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ON (T, f)-CONNECTIONS OF MATRICES AND GENERALIZED INVERSES OF LINEAR OPERATORS*

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Abstract. In this note, generalized connections $\sigma_{T,f}$ are investigated, where $A\sigma_{T,f}B = T_A f T_A^-(B)$ for positive semidefinite matrix A and hermitian matrix B, and operator monotone function $f : J \to \mathbb{R}$ on an interval $J \subset \mathbb{R}$. Here the symbol T_A^- denotes a reflexive generalized inverse of a positive bounded linear operator T_A. The problem of estimating a given generalized connection by other ones is studied. The obtained results are specified for special cases of α-arithmetic, α-geometric and α-harmonic operator means.

Key words. Positive definite matrix, Positive linear map, Operator monotone function, Connection, α-Arithmetic (α-Geometric, α-Harmonic) operator mean, Generalized inverse of linear operator.

AMS subject classifications. 15A45, 47A63, 47A64.

1. Motivation. We begin with some notation. By $\mathbb{M}_n$ we denote the C^*-algebra of $n \times n$ complex matrices. The symbol $\mathbb{B}(\mathbb{M}_n)$ stands for the C^*-algebra of all bounded linear operators from $\mathbb{M}_n$ into $\mathbb{M}_n$. We denote the (real) space of $n \times n$ Hermitian matrices by $\mathbb{H}_n$. The spectrum of an hermitian matrix $A \in \mathbb{H}_n$ is denoted by $\mathrm{Sp}\,(A)$.

For matrices $X, Y \in \mathbb{M}_n$, we write $Y \leq X$ (resp., $Y < X$) if $X - Y$ is positive semidefinite (resp., positive definite).

We say that a linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is *positive* if $0 \leq \Phi(X)$ for $0 \leq X \in \mathbb{M}_n$. If in addition $0 < \Phi(X)$ for $0 < X \in \mathbb{M}_n$ then we say that Φ is *strictly positive.*

A function $h : J \to \mathbb{R}$ with an interval $J \subset \mathbb{R}$ is said to be an *operator monotone function*, if for all Hermitian matrices A and B (of the same order) with spectra in J,

$$A \leq B \quad \text{implies} \quad h(A) \leq h(B)$$

(see [3, p. 112]).

Let $f : J \to \mathbb{R}$ be a continuous function on an interval $J \subset \mathbb{R}$. The f*-connection* of $n \times n$ positive definite matrices A and B such that $\mathrm{Sp}\,(A^{-1/2}BA^{-1/2}) \subset J$, is

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defined by

$$A\sigma_f B = A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2} \tag{1.1}$$

(cf. [6, p. 637]).

For $f(t)$ of the form $\alpha t + 1 - \alpha$, t^α and $(\alpha t^{-1} + 1 - \alpha)^{-1}$, one obtains the α-arithmetic mean, α-geometric mean and α-harmonic mean, respectively, defined as follows (see (1.2), (1.3) and (1.4)).

For $\alpha \in [0, 1]$, the *α-arithmetic mean* of $n \times n$ positive definite matrices A and B is defined by

$$A\nabla_\alpha B = (1 - \alpha)A + \alpha B. \tag{1.2}$$

For $\alpha \in [0, 1]$, the *α-geometric mean* of $n \times n$ positive definite matrices A and B is defined as follows:

$$A\sharp_\alpha B = A^{1/2}(A^{-1/2} B A^{-1/2})^\alpha A^{1/2} \tag{1.3}$$

(see [8, 13]).

For $\alpha \in [0, 1]$, the *α-harmonic mean* of $n \times n$ positive definite matrices A and B is defined by

$$A\,!_\alpha B = ((1 - \alpha)A^{-1} + \alpha B^{-1})^{-1}. \tag{1.4}$$

In [1] Ando showed that if $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is a positive linear map and $A, B \in \mathbb{M}_n$ are positive definite then

$$\Phi(A\sharp_\alpha B) \leq \Phi(A)\sharp_\alpha \Phi(B).$$

Aujla and Vasudeva [2] proved that

$$\Phi(A\sigma_f B) \leq \Phi(A)\sigma_f \Phi(B)$$

for an operator monotone function $f : (0, \infty) \to (0, \infty)$.

