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PSEUDO SCHUR COMPLEMENTS, PSEUDO PRINCIPAL PIVOT TRANSFORMS AND THEIR INHERITANCE PROPERTIES[*]

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Abstract. Extensions of the Schur complement and the principal pivot transform, where the usual inverses are replaced by the Moore-Penrose inverse, are revisited. These are called the pseudo Schur complement and the pseudo principal pivot transform, respectively. First, a generalization of the characterization of a block matrix to be an M-matrix is extended to the nonnegativity of the Moore-Penrose inverse. A comprehensive treatment of the fundamental properties of the extended notion of the principal pivot transform is presented. Inheritance properties with respect to certain matrix classes are derived, thereby generalizing some of the existing results. Finally, a thorough discussion on the preservation of left eigenspaces by the pseudo principal pivot transformation is presented.

Key words. Principal pivot transform, Schur complement, Nonnegative Moore-Penrose inverse, $P_\dagger$-Matrix, $R_\dagger$-Matrix, Left eigenspace, Inheritance properties.

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1. Introduction. Let $M \in \mathbb{R}^{m \times n}$ be a block matrix partitioned as

$$\left(\begin{array}{cc} A & B \\ C & D \end{array} \right),$$

where $A \in \mathbb{R}^{k \times k}$ is nonsingular. Then the classical Schur complement of A in M denoted by M/A is given by $D - CA^{-1}B \in \mathbb{R}^{(m-k) \times (n-k)}$. This notion has proved to be a fundamental idea in many applications like numerical analysis, statistics and operator inequalities, to name a few. We refer the reader to [27] for a comprehensive account of the Schur complement. In the formula for the Schur complement, apparently, Albert [1] was the first to replace A^{-1} by $A^\dagger$, the Moore-Penrose inverse of A. He studied positive definiteness and nonnegative definiteness for symmetric matrices using this formula. This expression for the Schur complement was fruther extended by Carlson [10] to include all matrices of the form $D - CA^{\{1\}}B$, where $A^{\{1\}}$ denotes any arbitrary {1}-inverse of A (A {1}-inverse of A is any matrix X which

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satisfies $AXA = A$). Carlson proved that this generalized Schur complement is invariant under the choice of $A^{\{1\}}$ if and only if $B = 0$ or $C = 0$ or $R(B) \subseteq R(A)$ and $R(C^T) \subseteq R(A^T)$. Here $R(X)$ denotes the range space of the matrix X. He also goes on to refer to this as a "natural setting for clean results in the area of Schur complements". Henceforth, we shall refer to these as natural conditions. He studies the relationship of the generalized Schur complements to certain optimal rank problems. The expression $D - CA^{\dagger}B$, is also referred to in the literature as the generalized Schur complement [11], where the Sylvester's determinantal formula and a quotient formula are proved, among other things. Nevertheless, since we will be concerned with the case of the Moore-Penrose inverse (which is also called the pseudo inverse), we shall refer to it as the pseudo Schur complement.

Let us now turn to the most important object that is being studied here. Again, consider M, partitioned as above. If A is nonsingular, then the principal pivot transform (PPT) of M [24] is the block matrix defined by

$$\begin{pmatrix} A^{-1} & -A^{-1}B \\ CA^{-1} & F \end{pmatrix},$$

where F is again, the Schur complement $F = D - CA^{-1}B$. This operation of obtaining the PPT arises in many contexts, namely mathematical programming, numerical analysis and statistics, to name a few. For an excellent survey of PPT we refer the reader to [23], and [20] for certain mapping and preserver properties of the PPT. This transformation has also received attention by researchers in linear complementarity theory for their inheritance properties with respect to many matrix classes. We refer to [22] for more details of these results which are presented for symmetric cones in Euclidean Jordan algebras. Just as in the case of the generalized Schur complement, it is natural to study the PPT when the usual inverses are replaced by generalized inverses. Meenakshi [17], was perhaps the first to study such a generalization for the Moore-Penrose inverse. We shall refer to this generalization of the PPT as the pseudo principal pivot transform (precise definition in Section 4). Recently, Rajesh Kannan and Bapat [19] obtained certain interesting extensions of some of the well known results on the principal pivot transform. They also studied almost skew-symmetric matrices.

We organize the contents of the paper as follows. In the next section, we provide a brief background for the rest of the material in the article. The main result in the subsequent section is an extension to the case of the Moore-Penrose inverse of an inheritance property rather well known for M-matrices. This is presented in Theorem 3.4. Section 4 collects certain basic properties of the pseudo principal pivotal transform. In the process, we point to a couple of errors in a recent work [19] and provide correct versions of these. Section 5 presents inheritance properties for the Schur complement and the pseudo principal pivot transform for two classes of matrices studied

recently. The main results here are Theorem 5.4, Theorem 5.5 and Theorem 5.10. In the concluding section, a thorough treatment of preservation of left eigenspaces by the principal pivot transform is undertaken, with Corollary 6.1 presenting a summary of the results there.

2. Preliminaries. Let $\mathbb{R}^n$ denote the n dimensional real Euclidean space and $\mathbb{R}^n_+$ denote the nonnegative orthant in $\mathbb{R}^n$. For a matrix $M \in \mathbb{R}^{m \times n}$, the set of all $m \times n$ matrices of real numbers, we denote the null space and the transpose of M by $N(M)$ and M^T, respectively. The Moore-Penrose inverse of a matrix $M \in \mathbb{R}^{m \times n}$, denoted by $M^\dagger$ is the unique solution $X \in \mathbb{R}^{n \times m}$ of the equations: $M = MXM$, $X = XMX$, $(MX)^T = MX$ and $(XM)^T = XM$. If M is nonsingular, then of course, we have $M^{-1} = M^\dagger$. Recall that $M \in \mathbb{R}^{n \times n}$ is called range-symmetric (or an EP matrix) if $R(M^T) = R(M)$. For this class of matrices, the Moore-Penrose inverse $M^\dagger$ commutes with M. Next, we collect some well known properties of $M^\dagger$ that will be frequently used in this paper [3]. For a subspace $S \subseteq \mathbb{R}^n$, let P_S denote the orthogonal projection of $\mathbb{R}^n$ onto the subspace S. If T is a subspace complementary to the subspace S, then $P_{S,T}$ stands for the projection of $\mathbb{R}^n$ onto the subspace S along the subspace T. We have: $R(M^T) = R(M^\dagger)$; $N(M^T) = N(M^\dagger)$; $MM^\dagger = P_{R(M)}$; $M^\dagger M = P_{R(M^T)}$. In particular, if $x \in R(M^T)$, then $x = M^\dagger M x$.

Next, we list certain results to be used in the sequel. The first result is well known, for instance one could refer to [26].

THEOREM 2.1. *Let $A \geq 0$ be a square matrix. If S is any principal square submatrix of A then $\rho(S) \leq \rho(A)$. The inequality is strict if A is irreducible.*

In the study of iterative schemes for systems of linear systems, the concept of a matrix splitting has proved to be very important. Specifically, for a nonsingular matrix A, a decomposition of the form $A = U - V$, when U is invertible, is referred to as a splitting. There are various types of splittings that have been studied in the literature. Let us only mention that the classical iterative techniques like the Gauss-Jordan method, the Gauss-Siedel method and the successive over-relaxation method are particular instances of a matrix splitting, as above [26]. When the matrix A is singular, the notion of a proper splitting was introduced by Berman and Plemmons [6]. For A, a decomposition of the form $A = U - V$ is called a proper splitting if $R(A) = R(U)$ and $N(A) = N(U)$. For many of the fundamental properties of a proper splitting and their role in the convergence of iterative methods for singular linear systems, we refer to the book [4]. In particular, the following will be useful in our discussion.

THEOREM 2.2. *(Theorem 1, [6]) Let $A = U - V$ be a proper splitting. Then:*

(a) $AA^\dagger = UU^\dagger$; $A^\dagger A = U^\dagger U$; $VU^\dagger U = V$.

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(b) $A = U(I - U^\dagger V)$.

(c) $I - U^\dagger V$ is invertible.

(d) $A^\dagger = (I - U^\dagger V)^{-1} U^\dagger$.

The next result gives two characterizations for the Moore-Penrose inverse to be nonnegative. The frame work is that of a proper splitting.

