

SIGNLESS LAPLACIAN SPECTRAL CHARACTERIZATION OF SOME JOINS*

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Abstract. The *join* of two disjoint graphs G and H , denoted by $G \vee H$, is the graph obtained by joining each vertex of G to each vertex of H . In this paper, the signless Laplacian characteristic polynomial of the join of two graphs is first formulated. And then, a lower bound for the i th largest signless Laplacian eigenvalue of a graph is given. Finally, it is proved that $G \vee K_m$, where G is an $(n-2)$ -regular graph on n vertices, and $\overline{K}_n \vee K_2$ except for $n = 3$, are determined by their signless Laplacian spectra.

Key words. Signless Laplacian spectrum, Q -cospectral, Join, Determined by the signless Laplacian spectrum.

AMS subject classifications. 05C50.

1. Introduction. All graphs considered in this paper are simple and undirected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$, where $v_1, v_2, \dots, v_n$ are indexed in the non-increasing order of degrees. Let $d_i = d_i(G) = d_G(v_i)$ be the degree of the vertex v_i , and $\deg(G) = (d_1, d_2, \dots, d_n)$ be the degree sequence of G . The *complement* of G , denoted by $\overline{G}$, is the graph with the same vertex set as G such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G . The *adjacency matrix* of G , denoted by $A(G)$, is the $n \times n$ matrix whose (i, j) -entry is 1 if v_i and v_j are adjacent and 0 otherwise. The matrix $Q(G) = D(G) + A(G)$ is called the *signless Laplacian matrix* of G , where $D(G)$ is the $n \times n$ diagonal matrix with $d_1, d_2, \dots, d_n$ as diagonal entries. Given an $n \times n$ symmetric matrix M , denote by $\phi(M; x) = \det(xI_n - M)$, or simply $\phi(M)$, the characteristic polynomial of M , where I_n is the identity matrix of size n . The roots of the equation $\phi(M; x) = 0$, denoted by $\theta_1(M) \geq \theta_2(M) \geq \dots \geq \theta_n(M)$, are called the eigenvalues of M . The eigenvalues of $A(G)$ and $Q(G)$ are called the *adjacency eigenvalues* and *signless Laplacian eigenvalues* of G , respectively. Denote

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by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and $\nu_1(G) \geq \nu_2(G) \geq \cdots \geq \nu_n(G)$ the adjacency eigenvalues and the signless Laplacian eigenvalues of G , respectively. The multi-set of eigenvalues of $A(G)$ (respectively, $Q(G)$) is called the *adjacency* (respectively, *signless Laplacian*) *spectrum* of G . Two graphs are said to be *A-cospectral* (respectively, *Q-cospectral*) if they have the same adjacency (respectively, signless Laplacian) spectrum. A graph is said to be *determined by its signless Laplacian* (respectively, *adjacency*) *spectrum* if there is no other non-isomorphic graph which is *Q-cospectral* (respectively, *A-cospectral*) with it.

Which graphs are determined by their spectra? This is a classical question in spectral graph theory, which was raised by Günthard and Primas [12] in 1956 with motivations from chemistry. For its background, please refer to [1, 5, 7, 9, 10]. It is well known that this question is still far from being completely solved, since it is often very challenging to check whether or not an arbitrary given graph is determined by its spectrum, even for some simple-looking graphs.

Let G and H be two disjoint graphs. Denote by $G \cup H$ the *disjoint union* of G and H . Especially, mG means the disjoint union of m copies of G . The *join* of two disjoint graphs G and H , denoted by $G \vee H$, is the graph obtained by joining each vertex of G to each vertex of H . Until now only several joins have been proven to be determined by their signless Laplacian spectra, stated as follows:

1. Any *star* $K_{1,m} \cong mK_1 \vee K_1$, except for $K_{1,3}$, is determined by its signless Laplacian spectrum [3, 18, 19], where K_n denotes a *complete graph* on n vertices.
2. K_n is determined by its signless Laplacian spectrum [9].
3. $K_1 \vee (cK_2 \cup (n - 2c - 1)K_1)$ with $n \geq 2c + 1$ and $c \geq 0$ is determined by its signless Laplacian spectrum [15].
4. Any *wheel graph* $K_1 \vee C_n$ is determined by its signless Laplacian spectrum [16], where C_n denotes a *cycle* on n vertices.
5. Let G be an r -regular graph on n vertices. For $r = 1$, $r = n - 2$, or $r = 2$ and $n \geq 11$, $G \vee K_1$ is determined by its signless Laplacian spectrum. For $r = n - 3$, $G \vee K_1$ is determined by its signless Laplacian spectrum if and only if $\overline{G}$ has no triangles [2].
6. Let G be an r -regular graph on n vertices. For $r = 1$ or $r = n - 2$, $G \vee K_2$ is determined by its signless Laplacian spectrum. For $r = n - 3$, $G \vee K_2$ is determined by its signless Laplacian spectrum if and only if $\overline{G}$ has no triangles [20].

The main purpose of this paper is to prove that some new families of joins are determined by their signless Laplacian spectra. Our paper is organized as follows. In Section 2, we formulate the signless Laplacian characteristic polynomial of the join of two graphs. Section 3 gives a lower bound for the i th largest signless Laplacian

eigenvalue of a graph. In Section 4, we prove that $G \vee K_m$ where G is an $(n-2)$ -regular graph on n vertices, and $\overline{K}_n \vee K_2$ except for $n = 3$, are determined by their signless Laplacian spectra.

2. The signless Laplacian characteristic polynomial of the join of two graphs. The M -coronal of an $n \times n$ square matrix M , denoted by $\Gamma_M(x)$, is defined [4, 17] to be the sum of the entries of the matrix $(xI_n - M)^{-1}$, that is,

$$\Gamma_M(x) = \mathbf{1}_n^T (xI_n - M)^{-1} \mathbf{1}_n,$$

where $\mathbf{1}_n$ denotes the column vector of size n with all the entries equal to one, and $\mathbf{1}_n^T$ means the transpose of $\mathbf{1}_n$. The characteristic polynomial of the signless Laplacian matrix of a graph G is called the *signless Laplacian characteristic polynomial* of G . In this section, we determine the signless Laplacian characteristic polynomial of the join of two graphs with the help of the coronal of the signless Laplacian matrix.

THEOREM 2.1. *Let G_i be an arbitrary graph on n_i vertices for $i = 1, 2$. Then*

$$\begin{aligned} \phi(Q(G_1 \vee G_2); x) &= \phi(Q(G_1); x - n_2) \cdot \phi(Q(G_2); x - n_1) \\ &\quad \cdot (1 - \Gamma_{Q(G_1)}(x - n_2) \cdot \Gamma_{Q(G_2)}(x - n_1)). \end{aligned}$$

Proof. Note that the signless Laplacian matrix of $G_1 \vee G_2$ can be written as

$$Q(G_1 \vee G_2) = \begin{pmatrix} n_2 I_{n_1} + Q(G_1) & J_{n_1 \times n_2} \\ J_{n_2 \times n_1} & n_1 I_{n_2} + Q(G_2) \end{pmatrix},$$

where $J_{s \times t}$ denotes the $s \times t$ matrix with all entries equal to one. Then the signless Laplacian characteristic polynomial of $G_1 \vee G_2$ is given by

$$\begin{aligned} \phi(Q(G_1 \vee G_2); x) &= \det \begin{pmatrix} (x - n_2)I_{n_1} - Q(G_1) & -J_{n_1 \times n_2} \\ -J_{n_2 \times n_1} & (x - n_1)I_{n_2} - Q(G_2) \end{pmatrix} \\ &= \det((x - n_2)I_{n_1} - Q(G_1)) \cdot \det(S) \\ &= \phi(Q(G_1); x - n_2) \cdot \det(S), \end{aligned}$$