A related result is due to Kaur et al. [7].

THEOREM A [7, Theorem 2.1] *Let A and B be $n \times n$ positive definite matrices such that $0 < b_1 \leq A \leq a_1$ and $0 < b_2 \leq B \leq a_2$ for some scalars $0 < b_i < a_i$, $i = 1, 2$.*

If $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is a strictly positive linear map, then for any operator mean σ with the representing function f, the following double inequality holds:

$$\omega^{1-\alpha}\left(\Phi(A)\sharp_\alpha \Phi(B)\right) \leq (\omega\Phi(A))\nabla_\alpha \Phi(B) \leq \frac{\alpha}{\mu}\Phi(A\sigma B),$$

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where $\mu = \frac{a_1 b_1 (f(b_2 a_1^{-1}) - f(a_2 b_1^{-1}))}{b_1 b_2 - a_1 a_2}$, $\nu = \frac{a_1 a_2 f(b_2 a_1^{-1}) - b_1 b_2 f(a_2 b_1^{-1})}{a_1 a_2 - b_1 b_2}$, $\omega = \frac{\alpha\nu}{(1-\alpha)\mu}$ *and* $\alpha \in (0,1)$.

The following result was proved in [11]. The symbol $\circ$ means the composition of maps.

Theorem B [11, Theorem 2.7] *Let f_1, f_2, g_1, g_2 be continuous real functions defined on an interval $J = [m, M] \subset \mathbb{R}_+$. Assume that $g_2 > 0$ and $g_2 \circ g_1^{-1}$ are operator monotone on intervals J and $J' = g_1(J)$, respectively, with invertible g_1 and concave g_2. Let A and B be $n \times n$ positive definite matrices such that $mA \leq B \leq MA$ with $0 < m < M$.*

If $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is a strictly positive linear map,

$$g_1(t) \leq f_1(t) \quad \text{and} \quad f_2(t) \leq g_2(t) \quad \text{for } t \in J,$$

and

$$\max_{t\in J} g_1(t) = \max_{t\in J} f_1(t),$$

then

$$c_{g_2}\, \Phi(A)\sigma_{f_2}\Phi(B) \leq \Phi(A\sigma_{g_2} B) \leq \Phi(A\sigma_{g_2\circ g_1^{-1}}(A\sigma_{f_1} B)),$$

where c_{g_2} is defined by

$$a_{g_2} = \frac{g_2(M) - g_2(m)}{M-m}\ , \quad b_{g_2} = \frac{Mg_2(m) - mg_2(M)}{M-m} \quad \text{and} \quad c_{g_2} = \min_{t\in J} \frac{a_{g_2} t + b_{g_2}}{g_2(t)}.$$

In the present note, we extend Theorems A and B from f-connections of type (1.1) to a class of (T, f)-connections of the form

$$A\sigma_{T,f} B = T_A f T_A^-(B)$$

for $A \geq 0$ and $B \in \mathbb{H}_n$, where T_A^- denotes a positive reflexive generalized inverse of a positive bounded map T_A, and in addition the map $T : X \to T_X$ is positive.

2. Results. A *generalized inverse* of a linear map $L : V \to W$ between linear spaces V and W is a linear map $L^- : W \to V$ satisfying $LL^-L = L$. If in addition $L^-LL^- = L^-$ then L^- is called a *reflexive* generalized inverse of L.

By $\operatorname{Ran}(L)$ we denote the *range* of a linear map L.

If L^- is a reflexive generalized inverse of L, then

$$LL^-(Y) = Y \quad \text{for } Y \in \operatorname{Ran}(L), \tag{2.1}$$

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and

$$L^- L(X) = X \quad \text{for } X \in \mathrm{Ran}\,(L^-).$$

It is known that if $L : H \to H$ is a bounded linear map with a Hilbert space H, then there exists a generalized inverse of L if and only if L has closed range [4].

Throughout this note, whenever the symbol L^- is used, it is assumed that there exists a generalized inverse L^- of a linear map L.

We denote the Loewner cone of all positive semidefinite $n \times n$ complex matrices by $\mathbb{L}_n$, i.e., $\mathbb{L}_n = \{X \in \mathbb{H}_n : X \geq 0\}$.