THEOREM 2.3. *(Theorem 3, [5]) Let $A = U - V$ be a proper splitting such that $U^\dagger \geq 0$ and $U^\dagger V \geq 0$. Then the following are equivalent:*

(a) $A^\dagger \geq 0$.

(b) $A^\dagger V \geq 0$.

(c) $\rho(U^\dagger V) < 1$.

If A and B are square invertible matrices, then $(AB)^{-1} = B^{-1}A^{-1}$. However, in case of generalized inverse this need not be true. The following result presents a characterization for the reverse order law to hold for the case of the Moore-Penrose inverse.

THEOREM 2.4. *(Theorem 2, [15]) Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$. Then $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $BB^* A^\dagger A$ and $A^* ABB^\dagger$ are Hermitian.*

3. The Moore-Penrose inverse and inheritance of nonnegativity. It is well known that the Schur complement and formulae for inverses of partitioned matrices go hand in hand. We proceed in the same spirit, where we first consider the Moore-Penrose inverse of partitioned matrices. The following result is quite well known. The necessary part was shown in Theorem 1 in [7] and Theorem 1 in [9]. In Corollary 2 in [8] and Theorem 2.10 in [25], a proof is given.

THEOREM 3.1. *Let $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times p}, C \in \mathbb{R}^{s \times n}, D \in \mathbb{R}^{s \times p}$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $F = D - CA^\dagger B$. Then $R(C^T) \subseteq R(A^T)$, $R(B) \subseteq R(A)$, $R(C) \subseteq R(F)$ and $R(B^T) \subseteq R(F^T)$ if and only if*

$$M^\dagger = \begin{pmatrix} A^\dagger + A^\dagger BF^\dagger CA^\dagger & -A^\dagger BF^\dagger \\ -F^\dagger CA^\dagger & F^\dagger \end{pmatrix}.$$

REMARK 3.1. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathbb{R}^{n \times n}$ and $D \in \mathbb{R}^{m \times m}$. Define $U = \begin{pmatrix} 0 & I_m \\ I_n & 0 \end{pmatrix}$, then U is orthogonal and $U^T = \begin{pmatrix} 0 & I_n \\ I_m & 0 \end{pmatrix}$. Then $N =$

$UMU^T = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$. The pseudo Schur complement of A in M is $F = D - CA^\dagger B$ and the pseudo Schur complement of D in N is $G = A - BD^\dagger C$.

Let us reiterate that the two inclusions $R(B) \subseteq R(A)$ and $R(C^T) \subseteq R(A^T)$ are the natural conditions that we had referred to, in the introduction. Next, we state a complementary result. This does not seem to be as well known as the previous result. However, we skip its proof. Note that this result uses the pseudo Schur complement $G = A - BD^\dagger C$, which is called the complementary Schur complement in [2]. This time, the natural conditions are $R(B^T) \subseteq R(D^T)$ and $R(C) \subseteq R(D)$. These conditions guarantee that the complementary Schur complement $G = A - BD^{\{1\}}C$ is invariant under any $\{1\}$-inverse $D^{\{1\}}$ of D. Again, we state the result and for a proof, the reader may refer to Theorem 2.1 in [13].

THEOREM 3.2. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with the blocks defined as in Theorem 3.1. Suppose that $R(B^T) \subseteq R(D^T)$, $R(C) \subseteq R(D)$, $R(B) \subseteq R(G)$ and $R(C^T) \subseteq R(G^T)$, where $G = A - BD^\dagger C$. Then

$$M^\dagger = \begin{pmatrix} G^\dagger & -G^\dagger BD^\dagger \\ -D^\dagger CG^\dagger & D^\dagger + D^\dagger CG^\dagger BD^\dagger \end{pmatrix}.$$

By comparing the two expressions for $M^\dagger$, we obtain the formulae:

$$G^\dagger = A^\dagger + A^\dagger BF^\dagger CA^\dagger \quad \text{and} \quad F^\dagger = D^\dagger + D^\dagger CG^\dagger BD^\dagger,$$

in the presence of all the eight inclusions of Theorem 3.1 and Theorem 3.2. Using these formulae, next we derive another expression for the Moore-Penrose inverse of M involving the pseudo Schur complements of A and D. This will be used in the subsequent result.

THEOREM 3.3. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Suppose that $R(C^T) \subseteq R(A^T)$, $R(B) \subseteq R(A)$, $R(B^T) \subseteq R(F^T)$, $R(C) \subseteq R(F)$, $R(C) \subseteq R(D)$, $R(B^T) \subseteq R(D^T)$, $R(B) \subseteq R(G)$ and $R(C^T) \subseteq R(G^T)$, where $F = D - CA^\dagger B$ and $G = A - BD^\dagger C$. Then

$$M^\dagger = \begin{pmatrix} G^\dagger & -A^\dagger BF^\dagger \\ -D^\dagger CG^\dagger & F^\dagger \end{pmatrix}.$$

REMARK 3.2. Let M be as above such that $R(C^T) \subseteq R(A^T)$, $R(B) \subseteq R(A)$, $R(B^T) \subseteq R(F^T)$, $R(C) \subseteq R(F)$, $R(C) \subseteq R(D)$, $R(B^T) \subseteq R(D^T)$, $R(B) \subseteq R(G)$ and $R(C^T) \subseteq R(G^T)$. Then it may be verified that

$$M^\dagger M = \begin{pmatrix} A^\dagger A & 0 \\ 0 & D^\dagger D \end{pmatrix}.$$

Our first main result uses the theorem above to obtain a necessary and sufficient condition for the nonnegativity of the Moore-Penrose inverse of a block matrix M. This is expressed in terms of the nonnegativity of the Moore-Penrose inverse of the principal subtransformations A and D and the nonnegativity of the Moore-Penrose inverse of the pseudo Schur complements F and G. Let us recall the classical result for Z-matrices. First, a square matrix S is called a Z-matrix, if all its off-diagonal entries are nonpositive. If S is a Z-matrix, it then follows that we can write $S = tI - T$ where $t \geq 0$ and $T \geq 0$. Let $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a square matrix. Suppose that A and D are square matrices of the same order and also are Z-matrices. Further, let B and C be nonpositive matrices so that L itself is a Z-matrix. Markham, (Theorem 3, [16]) has shown that L is (an invertible) M-matrix if and only if A, D and the two Schur complements, $F = D - CA^{-1}B$ and $G = A - BD^{-1}C$ are (invertible) M-matrices. It is well known that an invertible M-matrix has the property that its inverse is nonnegative [4]. Thus, L is an invertible M-matrix if and only if $A^{-1} \geq 0, D^{-1} \geq 0, F^{-1} \geq 0$ and $G^{-1} \geq 0$. In what follows, we enlarge the applicability of this result by considering the case of the Moore-Penrose inverse. To place this result in a proper perspective, let us observe the following. Let us say that a decomposition $S = U - V$ is a pseudo splitting if $U^\dagger \geq 0$ and $U^\dagger V \geq 0$. If S is a Z-matrix with the representation $S = tI - T$ with $t \neq 0$, as above, we may set $U = tI$ and $V = T$, so that $U^\dagger = U^{-1} \geq 0$ and since $V \geq 0$, we also have $U^\dagger V \geq 0$. Thus, a Z-matrix has always a pseudo splitting. Below, a splitting which is both a proper splitting and a pseudo splitting will be called a pseudo proper splitting.

THEOREM 3.4. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *with* A, B, C *and* $D \in \mathbb{R}^{n \times n}$ *such that* $B \leq 0$ *and* $C \leq 0$. *Let* A *and* D *possess pseudo proper splittings. Further, suppose that the inclusions in Theorem 3.3 hold. Then* $M^\dagger \geq 0$ *if and only if* $A^\dagger \geq 0, D^\dagger \geq 0,$ $F^\dagger \geq 0$ *and* $G^\dagger \geq 0$.

Proof. Sufficiency: Let $A^\dagger, D^\dagger, F^\dagger$ and $G^\dagger$ all be nonnegative. Since B and C are nonpositive, it follows that $M^\dagger = \begin{pmatrix} G^\dagger & -A^\dagger B F^\dagger \\ -D^\dagger C G^\dagger & F^\dagger \end{pmatrix}$ is nonnegative.

