.3 and Remark 2.5, it follows that M_C is relatively regular, which is equivalent with the existence of the Moore-Penrose inverse. Now, we will show that the Moore-Penrose inverse of M_C has the mentioned form if and only if one of the conditions (1) – (2) is satisfied.

First, suppose that for some $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$, there exist X such that the operator matrix M_C has the Moore-Penrose of the form $\begin{pmatrix} A^\dagger & X \\ 0 & B^\dagger \end{pmatrix}$. By the four Penrose equations, this is equivalent with the fact that the following system of the equations is solvable:

$$\begin{aligned} &AX + CB^\dagger = 0, \\ &A^\dagger C + XB = 0, \\ &AA^\dagger C + AXB + CB^\dagger B = 0, \\ &A^\dagger AX + A^\dagger CB^\dagger + XBB^\dagger = X. \end{aligned} \tag{3.17}$$

We can check that the system (3.17) has a solution X if and only if $B^\dagger BC^* = C^*$ and $AA^\dagger C = C$ which is equivalent with $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*)$, $\mathcal{R}(C) \subseteq \mathcal{R}(A)$.

On the other hand, the fact that for some $C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$, there exist Y such that the operator matrix M_C has the Moore-Penrose inverse of the form $\begin{pmatrix} A^\dagger & 0 \\ Y & B^\dagger \end{pmatrix}$ is equivalent with the fact that the following system of the equations is solvable:

$$\begin{aligned} &(CB^\dagger)^* = BY,\ CY = (CY)^*, YC = (YC)^*, \\ &(A^\dagger C)^* = YA,\ CYA = 0,\ BYC = 0,\ A^\dagger CB^\dagger = 0, \\ &AA^\dagger C + CYC + CB^\dagger B = C, \\ &YAA^\dagger + YCY + B^\dagger BY = Y. \end{aligned} \tag{3.18}$$

Suppose that there exists Y which satisfies (3.18). Then, using that $(CB^\dagger)^* = BY$, we get that $BYC = 0$ is equivalent with $CB^\dagger = 0$ which implies that $B^\dagger BY = 0$. Similarly, using that $(A^\dagger C)^* = YA$, we get that $CYA = 0$ is equivalent with $A^\dagger C = 0$ which implies that $YAA^\dagger = 0$. Now, the system (3.18) has the form:

$$\begin{aligned} &BY = 0,\ CY = (CY)^*, YC = (YC)^*, \\ &YA = 0,\ A^\dagger C = 0,\ CB^\dagger = 0, \\ &CYC = C, \\ &YCY = Y. \end{aligned} \tag{3.19}$$

It is clear that C must be relatively regular and that $A^\dagger C = 0$ and $CB^\dagger = 0$. Fur-

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thermore, by (3.19) we get that $Y = C^{\dagger}$. $\square$

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