Let $T : X \to T_X$ be a map from $\mathbb{L}_n$ into $\mathbb{B}(\mathbb{H}_n)$ with $T_X : \mathbb{H}_n \to \mathbb{H}_n$ satisfying

$$T_X \text{ is positive for } X \geq 0.$$

Let $f : J \to \mathbb{R}$ be a continuous function on an interval $J \subset \mathbb{R}$. The (T, f)-*connection* $\sigma_{T,f}$ of an $n \times n$ positive semidefinite matrix A and an $n \times n$ hermitian matrix B such that $\mathrm{Sp}\,(T_A^- B) \subset J$, is defined by

$$A \sigma_{T,f} B = T_A f T_A^-(B) \tag{2.2}$$

(cf. [6, p. 637]). For some applications of connections of the form (2.2), see [10, 11, 12].

Example 2.1. Let $A \geq 0$ be positive semidefinite (not necessarily positive definite) with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. By spectral decomposition $A = U \mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_n) U^*$ for some unitary matrix $U \in \mathbb{M}_n$. We consider a generalized inverse A^- of A given by $A^- = U \mathrm{diag}\,(\mu_1, \mu_2, \ldots, \mu_n) U^*$, where $\mu_i = \frac{1}{\lambda_i}$ if $\lambda_i > 0$, and $\mu_i = 0$ if $\lambda_i = 0$, $i = 1, 2, \ldots, n$. Thus, A^- is positive semidefinite, and $A^- = (A^-)^{1/2} (A^-)^{1/2}$ with

$$(A^-)^{1/2} = (A^{1/2})^- = U \mathrm{diag}\,(\sqrt{\mu_1}, \sqrt{\mu_2}, \ldots, \sqrt{\mu_n}) U^*$$

We are now in a position to set

$$T_A = A^{1/2} (\cdot) A^{1/2} \quad \text{and} \quad T_A^- = (A^-)^{1/2} (\cdot) (A^-)^{1/2}.$$

Then (2.2) takes the form (cf. (1.1))

$$A \sigma_{T,f} B = A^{1/2} f \left((A^-)^{1/2} B (A^-)^{1/2} \right) A^{1/2} \quad \text{for } B \in \mathbb{H}_n.$$

Example 2.2. (Cf. [12, Example 3.5].) By E we denote the $n \times n$ matrix of ones.

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Let $A = (a_{ij})$ be an $n \times n$ positive semidefinite matrix with $0 \leq a_{ij} < 2$, $i, j = 1, 2, \dots, n$, and such that $0 \leq A \leq E$. We define

$$T_A(X) = A \odot X = (a_{ij} x_{ij}) \quad \text{for } X = (x_{ij}) \in \mathbb{M}_n,$$

where $\odot$ stands for the *Schur product* of $n \times n$ matrices.

By Schur Product Theorem (see [5, Theorem 5.2.1]), the map T_A is positive, i.e.,

$$X \geq 0 \quad \text{implies} \quad T_A(X) \geq 0.$$

We consider the generalized inverse of T_A defined by

$$T_A^-(Y) = A^{[-1]} \odot Y \quad \text{for} \quad Y = (y_{ij}) \in \mathbb{M}_n \ , \tag{2.3}$$

where $A^{[-1]} = (c_{ij})$ with $c_{ij} = \frac{1}{a_{ij}}$ if $a_{ij} \neq 0$, and $c_{ij} = 0$ if $a_{ij} = 0$.

Under the hypothesis that $A = (a_{ij})$ with $0 \leq a_{ij} < 2$, $i, j = 1, 2, \dots, n$, we have

$$A^{[-1]} = E + (E - A) + (E - A)^{[2]} + (E - A)^{[3]} + \cdots,$$

the convergent Schur-power series (see [5, pp. 449–450]). Here the m-th Schur-power of $E - A$ is defined by

$$(E - A)^{[m]} = \underbrace{(E - A) \odot \cdots \odot (E - A)}_{m \text{ times}}, \quad m = 1, 2, \dots,$$

with $(E - A)^{[0]} = E$.