Necessity: Let $M^\dagger \geq 0$. Then $G^\dagger \geq 0$ and $F^\dagger \geq 0$. Next, we prove that $A^\dagger \geq 0$. Since A has a pseudo proper splitting, there exist matrices U_A and V_A such that $A = U_A - V_A$, $N(A) = N(U_A)$, $R(A) = R(U_A)$, $U_A^\dagger \geq 0$ and $U_A^\dagger V_A \geq 0$. Similarly, there exist matrices U_D and V_D such that $D = U_D - V_D$ with $N(D) = N(U_D)$, $R(D) = R(U_D)$, $U_D^\dagger \geq 0$ and $U_D^\dagger V_D \geq 0$. Set $U_M = \begin{pmatrix} U_A & 0 \\ 0 & U_D \end{pmatrix}$ and $V_M = \begin{pmatrix} V_A & -B \\ -C & V_D \end{pmatrix}$. Then $M = U_M - V_M$. Also, $U_M^\dagger = \begin{pmatrix} U_A^\dagger & 0 \\ 0 & U_D^\dagger \end{pmatrix} \geq 0$ and $U_M^\dagger V_M =$

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$\begin{pmatrix} U_A^\dagger V_A & -U_A^\dagger B \\ -U_D^\dagger C & U_D^\dagger V_D \end{pmatrix} \geq 0$. Next, we claim that $M = U_M - V_M$ is a proper splitting, i.e., $N(M) = N(U_M)$ and $R(M) = R(U_M)$.

Let $x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \in N(U_M)$ so that $U_M x = 0$. So, $U_A x^1 = 0$ and $U_D x^2 = 0$ which in turn give $x^1 \in N(A)$ and $x^2 \in N(D)$, since $N(A) = N(U_A)$ and $N(D) = N(U_D)$. Since $N(D) \subseteq N(B)$ and $N(A) \subseteq N(C)$ we then have $Mx = \begin{pmatrix} Ax^1 + Bx^2 \\ Cx^1 + Dx^2 \end{pmatrix} = 0$. This proves that $x \in N(M)$, and hence, $N(U_M) \subseteq N(M)$. On the other hand, let $x \in N(M)$, so that $M^\dagger M x = 0$. By Remark 3.2, we have $M^\dagger M = \begin{pmatrix} A^\dagger A & 0 \\ 0 & D^\dagger D \end{pmatrix}$, we then have $A^\dagger A x^1 = 0$ and $D^\dagger D x^2 = 0$. Thus, $x^1 \in N(A) = N(U_A)$ and $x^2 \in N(D) = N(U_D)$. Thus, $U_M x = \begin{pmatrix} U_A x^1 \\ U_D x^2 \end{pmatrix} = 0$ showing that $x \in N(U_M)$. Hence, $N(M) \subseteq N(U_M)$. So, $N(M) = N(U_M)$. In order to show that $R(M) = R(U_M)$, we show instead, that $N(M^T) = N(U_M^T)$. Let $y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \in N(U_M^T)$, so that $U_A^T y^1 = 0$ and $U_D^T y^2 = 0$. So, $y^1 \in N(A^T)$ (since $N(A^T) = N(U_A^T)$) and $y^2 \in N(D^T)$ (since $N(D^T) = N(U_D^T)$). Also, $N(A^T) \subseteq N(B^T)$ and $N(D^T) \subseteq N(C^T)$, so that $M^T y = \begin{pmatrix} A^T y^1 + C^T y^2 \\ B^T y^1 + D^T y^2 \end{pmatrix} = 0$. This proves that $y \in N(M^T)$ and so $N(U_M^T) \subseteq N(M^T)$. By the rank-nullity dimension theorem, we conclude that $N(U_M^T) = N(M^T)$ and so we have $R(U_M) = R(M)$. Thus, $M = U_M - V_M$ is a pseudo proper splitting. Since $M^\dagger \geq 0$, by Theorem 2.3, we have $\rho(U_M^\dagger V_M) < 1$. By Theorem 2.1, we then have $\rho(U_A^\dagger V_A) \leq \rho(U_M^\dagger V_M) < 1$. By Theorem 2.3 again, it follows that $A^\dagger \geq 0$. Similarly, $D^\dagger \geq 0$. $\square$

Next, we give an example to illustrate the above result.

EXAMPLE 3.1. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$, $C = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then the inclusions of Theorem 3.3 are satisfied. Set $U_A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $V_A = U_A - A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $U_D = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ and $V_D = U_D - D = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then it is easy to verify that these are pseudo proper splittings for A and

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D, respectively. Observe that $A^\dagger \geq 0$, $D^\dagger \geq 0$, $F^\dagger \geq 0$, $G^\dagger \geq 0$ and $M^\dagger \geq 0$. In fact,

$$M^\dagger = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{8} & \frac{3}{8} \end{pmatrix}.$$

4. The pseudo principal pivot transform. As mentioned in the introduction, the principal pivot transform involving the Moore-Penrose inverse has been studied in the literature. In what follows, we consider it once again, albeit with a different name, the pseudo principal pivot transform. We also find it natural to consider the complementary pseudo principal pivot transform.

DEFINITION 4.1. Let M be defined as in Theorem 3.1. Then the *pseudo principal pivot transform of M relative to A* is defined by

$$H := pppt(M, A)_\dagger = \begin{pmatrix} A^\dagger & -A^\dagger B \\ CA^\dagger & F \end{pmatrix},$$

where $F = D - CA^\dagger B$. The *complementary pseudo principal pivot transform of M relative to D* is defined by

$$J := cpppt(M, D)_\dagger = \begin{pmatrix} G & BD^\dagger \\ -D^\dagger C & D^\dagger \end{pmatrix},$$

where $G = A - BD^\dagger C$.

Both the operations of pseudo principal transforms defined here are involutions. Specifically, we have the following result. It is interesting to observe that in the next results which are of a fundamental nature, the natural conditions provide the frame work (see also Theorem 4.1).

LEMMA 4.1. *Let M be the same as above, and let $H = pppt(M, A)_\dagger$ and $J = cpppt(M, D)_\dagger$.*

(i) Suppose that $R(B) \subseteq R(A)$ and $R(C^T) \subseteq R(A^T)$. Then $pppt\left(H, A^\dagger\right)_\dagger = M$.

(ii) Suppose that $R(C) \subseteq R(D)$ and $R(B^T) \subseteq R(D^T)$. Then $cpppt\left(J, D^\dagger\right)_\dagger = M$.

Proof. We prove (i). The proof for (ii) is similar. Set $W = A^\dagger$, $X = -A^\dagger B$ and $Y = CA^\dagger$. Then $H = \begin{pmatrix} W & X \\ Y & F \end{pmatrix}$. So,

$$pppt(H, A^\dagger)_\dagger = pppt(H, W)_\dagger = \begin{pmatrix} W^\dagger & -W^\dagger X \\ YW^\dagger & F - YW^\dagger X \end{pmatrix}$$

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$$= \begin{pmatrix} A & -A(-A^\dagger B) \\ CA^\dagger A & D - CA^\dagger B + CA^\dagger AA^\dagger B \end{pmatrix}$$

$$= \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$$= M. \quad \square$$

We now prove two extensions of the domain-range exchange property, well known in the nonsingular case.

LEMMA 4.2. *(i) Suppose that $R(B) \subseteq R(A)$ and $R(C^T) \subseteq R(A^T)$. Then M and $H = pppt(M, A)_\dagger$ are related by the formula:*

$$M \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} AA^\dagger y^1 \\ y^2 \end{pmatrix} \text{ if and only if } H \begin{pmatrix} y^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} A^\dagger A x^1 \\ y^2 \end{pmatrix}.$$

(ii) Suppose that $R(C) \subseteq R(D)$, $R(B^T) \subseteq R(D^T)$ and $J = cpppt(M, D)_\dagger$. Then M and J are related by the formula:

$$M \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} y^1 \\ DD^\dagger y^2 \end{pmatrix} \text{ if and only if } J \begin{pmatrix} y^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} x^1 \\ D^\dagger D y^2 \end{pmatrix}.$$

Proof. We prove (i). The proof for (ii) is similar. Suppose that $M \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} AA^\dagger y^1 \\ y^2 \end{pmatrix}$. Then

(4.1) $$Ax^1 + Bx^2 = AA^\dagger y^1$$

and

(4.2) $$Cx^1 + Dx^2 = y^2.$$

Premultipling (4.1) by $A^\dagger$ (and rearranging) we get $A^\dagger y^1 - A^\dagger B x^2 = A^\dagger A x^1$. Premultiplying this equation by C, we then have $CA^\dagger y^1 - CA^\dagger B x^2 = CA^\dagger A x^1 = Cx^1$. So, $CA^\dagger y^1 + Fx^2 = CA^\dagger y^1 + Dx^2 - CA^\dagger B x^2 = Cx^1 + Dx^2 = y^2$. Thus, $H \begin{pmatrix} y^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} A^\dagger y^1 - A^\dagger B x^2 \\ CA^\dagger y^1 + Fx^2 \end{pmatrix} = \begin{pmatrix} A^\dagger A x^1 \\ y^2 \end{pmatrix}.$

Conversely, let $H \begin{pmatrix} y^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} A^\dagger A x^1 \\ y^2 \end{pmatrix}$. Then $A^\dagger y^1 - A^\dagger B x^2 = A^\dagger A x^1$ and $CA^\dagger y^1 + (D - CA^\dagger B)x^2 = y^2$. Premultiplying (4.1) by A, we have $AA^\dagger y^1 - Bx^2 = Ax^1$ so that $Ax^1 + Bx^2 = AA^\dagger y^1$. Again, premultiplying the (4.1) by C, we get $CA^\dagger y^1 - CA^\dagger B x^2 = Cx^1$. Hence, using (4.2) we have, $Cx^1 + Dx^2 = CA^\dagger y^1 - CA^\dagger B x^2 + Dx^2 =$

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y^2, proving that $M \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} AA^\dagger y^1 \\ y^2 \end{pmatrix}$. □

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Let T_1 be the block matrix with the same block sizes as M such that $T_1 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$ and $T_2 = I - T_1$. Let $C_1 = T_2 + T_1 M$ and $C_2 = T_1 + T_2 M$. In Theorem 3.2 [19], it is claimed that $C_1 C_2^\dagger = pppt(M, A)_\dagger$. First, we give a counter example to show that this is not true and then present a correct result with a short proof.

EXAMPLE 4.1. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, C = \begin{pmatrix} -1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 0 \end{pmatrix}$. Then $M = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ and $H = pppt(M, A)_\dagger = \frac{1}{10} \begin{pmatrix} 1 & 2 & -3 \\ -1 & -2 & 3 \\ -2 & -4 & 6 \end{pmatrix}$.

Now, $C_1 = \begin{pmatrix} I & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $C_2^\dagger = \begin{pmatrix} 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & -\frac{1}{6} & \frac{5}{6} \end{pmatrix}$. Here $C_1 C_2^\dagger = \begin{pmatrix} 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & -\frac{1}{4} & \frac{1}{4} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \neq H$.

LEMMA 4.3. Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, such that $R(B) \subseteq R(A)$. Let T_1, T_2, C_1 and C_2 be as defined earlier. Then $pppt(M, A)_\dagger = C_1 C_2^\dagger$.

Proof. We have $C_1 = \begin{pmatrix} I & 0 \\ C & D \end{pmatrix}, C_2 = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$ and $C_2^\dagger = \begin{pmatrix} A^\dagger & -A^\dagger B \\ 0 & I \end{pmatrix}$ so that $C_1 C_2^\dagger = \begin{pmatrix} A^\dagger & -A^\dagger B \\ CA^\dagger & D - CA^\dagger B \end{pmatrix} = pppt(M, A)_\dagger$. □

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $R(B) \subseteq R(A)$ and $R(C^T) \subseteq R(A^T)$. In Theorem 3.3, [19] the formula $H^\dagger = J$ is claimed to hold. First, in the following example, we show that this is not true.

EXAMPLE 4.2. Let $A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, C = \begin{pmatrix} -1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 0 \end{pmatrix}$. Then $M = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ -1 & 1 & 0 \end{pmatrix}$, $R(B) \subseteq R(A)$, $R(C^T) \subseteq R(A^T)$, $H =$

$$pppt(M, A)_\dagger = \frac{1}{10} \begin{pmatrix} 1 & 2 & -5 \\ -1 & -2 & 5 \\ -2 & -4 & 10 \end{pmatrix}, \ J = cpppt(M, D)_\dagger = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and}$$

$$H^\dagger = \frac{1}{18} \begin{pmatrix} 1 & -1 & -2 \\ 2 & -2 & -4 \\ -5 & 5 & 10 \end{pmatrix} \neq J.$$

Now, we prove a correct version. Once again, the natural conditions are handy.

THEOREM 4.1. *Let $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times p}, C \in \mathbb{R}^{s \times n}, D \in \mathbb{R}^{s \times p}$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Suppose that $R(B) \subseteq R(A)$, $R(C^T) \subseteq R(A^T)$, $R(C) \subseteq R(D)$ and $R(B^T) \subseteq R(D^T)$. Then $H^\dagger = J$, where $H = pppt(M, A)_\dagger$ and $J = cpppt(M, D)_\dagger$.*

Proof. In Lemma 4.3, it is shown that $pppt(M, A)_\dagger = C_1 C_2^\dagger$. Also, $pppt(M, D)_\dagger = \begin{pmatrix} A - BD^\dagger C & BD^\dagger \\ -D^\dagger C & D^\dagger \end{pmatrix} = C_2 C_1^\dagger$. So, it is enough to prove that $(C_1 C_2^\dagger)^\dagger = C_2 C_1^\dagger$. Now,

$$C_1^T C_1 C_2^\dagger C_2 = \begin{pmatrix} A^\dagger A + C^T C & C^T D \\ D^T C & D^T D \end{pmatrix}$$

and

$$C_2^\dagger (C_2^\dagger)^T C_1^\dagger C_1 = \begin{pmatrix} A^\dagger (A^\dagger)^T + A^\dagger B B^T (A^\dagger)^T & -A^\dagger B \\ -B^T (A^\dagger)^T & D^\dagger D \end{pmatrix}.$$

Observe that $C_1^T C_1 C_2^\dagger C_2$ and $C_2^\dagger (C_2^\dagger)^T C_1^\dagger C_1$ both are symmetric. From Theorem 2.4, it follows that $(C_1 C_2^\dagger)^\dagger = C_2 C_1^\dagger$. $\square$

The next result is a generalization of Proposition 3.3, [23].

THEOREM 4.2. *Let $M = (A|B)$ be a partitioned matrix with $A \in \mathbb{R}^{n \times r}$ and $B \in \mathbb{R}^{n \times (n-r)}$. Suppose that $R(B) \subseteq R(A)$. Then $pppt(M^T M, A^T A)$ is a $\{1\}$-inverse of $M^T M$.*

Proof. Let $M^T M = \begin{pmatrix} A^T A & A^T B \\ B^T A & B^T B \end{pmatrix}$. First, we observe that $AA^\dagger B = B$ (since $R(B) \subseteq R(A)$). Set $P = \begin{pmatrix} I & 0 \\ B^T (A^T)^\dagger & I \end{pmatrix}$, $Q = \begin{pmatrix} I & A^\dagger B \\ 0 & I \end{pmatrix}$ and $N = \begin{pmatrix} A^T A & 0 \\ 0 & 0 \end{pmatrix}$. Then the expression for $M^T M$ can be written as

$$M^T M = \begin{pmatrix} I & 0 \\ B^T (A^T)^\dagger & I \end{pmatrix} \begin{pmatrix} A^T A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & A^\dagger B \\ 0 & I \end{pmatrix} = PNQ.$$

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Let $N^{(1)}$ denote a $\{1\}$-inverse of N. Then a $\{1\}$-inverse of $M^T M$ is given by

$$
\begin{aligned}
Q^{-1} N^{(1)} P^{-1} &= \begin{pmatrix} I & -A^\dagger B \\ 0 & I \end{pmatrix} \begin{pmatrix} (A^T A)^\dagger & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -B^T (A^T)^\dagger & I \end{pmatrix} \\
&= \begin{pmatrix} (A^T A)^\dagger & -A^\dagger B \\ (A^\dagger B)^T & 0 \end{pmatrix} \\
&= pppt(M^T M, A^T A). \quad \square
\end{aligned}
$$