It is evident by Schur Product Theorem that $(E - A)^{[m]} \geq 0$, $m = 0, 1, 2, 3, \dots$, because $E - A \geq 0$. Therefore,

$$A^{[-1]} \geq 0 \tag{2.4}$$

(see [5, Theorem 6.3.5]). So, the map T_A^- is positive by (2.3) and (2.4).

In summary, in this situation (T, f)-connection is given by

$$A \sigma_{T,f} B = A \odot f\left(A^{[-1]} \odot B\right) \quad \text{for } B \in \mathbb{H}_n.$$

For a function $g : J \to \mathbb{R}_+$ defined on an interval $J = [m, M]$ with $m < M$, we define (see [9])

$$a_g = \tfrac{g(M) - g(m)}{M - m} \ , \quad b_g = \tfrac{M g(m) - m g(M)}{M - m} \quad \text{and} \quad c_g = \min_{t \in J} \tfrac{a_g t + b_g}{g(t)}. \tag{2.5}$$

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In the forthcoming theorem, we extend [9, Corollary 3.4] from the classical map $X \to T_X(\cdot) = X^{1/2}(\cdot)X^{1/2}$, $X > 0$, with the inverse $T_X^{-1}(\cdot) = X^{-1/2}(\cdot)X^{-1/2}$, to an arbitrary positive map $X \to T_X$ with a reflexive generalized inverse T_X^- of T_X.

THEOREM 2.3. *Assume that*

(i) *for any $X \geq 0$, $T_X : \mathbb{H}_n \to \mathbb{H}_n$ is a bounded linear operator, and $T_X^- : \mathbb{H}_n \to \mathbb{H}_n$ is a reflexive generalized inverse of T_X, satisfying the following conditions:*

$$T_X \text{ and } T_X^- \text{ are positive,} \tag{2.6}$$

$$T_X(I_n) = X, \tag{2.7}$$

$$I_n \in \operatorname{Ran}(T_X^-), \tag{2.8}$$

(ii) *$\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is a strictly positive linear map,*

(iii) *A is an $n \times n$ positive semidefinite matrix and B is an $n \times n$ Hermitian matrix with $\operatorname{Sp}(T_A^- B) \subset J = [m, M]$, $m < M$, such that*

$$B \in \operatorname{Ran} T_A \quad \text{and} \quad \Phi(B) \in \operatorname{Ran} T_{\Phi(A)}. \tag{2.9}$$

If $g : J \to \mathbb{R}_+$ is a continuous concave operator monotone function, then

$$c_g\, \Phi(A)\sigma_{T,g}\Phi(B) \leq \Phi(A\sigma_{T,g}B),$$

where a_g, b_g and c_g are defined by (2.5).

Proof. The proof of inequality (2.12) is based on the proof of [6, Theorem 1] (cf. also [9, Corollary 3.4]).

Since $g : J \to (0, \infty)$ is concave, it is not hard to verify that

$$a_g t + b_g \leq g(t) \quad \text{for } t \in J. \tag{2.10}$$

It follows from (2.5) that

$$c_g \leq \frac{a_g t + b_g}{g(t)} \quad \text{for } t \in J.$$

However, g is positive, so $a_g t + b_t > 0$ for $t \in J$. Thus, c_g is positive. Therefore,

$$g(t) \leq \frac{a_g}{c_g} t + \frac{b_g}{c_g}. \tag{2.11}$$

In consequence, (2.10) and (2.11) give

$$a_g t + b_g \leq g(t) \leq \frac{a_g}{c_g} t + \frac{b_g}{c_g} \quad \text{for } t \in [m, M], \tag{2.12}$$

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which is the required estimation.

We now denote

$$l_1(t) = a_g t + b_g \quad \text{and} \quad l_2(t) = \frac{a_g}{c_g} t + \frac{b_g}{c_g} \quad \text{for } t \in J = [m, M].$$

Consider $C = \Phi(A)$ and $D = \Phi(B)$. Because $\mathrm{Sp}\,(T_A^- B) \subset [m, M]$, we establish $mI_n \leq T_A^- B \leq MI_n$, and next $mT_A I_n \leq T_A T_A^- B \leq MT_A I_n$, because T_A is positive. In addition, $B \in \mathrm{Ran}\, T_A$ and $A = T_A(I_n)$ (see (2.9) and (2.7)). Consequently, $mA \leq B \leq MA$ (see (2.1)). From this $m\Phi(A) \leq \Phi(B) \leq M\Phi(A)$, i.e., $mC \leq D \leq MC$. But T_C^- is positive, so $mT_C^- C \leq T_C^- D \leq MT_C^- C$. Furthermore, $T_C^- C = I_n$, because $T_C I_n = C$ (see (2.7)) and hence $I_n = T_C^- T_C I_n = T_C^- C$ by $I_n \in \mathrm{Ran}\, T_C^-$ (see (2.8)). Thus, we derive $mI_n \leq T_C^- D \leq MI_n$ and $\mathrm{Sp}\,(T_C^- D) \subset [m, M]$.

Since $\mathrm{Sp}\,(T_C^- D) \subset [m, M]$ and $g(t) \leq l_2(t)$ for $t \in [m, M]$ (see (2.12)), we find that

$$g(T_C^- D) \leq l_2(T_C^- D).$$

Now, by making use of positivity of T_C we obtain

$$T_C g(T_C^- D) \leq T_C l_2(T_C^- D).$$

But $l_2(t) = \frac{1}{c_g} l_1(t)$ for $t \in [m, M]$. So, we have

$$c_g T_C g(T_C^- D) \leq T_C l_1(T_C^- D).$$

Therefore,

$$c_g C \sigma_{T,g} D \leq C \sigma_{T,l_1} D,$$

i.e.,

$$c_g \Phi(A) \sigma_{T,g} \Phi(B) \leq \Phi(A) \sigma_{T,l_1} \Phi(B). \tag{2.13}$$

Simultaneously, it is clear that

$$C \sigma_{T,l_1} D = T_C l_1 T_C^- D = T_C(a_g T_C^- D + b_g I_n) = a_g T_C T_C^- D + b_g T_C I_n. \tag{2.14}$$

However, $T_C(I_n) = C$ and $D \in \mathrm{Ran}\, T_C$ (see (2.7) and (2.9)). Hence, $T_C T_C^-(D) = D$ (see (2.1)). Finally, from (2.14) we obtain

$$C \sigma_{T,l_1} D = a_g D + b_g C, \tag{2.15}$$

which means

$$\Phi(A) \sigma_{T,l_1} \Phi(B) = a_g \Phi(B) + b_g \Phi(A) = \Phi(a_g B + b_g A). \tag{2.16}$$

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By virtue of (2.13) and (2.16), we get

$$c_g \Phi(A)\sigma_{T,g}\Phi(B) \le \Phi(a_g B + b_g A).$$

On the other hand, since $\mathrm{Sp}\,(T_A^- B) \subset [m, M]$ and $l_1(t) \le g(t)$ for $t \in [m, M]$ (see (2.12)), we have

$$l_1(T_A^- B) \le g(T_A^- B),$$

and further

$$T_A l_1(T_A^- B) \le T_A g(T_A^- B).$$

Therefore,

$$A\sigma_{T,l_1} B \le A\sigma_{T,g} B.$$

Hence,

$$\Phi(A\sigma_{T,l_1} B) \le \Phi(A\sigma_{T,g} B). \tag{2.17}$$

Moreover, it follows that

$$A\sigma_{T,l_1} B = T_A(a_g T_A^- B + b_g I_n) = a_g B + b_g A,$$

since $B \in \mathrm{Ran}\, T_A$, $T_A T_A^-(B) = B$ and $T_A(I_n) = A$ (see (2.9), (2.1) and (2.7)).

So, in light of (2.17) we see that

$$\Phi(a_g B + b_g A) \le \Phi(A\sigma_{T,g} B). \tag{2.18}$$

In summary, combining (2.13), (2.15) and (2.18) leads to

$$c_g \Phi(A)\sigma_{T,g}\Phi(B) \le \Phi(A)\sigma_{T,l_1}\Phi(B) = \Phi(a_g B + b_g A) \le \Phi(A\sigma_{T,g} B). \quad \square$$

Remark 2.4. In the case $T_A(\cdot) = A^{1/2}(\cdot)A^{1/2}$ with $A > 0$, Theorem 2.3 reduces to [6, Theorem 1], cf. also [9, Corollary 3.4].