5. Inheritance properties of matrix classes. In this section, we consider inheritance properties of the pseudo principal pivot transform with regard to two matrix classes. These classes are generalizations of the corresponding classes of matrices which are relevant in the context of the linear complementarity problem. First, we consider $P_\dagger$-matrices, recently studied in [18]. Let us add that these matrices have not been fully understood. In particular their relationship with solutions of linear complementarity problems has not been explored. Nevertheless, certain interesting generalizations of connections between $P_\dagger$-matrices and interval matrices have been derived in [18]. To begin with, let us first recall the definition of a P-matrix. A square matrix A is called a P-matrix if all its principal minors are positive [14]. P-matrices have been studied widely in the literature due to many of their interesting properties and important applications. We refer the reader to [14] for many of these properties and [12] for their applications to the linear complementarity problem. Given $Q \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, the linear complementarity problem denoted by $LCP(Q, q)$ is to determine if there exists $x \in \mathbb{R}^n$ such that $x \geq 0$, $y = Qx + q \geq 0$ and $\langle x, y \rangle = x^T y = 0$. A well known result [12] states that Q is a P-matrix iff the problem above has a unique solution for every $q \in \mathbb{R}^n$. This property is referred to as the globally uniquely solvable property.

Let us now recall the definition of a $P_\dagger$-matrix.

DEFINITION 5.1. (Definition 2.1, [18]) A square matrix A is said to be a $P_\dagger$-*matrix* if for each non zero $x \in R(A^T)$ there is an $i \in \{1, 2, \ldots, n\}$ such that $x_i(Ax)_i > 0$. Equivalently, for any $x \in R(A^T)$ the inequalities $x_i(Ax)_i \leq 0$ for $i = 1, 2, \ldots, n$ imply that $x = 0$.

Let us just include a fundamental property of a $P_\dagger$-matrix, similar to that of a P-matrix.

THEOREM 5.1. (Theorem 2.3, [18]) *A is a $P_\dagger$-matrix if and only if $A^\dagger$ is a $P_\dagger$-matrix.*

In [18], the relationships of $P_\dagger$-matrices with certain subsets of intervals of matrices were studied. Let $A, B \in \mathbb{R}^{m \times n}$ with $A \leq B$. Set $J(A, B) := \{C \in \mathbb{R}^{m \times n} : A \leq C \leq B\}$. Let $r(A, B)$ denote the set of all matrices whose rows are convex linear

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combinations of the corresponding rows of A and B. Similarly, let $c(A, B)$ denote the set of all matrices whose columns are convex linear combinations of the corresponding columns of A and B. Then $r(A, B)$ and $c(A, B)$ are subsets of $J(A, B)$. Now, suppose that A, B are such that $R(A) = R(B)$ and $N(A) = N(B)$. Finally, let $K(A, B)$ denote the set

$$K(A, B) := \{C \in J(A, B) : R(C) = R(A) \text{ and } N(C) = N(A)\}.$$

Next, we state some results which characterize the inclusions $r(A, B) \subseteq K(A, B)$ and $c(A, B) \subseteq K(A, B)$ in terms of certain $P_\dagger$-matrices.

THEOREM 5.2. (Theorem 3.2, [18]) *Let A, $B \in \mathbb{R}^{n \times n}$ be such that $R(A) = R(B)$ and $N(A) = N(B)$. Then $r(A, B) \subseteq K(A, B)$ if and only if $BA^\dagger$ is a $P_\dagger$-matrix.*

THEOREM 5.3. (Theorem 3.3, [18]) *Let $A, B \in \mathbb{R}^{n \times n}$ be such that $R(A) = R(B)$ and $N(A) = N(B)$. Then $c(A, B) \subseteq K(A, B)$ if and only if $B^\dagger A$ is a $P_\dagger$-matrix.*

It is quite well known that if M is a P-matrix, then the principal pivot transform is also a P-matrix (Theorem 5.2, [23]). More generally, for symmetric cones in Euclidean Jordan algebras, it has been shown that M is a P-matrix if and only if the (principal subtransformations) A and D and the principal pivot transform are P-matrices (Theorem 1 and Theorem 3, [22]). In what follows, we prove extensions of these results to the case of $P_\dagger$-matrices.

First, we show that if M is a $P_\dagger$-matrix it does not necessarily follow that $pppt(M, A)_\dagger$ is a $P_\dagger$-matrix.

EXAMPLE 5.1. Let $M = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ -1 & 1 & -0.5 \end{pmatrix}$, with $A = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $C = \begin{pmatrix} -1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -0.5 \end{pmatrix}$. Let $x \in R(M^T)$. Then $x = \alpha(2, -2, 1)^T$, $\alpha \in \mathbb{R}$. Suppose that $x_i(Mx)_i \leq 0$ for $i = 1, 2, 3$. Then $18\alpha^2 \leq 0$ so that $x = 0$. Thus, M is a $P_\dagger$-matrix. It can be verified that

$$H = pppt(M, A)_\dagger = \begin{pmatrix} 0.125 & 0.125 & -0.25 \\ -0.125 & -0.125 & 0.25 \\ -0.25 & -0.25 & 0 \end{pmatrix}.$$

Let $x^0 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$. Then $0 \neq x^0 \in R(H^T)$ and $x_i^0(Hx^0)_i \leq 0$ for $i = 1, 2, 3$. This

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shows that H is not a $P_\dagger$-matrix.

THEOREM 5.4. *For* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$*, suppose that the four inclusions of Theorem 3.1 hold. If M is a $P_\dagger$-matrix, then $pppt(M, A)_\dagger$, A and D are $P_\dagger$-matrices.*

Proof. First, we show that $H = pppt(M, A)_\dagger$ is a $P_\dagger$-matrix. Let $z = \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} \in R(H^T)$. There exists $v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ such that $H^T v = z$. Then $z^1 = (A^\dagger)^T v^1 + (CA^\dagger)^T v^2$ and $z^2 = (-A^\dagger B)^T v^1 + F^T v^2$. Thus, $z^1 \in R(A)$ and so $z^1 = AA^\dagger z^1$. Also, since $R((A^\dagger B)^T) \subseteq R(F^T)$, we have $z^2 \in R(F^T)$ so that $z^2 = F^\dagger F z^2$. Set $w = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = Hz$. Then $w^1 = A^\dagger z^1 - A^\dagger B z^2$ and $w^2 = CA^\dagger z^1 + F z^2$. Observe that $w^1 \in R(A^\dagger)$ and so $w^1 = A^\dagger A w^1$. So, $H \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = \begin{pmatrix} A^\dagger A w^1 \\ w^2 \end{pmatrix}$. By (i) of Lemma 4.2, we have $M \begin{pmatrix} w^1 \\ z^2 \end{pmatrix} = \begin{pmatrix} AA^\dagger z^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} z^1 \\ w^2 \end{pmatrix}$. Set $x = \begin{pmatrix} w^1 \\ z^2 \end{pmatrix}$. Now, let us suppose that $z_i(Hz)_i \leq 0$ for all i. Then $z_i^1 w_i^1 \leq 0$ for all i and $z_j^2 w_j^2 \leq 0$ for all j. It then follows that $x_i(Mx)_i \leq 0$ for all i. Further by Theorem 3.1, $M^\dagger M x = \begin{pmatrix} A^\dagger A & 0 \\ 0 & F^\dagger F \end{pmatrix} x = x$, proving that $x \in R(M^T)$. Since M is a $P_\dagger$-matrix, we then have $x = 0$. That is, $w^1 = 0$ and $z^2 = 0$. So $z^1 = 0$, and hence, $z = 0$. This proves that H is a $P_\dagger$-matrix.

Next, we show that A is a $P_\dagger$-matrix. Let $x^1 \in R(A^T)$ (so that $x^1 = A^\dagger A x^1$) and suppose that $(x^1)_i(Ax^1)_i \leq 0$ for all i. Define $x = \begin{pmatrix} x^1 \\ 0 \end{pmatrix}$. Then $M^\dagger M x = x$ so that $x \in R(M^T)$. Also, $x_j(Mx)_j = \begin{pmatrix} x_j^1(Ax^1)_j \\ 0 \end{pmatrix} \leq 0$ for all j. Since M is a $P_\dagger$-matrix we then have $x = 0$ so that $x^1 = 0$. Hence, A is a $P_\dagger$-matrix.

Finally, we show that D is a $P_\dagger$-matrix. Let $x^2 \in R(D^T)$ so that $x^2 = D^\dagger D x^2$. Then $x^2 = F^\dagger F x^2$ (since $F = D - CA^\dagger B$, and hence, $R(D^T) \subseteq R(F^T)$). Suppose that $(x^2)_i(Dx^1)_i \leq 0$ for all i. Define $x = \begin{pmatrix} 0 \\ x^2 \end{pmatrix}$. Then $M^\dagger M x = x$, so that $x \in R(M^T)$. Also, $x_j(Mx)_j = \begin{pmatrix} 0 \\ x_j^2(Dx^2)_j \end{pmatrix} \leq 0$ for all j. Since M is a $P_\dagger$-matrix we then have $x = 0$ so that $x^2 = 0$. Hence, D is a $P_\dagger$-matrix. $\square$

Again, it may be shown that if $H = pppt(M, A)_\dagger$ is a $P_\dagger$-matrix, then M need not be a $P_\dagger$-matrix. We present sufficient conditions which guarantee that such an

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implication holds.

THEOREM 5.5. *For $H = pppt(M, A)_\dagger$, suppose that $R(B) \subseteq R(A)$, $R(C^T) \subseteq R(A^T), R(C) \subseteq R(D)$ and $R(B^T) \subseteq R(D^T)$. If H is a $P_\dagger$-matrix, then M is a $P_\dagger$-matrix.*

Proof. Set $W = A^\dagger$, $X = -A^\dagger B$ and $Y = CA^\dagger$. Then $H = \begin{pmatrix} W & X \\ Y & F \end{pmatrix}$. Since $R(B) \subseteq R(A)$ and $R(C^T) \subseteq R(D^T)$, by (i) of Lemma 4.1, we have $pppt(H, A^\dagger)_\dagger = M$. Now $R(X) = R(-A^\dagger B) \subseteq R(A^\dagger) = R(W)$, $R(Y^T) = R((CA^\dagger)^T) \subseteq R((A^\dagger)^T) = R(W^T)$. Note that the pseudo Schur complement of H in W is $F - YW^\dagger X = F + CA^\dagger AA^\dagger B = D$. We also have $R(YW^\dagger) = R(CA^\dagger A) = R(C) \subseteq R(D)$ and $R((W^\dagger X)^T) = R((AA^\dagger B)^T) = R(B^T) \subseteq R(D^T)$. So, the assumptions of Theorem 5.4 for the matrix H are satisfied. We conclude that M is a $P_\dagger$-matrix. □

A similar result also holds for the complementary pseudo principal pivot transform. We simply state it and skip its proof.

THEOREM 5.6. *For $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, suppose that the four inclusions of Theorem 3.2 hold. If $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a $P_\dagger$-matrix, then $cpppt(M, D)_\dagger$, D and A are $P_\dagger$-matrices.*

Next, we consider a converse of the result above. Its proof is similar, and hence it is omitted.

THEOREM 5.7. *For $J = cpppt(M, D)_\dagger$, suppose that $R(B) \subseteq R(A)$, $R(C^T) \subseteq R(A^T)$, $R(C) \subseteq R(D)$ and $R(B^T) \subseteq R(D^T)$. If J is a $P_\dagger$-matrix, then M is a $P_\dagger$-matrix.*

In the last part of this section, we consider another class of matrices which remains invariant under the pseudo principal pivot transform. First, we recall the concept of R_0-matrices.

DEFINITION 5.2. *$M \in \mathbb{R}^{n \times n}$ is said to be an R_0-matrix if $LCP(M, 0)$ has zero as the only solution.*

The importance of R_0-matrices is due to the following result. The solution set of $LCP(M, q)$ is the set of all $x \in \mathbb{R}^n$ such that $x \geq 0$ and $Mx + q \geq 0$.

THEOREM 5.8. *(Part of Proposition 3.9.23, [12]) Let $M \in \mathbb{R}^{n \times n}$. Then M is an R_0-matrix if and only if for every $q \in \mathbb{R}^n$, the solution set of $LCP(M, q)$ is bounded.*

Recently, an extension of R_0-matrices was studied, typically for singular matrices.

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We recall its definition.

DEFINITION 5.3. (Definition 3.4, [21]) Let $M \in \mathbb{R}^{n \times n}$ and S be a subspace of $\mathbb{R}^n$. Then M is called an R_0-*matrix relative to* S if the only solution for $LCP(M,0)$ in S is the zero solution. In other words, M is an R_0-matrix relative to S if $x = 0$ is the only vector $x \in S$ such that $x \geq 0$, $y = Mx \geq 0$ and $x^T y = 0$. In particular, $M \in \mathbb{R}^{n \times n}$ is called an $R_\dagger$-*matrix* if M is an R_0-matrix relative to $R(M^T)$.

Recall that $M \in \mathbb{R}^{n \times n}$ is *copositive* if $\langle x, Mx \rangle \geq 0$ for all $x \geq 0$ and *strictly copositive* if $\langle x, Mx \rangle > 0$ for all $x \geq 0$, $x \neq 0$. The following result presents sufficient conditions under which a matrix is an $R_\dagger$-matrix.

THEOREM 5.9. (Theorem 3.7, [21]) *For* $M \in \mathbb{R}^{n \times n}$, *let* $M = U - V$ *be a pseudo proper splitting with* $U^\dagger$ *strictly copositive. If* $\rho(U^\dagger V) < 1$, *then* M *is an* $R_\dagger$-*matrix.*

Now, we present inheritance results on $R_\dagger$-matrices.

THEOREM 5.10. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where* A, B, C *and* D *are square matrices of the same order satisfying the conditions of Theorem 3.1. We have the following:*

(a) If M *is an* $R_\dagger$-*matrix, then* $pppt(M, A)_\dagger$ *is also an* $R_\dagger$-*matrix.*

(b) If M *is an* $R_\dagger$-*matrix and* $CA^\dagger \geq 0$, *then* A *is an* $R_\dagger$-*matrix.*

Proof. (a) As before, we denote $pppt(M, A)_\dagger$ by H. Let $z \in R(H^T)$ such that $z \geq 0$, $\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = v = Hz \geq 0$ and $\langle z, v \rangle = 0$. We then have $v^1 = A^\dagger z^1 - A^\dagger B z^2$ and $v^2 = CA^\dagger z^1 + F z^2$. Thus, $v^1 \in R(A^\dagger)$ so that $v^1 = A^\dagger A v^1$. Since $z \in R(H^T)$, we have $z^1 = (A^\dagger)^T u^1 + (CA^\dagger)^T u^2$ and $z^2 = (-A^\dagger B)^T u^1 + F^T u^2$ for some u^1 and u^2. Note that $z^1 \in R((A^\dagger)^T) = R(A)$ and so, $z^1 = AA^\dagger z^1$. Also, since $R((A^\dagger B)^T) \subseteq R(F^T)$, we obtain that $z^2 \in R(F^T)$ and so $z^2 = F^\dagger F z^2$.

Now $H \begin{pmatrix} z^1 \\ z^2 \end{pmatrix} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} A^\dagger A v^1 \\ v^2 \end{pmatrix}$. By (i) of Lemma 4.2, we have

$$M \begin{pmatrix} v^1 \\ z^2 \end{pmatrix} = \begin{pmatrix} AA^\dagger z^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} z^1 \\ v^2 \end{pmatrix}.$$

Set $x = \begin{pmatrix} v^1 \\ z^2 \end{pmatrix}$ and $y = \begin{pmatrix} z^1 \\ v^2 \end{pmatrix}$. Clearly, $x \geq 0$ and $Mx = y \geq 0$. Also $\langle x, y \rangle = \langle z, v \rangle = 0$. Next, we show that $x \in R(M^T)$. Since, $v^1 = A^\dagger A v^1$ and $z^2 = F^\dagger F z^2$, we have $M^\dagger M x = \begin{pmatrix} A^\dagger A & 0 \\ 0 & F^\dagger F \end{pmatrix} x = x$, proving that $x \in R(M^T)$. Since M is an $R_\dagger$-matrix, we conclude that $x = 0$. That is, $v^1 = 0$ and $z^2 = 0$. In turn, we then have $z^1 = 0$ and hence $z = 0$, proving that H is an $R_\dagger$-matrix.