Remark 2.5. It is evident that Theorem 2.3 simplifies if T_X is invertible. In fact, then the condition (2.9) is automatically fulfilled, and therefore, it can be dropped off. For the same reason, condition (2.8) can be deleted.

Remark 2.6. In Theorem 2.3, condition (i) can be assumed to hold for $X = A$ and $X = \Phi(A)$, only.

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The next result is an extension of [7, Theorem 2.1].

THEOREM 2.7. *Under the assumptions* (i)–(iii) *of Theorem 2.3 for X, T_X, Φ, A and B, let f_1, f_2, g_1, g_2 be continuous real functions defined on an interval $J = [m, M]$, $m < M$. Suppose that g_1 is invertible on J, g_2 is positive concave and operator monotone on J, and $g_2 \circ g_1^{-1}$ is operator monotone on $J' = g_1(J)$, and*

$$g_1(t) \le f_1(t) \quad \text{and} \quad f_2(t) \le g_2(t) \quad \text{for } t \in J, \tag{2.19}$$

$$\max_{t \in J} g_1(t) = \max_{t \in J} f_1(t), \tag{2.20}$$

$$f_1 \mathrm{Ran}\,(T_A^-) \subset \mathrm{Ran}\,(T_A^-) \quad \text{and} \quad g_1 \mathrm{Ran}\,(T_A^-) \subset \mathrm{Ran}\,(T_A^-)\ . \tag{2.21}$$

Then

$$c_{g_2}\, \Phi(A) \sigma_{T,f_2} \Phi(B) \le \Phi(A \sigma_{T,g_2} B) \le \Phi(A \sigma_{T, g_2 \circ g_1^{-1}} (A \sigma_{T,f_1} B)), \tag{2.22}$$

where c_{g_2} is defined by (2.5) with $g = g_2$.

Proof. As in the proof of Theorem 2.3, we obtain $mA \le B \le MA$, and further $m\Phi(A) \le \Phi(B) \le M\Phi(A)$ by the positivity of Φ. Hence, by the positivity of $T_{\Phi(A)}^-$, we establish

$$m T_{\Phi(A)}^- \Phi(A) \le T_{\Phi(A)}^- \Phi(B) \le M T_{\Phi(A)}^- \Phi(A).$$

Now, from (2.7)-(2.8) we deduce that

$$m I_n \le T_{\Phi(A)}^- \Phi(B) \le M I_n.$$

Therefore, $\mathrm{Sp}\,(T_{\Phi(A)}^- \Phi(B)) \subset [m, M]$.

In light of the second inequality of (2.19), we have

$$f_2 T_{\Phi(A)}^- \Phi(B) \le g_2 T_{\Phi(A)}^- \Phi(B),$$

and next, by the positivity of $T_{\Phi(A)}$,

$$T_{\Phi(A)} f_2 T_{\Phi(A)}^- \Phi(B) \le T_{\Phi(A)} g_2 T_{\Phi(A)}^- \Phi(B).$$

That is,

$$\Phi(A) \sigma_{T,f_2} \Phi(B) \le \Phi(A) \sigma_{T,g_2} \Phi(B). \tag{2.23}$$

It follows from Theorem 2.3 applied to the function $g = g_2$ that

$$c_{g_2}\, \Phi(A) \sigma_{T,g_2} \Phi(B) \le \Phi(A \sigma_{T,g_2} B).$$

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For this reason, (2.23) implies

$$c_{g_2}\,\Phi(A)\sigma_{T,f_2}\Phi(B) \le \Phi(A\sigma_{T,g_2}B). \tag{2.24}$$

This proves the left-hand side inequality of (2.22).

Furthermore, we find that

$$A\sigma_{T,g_2}B = A\sigma_{T,h\circ g_1}B = A\sigma_{T,h}(A\sigma_{T,g_1}B), \tag{2.25}$$

where $h = g_2 \circ g_1^{-1}$ and $\circ$ means composition. Indeed, by (2.21) we get $g_1T_A^-B \in \operatorname{Ran} T_A^-$. So, we have

$$g_1T_A^-B = T_A^-T_Ag_1T_A^-B. \tag{2.26}$$

Hence,

$$A\sigma_{T,h\circ g_1}B = T_Ahg_1T_A^-B = T_AhT_A^-T_Ag_1T_A^-B$$

$$= T_AhT_A^-(A\sigma_{T,g_1}B) = A\sigma_{T,h}(A\sigma_{T,g_1}B),$$

which yields (2.25).

On the other hand, from the first inequality of (2.19) we obtain

$$g_1T_A^-B \le f_1T_A^-B,$$

and further

$$T_Ag_1T_A^-B \le T_Af_1T_A^-B$$

by the positivity of T_A. Thus, we have

$$A\sigma_{T,g_1}B \le A\sigma_{T,f_1}B. \tag{2.27}$$

From (2.19) we see that

$$\min_{t\in J} g_1(t) \le \min_{t\in J} f_1(t).$$

This together with (2.20) implies

$$f_1(J) \subset g_1(J). \tag{2.28}$$

We now introduce

$$Z_0 = T_A^-(A\sigma_{T,g_1}B) \quad \text{and} \quad W_0 = T_A^-(A\sigma_{T,f_1}B).$$

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Clearly, by (2.26) and (2.21),

$$Z_0 = g_1 T_A^-(B) \quad \text{and} \quad W_0 = f_1 T_A^-(B).$$

Then $\mathrm{Sp}\,(Z_0) \subset g_1(J)$ and $\mathrm{Sp}\,(W_0) \subset f_1(J) \subset g_1(J)$, because $\mathrm{Sp}\,(T_A^- B) \subset J$.

So, from (2.27) and (2.28), we obtain

$$A\sigma_{T,h}(A\sigma_{T,g_1}B) \le A\sigma_{T,h}(A\sigma_{T,f_1}B), \tag{2.29}$$

because h is operator monotone on $J' = g_1(J)$, and T_A and T_A^- are positive.

From (2.25) and (2.29), it follows that

$$\Phi(A\sigma_{T,g_2}B) \le \Phi(A\sigma_{T,g_2\circ g_1^{-1}}(A\sigma_{T,f_1}B)). \tag{2.30}$$

Now, according to (2.24) and (2.30), we infer that (2.22) is satisfied. ◻

The discussion of inequality (2.22) for the cases $f_1 = f_2$ and $g_1 = g_2$ with $T_A = A^{-1/2}(\cdot)A^{-1/2}$, $A > 0$, can be found in [11].

We now consider the case $g_1 = g_2$ of (2.22) for arbitrary T_A.

Corollary 2.8. *Under the assumptions* (i)–(iii) *of Theorem 2.3 for X, T_X, Φ, A and B, let f_1, f_2, g be continuous real functions defined on an interval $J = [m, M]$, $m < M$. Suppose that g is invertible positive concave and operator monotone on J, and*

$$f_2(t) \le g(t) \le f_1(t) \quad \text{for } t \in J,$$

$$\max_{t\in J} g(t) = \max_{t\in J} f_1(t),$$

$$f_1 \mathrm{Ran}\,(T_A^-) \subset \mathrm{Ran}\,(T_A^-) \quad \text{and} \quad g\mathrm{Ran}\,(T_A^-) \subset \mathrm{Ran}\,(T_A^-)\ .$$

Then

$$c_g\, \Phi(A)\sigma_{T,f_2}\Phi(B) \le \Phi(A\sigma_{T,g}B) \le \Phi(A\sigma_{T,f_1}B),$$

where c_g is defined by (2.5).

Proof. It is enough to apply Theorem 2.7 with $g_1 = g_2 = g$ and $g_2 \circ g_1^{-1} = \mathrm{id}$ and $A\sigma_{T,\mathrm{id}}\,(A\sigma_{T,f_1}B) = A\sigma_{T,f_1}B$. ◻

Some concrete versions of inequalities (2.22) of Theorem 2.7 which depend on the form of $g_1 \circ g_2^{-1}$ are included in Corollary 2.9. By making use of affine, power and inverse-affine functions we obtain arithmetic, geometric and harmonic operator means of A and $A\sigma_{T,f_1}B$, respectively, on the right-hand side of (2.22).