(b) Let $x \in R(A^T)$ be such that $x \geq 0$, $y = Ax \geq 0$ and $\langle x, y \rangle = 0$. Define $z = \begin{pmatrix} x \\ 0 \end{pmatrix} \geq 0$. Then $M^\dagger Mz = z$, since $A^\dagger Ax = x$. Since $C^\dagger A \geq 0$ and $Ax \geq 0$, we have $Cx = (CA^\dagger)(Ax) \geq 0$. Set $v = Mz = \begin{pmatrix} Ax \\ Cx \end{pmatrix}$. Then $v \geq 0$ and $\langle z, v \rangle = \langle x, y \rangle = 0$. Since M is an $R_\dagger$ matrix, we have $z = 0$ and hence $x = 0$. This proves that A is an $R_\dagger$-matrix. $\square$

Again, an analogous result holds for the complementary pseudo principal pivot transform. The proof is similar to the proof of the theorem above and hence, it is omitted.

THEOREM 5.11. *Let* $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, *where A, B, C and D are square matrices of the same order satisfying the conditions of Theorem 3.2. Then the following hold:*

(a) *If M is an $R_\dagger$-matrix then so is the matrix* $cpppt(M, D)_\dagger$.

(b) *If M is an $R_\dagger$-matrix and $B^\dagger D \geq 0$ then so is D.*

6. Eigenvalue and eigenvector relationships. In this concluding section, we discuss relationships between the spectra and the left eigenspaces of the pseudo principal pivot transform $pppt(M, A)_\dagger$ and $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Almost the entire theory corresponding to the invertible case, as discussed in [20], is recovered for the singular case in a suitable frame work. In the rest of the section, we suppose that A is range-symmetric and that the inclusions in Theorem 3.1 hold. Let T_1 be the block matrix with the same block sizes as M such that $T_1 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$ and $T_2 = I - T_1$. Let

$$(6.1) \qquad C_1 = T_2 + T_1 M \quad and \quad C_2 = T_1 + T_2 M.$$

Suppose that $R(B) \subseteq R(A)$ then, by Lemma 4.3 $C_1 C_2^\dagger = pppt(M, A)_\dagger$. We investigate the question of when M and $H = pppt(M, A)_\dagger$ share a left eigenvector. In other words, we ask when there exists a nonzero $x \in \mathbb{R}^n$ such that

$$(6.2) \qquad x^T M = \lambda x^T \quad and \quad x^T H = \mu x^T$$

for $\lambda, \mu \in \mathbb{R}$. First, we consider the case $\lambda = \mu = -1$. The following result may be compared with observation 5.1 in [20].

THEOREM 6.1. *Let M be defined as above such that A is a range-symmetric. Then $-1 \in \sigma(M)$ if and only if $-1 \in \sigma(H)$. Also, the corresponding left eigenvectors are common to M and H.*

Proof. We have, $x^T M = -x^T \Leftrightarrow x^T(C_1 + C_2 - I) = -x^T \Leftrightarrow x^T(C_1 + C_2) = 0 \Leftrightarrow x^T C_1 = -x^T C_2$. Post multiplying by $C_2^\dagger$, we then have $x^T H = -x^T C_2 C_2^\dagger$. Since A is range-symmetric, we then have $AA^\dagger = A^\dagger A$ which further implies that $C_2 C_2^\dagger = C_2^\dagger C_2$. Also, using the facts that $MC_2^\dagger C_2 = M$ and $x^T M = -x^T$, we conclude that $-x^T C_2 C_2^\dagger = -x^T C_2^\dagger C_2 = -x^T$.

Conversely, let $x^T H = -x^T$. Then $x^T C_1 C_2^\dagger = -x^T$ so that $x^T C_1 C_2^\dagger C_2 = -x^T C_2$. Now, $C_1 C_2^\dagger C_2 = \begin{pmatrix} AA^\dagger & 0 \\ C & D \end{pmatrix}$ and so we have, $(C_1 C_2^\dagger C_2)^T x = \begin{pmatrix} AA^\dagger u & C^T v \\ 0 & D^T v \end{pmatrix} = \begin{pmatrix} u & C^T v \\ 0 & D^T v \end{pmatrix} = C_1^T x$, where $x = \begin{pmatrix} u \\ v \end{pmatrix}$. Thus, $x^T C_1 C_2^\dagger C_2 = x^T C_1$. So, we have $x^T C_1 = -x^T C_2$ which implies that $x^T M = -x^T$. Thus, $-1 \in \sigma(H)$ implies that $-1 \in \sigma(M)$. $\square$

The next result is an analogue of Theorem 5.2, [20].

THEOREM 6.2. *Let M be defined as above. Suppose that $x^T M = \lambda x^T$ and $x^T H = \mu x^T$ for some nonzero $x \in \mathbb{R}^n$. Then both the following statements hold:*

(a) *If $\lambda \neq \mu$, then $\frac{1+\lambda}{1+\mu} \in \sigma(A)$.*

(b) *If $\lambda \neq \frac{1}{\mu}$, then $\mu \frac{1+\lambda}{1+\mu} \in \sigma(D)$.*

Proof. (a) We have $x^T(C_1 + C_2 - I) = \lambda x^T$ and $x^T C_1 C_2^\dagger = \mu x^T$. Postmultiplying the second equation by C_2 we have $x^T C_1 C_2^\dagger C_2 = \mu x^T C_2$. As before, since $x^T C_1 C_2^\dagger C_2 = x^T C_1$, we have $x^T C_1 = \mu x^T C_2$. Substituting this in the first equation, we get $x^T[(1+\mu)C_2 - (1+\lambda)I] = 0$. Thus, every common left eigenvector of M and H is a left eigenvector of $X := (1+\mu)C_2 - (1+\lambda)I$, corresponding to the eigenvalue zero. Then X is a singular matrix. Similarly, it follows that every common left eigenvector of M and H is a left eigenvector of $Y := (1+\mu)C_1 - \mu(1+\lambda)I$, corresponding to the eigenvalue zero. Rewriting X in the block form, we have

$$X = \begin{pmatrix} (1+\mu)A - (1+\lambda)I & (1+\mu)B \\ 0 & (\mu - \lambda)I \end{pmatrix}.$$

Let $\lambda \neq \mu$. If $\mu = -1$, then $\lambda \neq -1$ and so the diagonal blocks of X are non-singular so that X would be non-singular, a contradiction. Hence $\mu + 1 \neq 0$. Let $x^T X = 0$, with $x = \begin{pmatrix} u \\ v \end{pmatrix}$. Then $(1+\mu)A^T u = (1+\lambda)u$ and $(1+\mu)B^T u + (\mu - \lambda)v = 0$. If $u = 0$, then $v \neq 0$ so that $\lambda = \mu$, a contradiction. Hence, $u \neq 0$. Thus, $\frac{1+\lambda}{1+\mu} \in \sigma(A)$, proving (a).

(b) As in part (a), the block matrix

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$$Y = \begin{pmatrix} (1-\mu\lambda)I & 0 \\ (1+\mu)C & (1+\mu)D - \mu(1+\lambda)I \end{pmatrix}$$

is a singular matrix. Let $1 - \mu\lambda \neq 0$. If $\mu = -1$ then $\lambda \neq -1$ and so the diagonal blocks of Y are non-singular so that Y would be non-singular, a contradiction. Hence, $1 + \mu \neq 0$. Let $x = \begin{pmatrix} u \\ v \end{pmatrix}$ such that $x^T Y = 0$. Then $(1+\mu)D^T v = \mu(1+\lambda)v$ and $(1-\mu\lambda)u + (1+\mu)C^T v = 0$. If $v = 0$, then $u \neq 0$ so that $1 - \mu\lambda = 0$, a contradiction. Hence, $v \neq 0$. Thus, $\mu\frac{1+\lambda}{1+\mu} \in \sigma(D)$, proving (b). $\square$

The following result extends Theorem 5.3 (a), [20].