Corollary 2.9. *Under the assumptions of Theorem 2.7 with $A, B > 0$:*

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(I) *If $g_2 \circ g_1^{-1}$ is an affine function, i.e., $g_2 \circ g_1^{-1}(s) = as + b$ for $s \in g_1(J)$, $a > 0$, then the right-hand side inequality of (2.22) reduces to*

$$\Phi(A\sigma_{T,g_2}B) \le a\,\Phi(A\sigma_{T,f_1}B) + b\,\Phi(A). \tag{2.31}$$

(II) *If $g_2 \circ g_1^{-1}$ is a power function, i.e., $g_2 \circ g_1^{-1}(s) = s^{\alpha}$ for $s \in g_1(J) \subset \mathbb{R}_+$, $\alpha \in [0,1]$, then the right-hand side inequality of (2.22) reduces to*

$$\Phi(A\sigma_{T,g_2}B) \le \Phi(T_A(T_A^{-}(A\sigma_{T,f_1}B))^{\alpha}). \tag{2.32}$$

(III) *If $g_2 \circ g_1^{-1}$ is an inverse function of the form $g_2 \circ g_1^{-1}(s) = (\alpha s^{-1} + 1 - \alpha)^{-1}$ for $s \in g_1(J) \subset \mathbb{R}_+$, $\alpha \in [0,1]$, then the right-hand side inequality of (2.22) reduces to*

$$\Phi(A\sigma_{T,g_2}B) \le \Phi([(1-\alpha)A^{-1} + \alpha(A\sigma_{T,f_1}B)^{-1}]^{-1}). \tag{2.33}$$

Proof. **(I)** Since $a > 0$, the function $g_2 \circ g_1^{-1}(s) = as + b$ is operator monotone (see [3, p. 113]). Moreover,

$$A\sigma_{T,h}(A\sigma_{T,f_1}B) = T_A h T_A^{-}(A\sigma_{T,f_1}B).$$

Simultaneously,

$$A\sigma_{T,f_1}B = T_A f_1 T_A^{-} B \in \mathrm{Ran}\,(T_A).$$

So, for $h(s) = as + b$, by (2.1) we have

$$A\sigma_{T,h}(A\sigma_{T,f_1}B) = T_A(aT_A^{-}(A\sigma_{T,f_1}B) + bI_n)$$

$$= aT_AT_A^{-}(A\sigma_{T,f_1}B) + bT_A(I_n) = aA\sigma_{T,f_1}B + bA.$$

Now, to see (2.31) it is enough to employ (2.22).

(II) The function $g_2 \circ g_1^{-1}(s) = s^{\alpha}$ with $\alpha \in [0,1]$ is operator monotone (see [3, p. 115]). So, to prove (2.32), it is sufficient to apply (2.22).

(III) Inequality (2.33) follows from (2.22) applied to the operator monotone function $g_2 \circ g_1^{-1}(s) = (\alpha s^{-1} + 1 - \alpha)^{-1}$ with $\alpha \in [0,1]$ (see [3, p. 114]). ◻

Theorem 2.7 simplifies if in addition T_X is invertible.

Corollary 2.10. *Assume that*

(i) *for any $X \ge 0$, $T_X : \mathbb{H}_n \to \mathbb{H}_n$ is an invertible bounded positive linear operator with positive inverse T_X^{-1} such that $T_X(I_n) = X$,*
(ii) *$\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is a strictly positive linear map,*

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(iii) *A and B are $n \times n$ positive definite matrices with* $\mathrm{Sp}\,(T_A^{-1}B) \subset J = [m, M]$, $0 < m < M$.

Let f_1, f_2, g_1, g_2 be continuous real functions defined on J satisfying conditions (2.19) and (2.20). Suppose that g_1 is invertible on J, g_2 is positive concave and operator monotone on J, and $g_2 \circ g_1^{-1}$ is operator monotone on interval $J' = g_1(J)$.

Then inequality (2.22) is satisfied with (2.5) for $g = g_2$.

Proof. With $T_X^- = T_X^{-1}$ conditions (2.8) and (2.21) hold automatically. Now it is sufficient to use Theorem 2.7. $\square$

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