THEOREM 6.3. *Let M be partitioned as above. Let M and H have a common eigenvector corresponding to the eigenvalues $\lambda \in \sigma(M)$ and $\mu \in \sigma(H)$, respectively. Let $\lambda = \mu \ (\neq -1)$. Then there exist vectors u and v with at least one of them being nonzero such that*

(6.3) $\qquad A^T u = u, \ \ B^T u = 0, \ \ C^T v + (1-\lambda)u = 0 \ \ and \ \ D^T v = \lambda v.$

In this case, either $\lambda = 1$ and $1 \in \sigma(A)$, or $\lambda \in \sigma(D)$.

Conversely, if there exist vectors u and v, not both zero, that satisfy the equation (6.3) then M and H have a common eigenvector corresponding to the eigenvalue $\lambda \in \sigma(M) \cap \sigma(H)$.

Proof. As argued earlier, if x is a common eigenvector for M and H corresponding to the eigenvalues λ and μ, respectively, then x satisfies the equations $X^T x = 0$ and $Y^T x = 0$. Here

$$X = (1+\lambda)\begin{pmatrix} A-I & B \\ 0 & 0 \end{pmatrix} \ \ \text{and} \ \ Y = (1+\lambda)\begin{pmatrix} (1-\lambda)I & 0 \\ C & D - \lambda I \end{pmatrix}.$$

Set $x = \begin{pmatrix} u \\ v \end{pmatrix}$. Then at least one of u, v is nonzero. It may be verified that u and v satisfy the equations as in the proof of Theorem 6.2. If $v = 0$, then $u \neq 0$ and so the equation $A^T u = u$ implies that $1 \in \sigma(A)$. The equation $C^T v + (1-\lambda)u = 0$ yields $\lambda = \mu = 1$. On the other hand, if $u = 0$, then $v \neq 0$ and so the equation $D^T v = \lambda v$ implies that $\lambda \in \sigma(D)$.

Conversely, suppose that u and v, not both zero, satisfy the four equations as above. With $0 \neq x = \begin{pmatrix} u \\ v \end{pmatrix}$ we then have

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$$C_2^T x = \left(\begin{array}{c} A^T u \\ B^T u + v \end{array} \right) = \left(\begin{array}{c} u \\ v \end{array} \right) = x.$$

We also have

$$C_1^T x = \left(\begin{array}{c} u + C^T v \\ D^T v \end{array} \right) = \lambda \left(\begin{array}{c} u \\ v \end{array} \right) = \lambda x.$$

Thus,

$$M^T x = (C_1^T + C_2^T - I)x = \lambda x.$$

Further,

$$H^T x = (C_1 C_2^\dagger)^T x = (C_2^\dagger)^T C_1^T x = \lambda (C_2^\dagger)^T x.$$

Since $x = C_2^T x$ and $C_2 C_2^\dagger = \left(\begin{array}{cc} AA^\dagger & 0 \\ 0 & I \end{array} \right)$, we have

$$H^T x = \lambda (C_2^\dagger)^T C_2^T x = \lambda (C_2 C_2^\dagger)^T x = \lambda C_2 C_2^\dagger x = \lambda \left(\begin{array}{c} AA^\dagger u \\ v \end{array} \right).$$

Since A is range-symmetric and $u \in R(A^T)$, we have $AA^\dagger u = A^\dagger A u = u$. Thus, $H^T x = \lambda x$, completing the proof of the converse part. $\square$

The case $\lambda = \frac{1}{\mu}$ is considered next.

THEOREM 6.4. *Let M and H have a common eigenvector corresponding to the eigenvalues $\lambda \in \sigma(M)$ and $\mu \in \sigma(H)$, respectively. Let $\lambda = \frac{1}{\mu} (\neq 1, -1)$. Then there exist vectors u and v with at least one of them being nonzero such that*

(6.4) $A^T u = \lambda u, \ B^T u + (1 - \lambda)v = 0, \ C^T v = 0 \ and \ D^T v = v.$

In this case, $\lambda \in \sigma(A)$.

Conversely, if there exist vectors u and v, not both zero, that satisfy the equations (6.4) then M and H have a common eigenvector corresponding to the eigenvalues $\lambda \in \sigma(M)$ and $\mu \in \sigma(H)$, respectively with $\lambda = \frac{1}{\mu}$.

Proof. Here $X = \frac{1+\lambda}{\lambda} \left(\begin{array}{cc} A - \lambda I & B \\ 0 & (1-\lambda)I \end{array} \right)$ and $Y = \frac{1+\lambda}{\lambda} \left(\begin{array}{cc} 0 & 0 \\ C & D - I \end{array} \right)$. Let x be a common eigenvector, with $x = \left(\begin{array}{c} u \\ v \end{array} \right)$. Then at least one of u, v is nonzero. It may now be verified that the equations as above, hold.

If $u = 0$, then $v \neq 0$. However, this contradicts the equation $B^T u + (1 - \lambda)v = 0$,

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as $\lambda \neq 1$. So, $u \neq 0$, showing that $\lambda \in \sigma(A)$.

For the converse part, we have

$$C_2^T x = \begin{pmatrix} A^T u \\ B^T u + v \end{pmatrix} = \begin{pmatrix} \lambda u \\ \lambda v \end{pmatrix} = \lambda x.$$

We also have

$$C_1^T x = \begin{pmatrix} u + C^T v \\ D^T v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} = x.$$

A calculation similar to the previous result shows that $M^T x = \lambda x$ and $H^T x = \frac{1}{\lambda} x$, completing the proof. □

The results above are summarized in the following result. This could be considered as an analogue of Corollary 5.4, [20].

COROLLARY 6.1. *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and H denote the pseudo principal pivot transform of M. Let A be range-symmetric and assume that the inclusion relations of Theorem 3.1 are satisfied, i.e., $R(C^T) \subseteq R(A^T)$, $R(B) \subseteq R(A)$, $R(C) \subseteq R(F)$ and $R(B^T) \subseteq R(F^T)$. Suppose that M and H have a common eigenvector corresponding to the eigenvalues $\lambda \in \sigma(M)$ and $\mu \in \sigma(H)$, respectively with neither being equal to -1. Then there exist numbers α, β such that $\alpha + \beta - 1 = \lambda$ and $\mu = \frac{\beta}{\alpha}$ with $\alpha \in \sigma(A)$ or $\beta \in \sigma(D)$.*

Proof. Case (i): Let $\lambda = \mu$. Set $\alpha = 1$ and $\beta = \lambda$. If $\lambda \neq 1$, by Theorem 6.3, we have $\beta \in \sigma(D)$. On the other hand, if $\beta \notin \sigma(D)$, again by Theorem 6.3, we have $\lambda = 1$ and $\alpha \in \sigma(A)$. Also, $\alpha + \beta - 1 = \lambda$ and $\frac{\beta}{\alpha} = \lambda = \mu$.

Case (ii): $\lambda \neq \mu$. Set $\alpha = \frac{1+\lambda}{1+\mu}$ and $\beta = \mu \frac{1+\lambda}{1+\mu}$. Then, by (a) of Theorem 6.2, we have $\alpha \in \sigma(A)$. Also, $\alpha + \beta - 1 = \lambda$ and $\frac{\beta}{\alpha} = \mu$.

Case (iii): $\lambda = \frac{1}{\mu}$. Define $\alpha = \lambda$ and $\beta = 1$. Then, by Theorem 6.4, we have $\alpha \in \sigma(A)$. Also, $\alpha + \beta - 1 = \lambda$ and $\frac{\beta}{\alpha} = \frac{1}{\lambda} = \mu$.

Case (iv): $\lambda \neq \frac{1}{\mu}$. Set $\beta = \mu \frac{1+\lambda}{1+\mu}$ and $\alpha = \frac{1+\lambda}{1+\mu}$. Then by (b) of Theorem 6.2, we have $\beta \in \sigma(D)$. Further, $\alpha + \beta - 1 = 1 + \lambda - 1 = \lambda$ and $\frac{\beta}{\alpha} = \mu$.

This completes the proof. □

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