where

$$\begin{aligned} S &= (x - n_1)I_{n_2} - Q(G_2) - J_{n_2 \times n_1}((x - n_2)I_{n_1} - Q(G_1))^{-1} J_{n_1 \times n_2} \\ &= (x - n_1)I_{n_2} - Q(G_2) - \Gamma_{Q(G_1)}(x - n_2) J_{n_2 \times n_2} \end{aligned}$$

is the Schur complement[22] of $(x - n_1)I_{n_2} - Q(G_2)$. Note that $J_{n_2 \times n_2} = \mathbf{1}_{n_2} \mathbf{1}_{n_2}^T$, whose rank is equal to 1. Then the result follows from

$$\begin{aligned}\det(S) &= \phi(Q(G_2); x - n_1) \cdot \left(1 - \Gamma_{Q(G_1)}(x - n_2) \cdot \mathbf{1}_{n_2}^T ((x - n_1)I_{n_2} - Q(G_2))^{-1} \mathbf{1}_{n_2}\right) \\ &= \phi(Q(G_2); x - n_1) \cdot \left(1 - \Gamma_{Q(G_1)}(x - n_2) \cdot \Gamma_{Q(G_2)}(x - n_1)\right).\end{aligned}$$

Here, in the penultimate step, we used the Sherman-Morrison formula for the determinant. $\square$

Theorem 2.1 enables us to construct infinitely many pairs of Q -cospectral graphs, as stated in the following corollary.

COROLLARY 2.2. *If G is an arbitrary graph, and H_1 and H_2 are Q -cospectral graphs with $\Gamma_{Q(H_1)}(x) = \Gamma_{Q(H_2)}(x)$, then $G \vee H_1$ and $G \vee H_2$ are Q -cospectral.*

EXAMPLE 2.3. (See [2, Theorem 3.6]) *It is known [6] that $K_{1,3}$ and $C_3 \cup K_1$ are Q -cospectral. By simple computations, we have $\Gamma_{Q(K_{1,3})}(x) = \Gamma_{Q(C_3 \cup K_1)}(x) = \frac{4(x-1)}{x(x-4)}$. Then, by Corollary 2.2, we obtain that $G \vee K_{1,3}$ and $G \vee (C_3 \cup K_1)$ are Q -cospectral, where G is an arbitrary graph.*

Let G be an r -regular with n vertices and let $K_{p,q}$ denote the complete bipartite graph with $p, q \geq 1$ vertices in the two parts of its bipartition. The following formulas come from [4]:

$$\begin{aligned}\Gamma_{Q(G)}(x) &= \frac{n}{x - 2r}; \\ \Gamma_{Q(K_{p,q})}(x) &= \frac{(p+q)x - (p-q)^2}{x^2 - (p+q)x}.\end{aligned}$$

Substituting these back into Theorem 2.1, we obtain the following results.

COROLLARY 2.4. (See [13, Theorem 2.1]) *For $i = 1, 2$, let G_i be an r_i -regular graph on n_i vertices. Then*

$$\phi(Q(G_1 \vee G_2); x) = \frac{\phi(Q(G_1); x - n_2)\phi(Q(G_2); x - n_1)}{(x - 2r_1 - n_2)(x - 2r_2 - n_1)}f(x),$$

where $f(x) = x^2 - (2(r_1 + r_2) + (n_1 + n_2))x + 2(2r_1r_2 + r_1n_1 + r_2n_2)$.

COROLLARY 2.5. *Let G be an r -regular graph on n vertices and $K_{p,q}$ a complete bipartite graph. Then*

$$\phi(Q(G \vee K_{p,q}); x) = \frac{\phi(Q(G); x - p - q)\phi(Q(K_{p,q}); x - n)}{(x - 2r - p - q)(x - n)(x - n - p - q)}g(x),$$

where $g(x) = x^3 - 2(n + p + q + r)x^2 + (n^2 + (p + q + 2r)(p + q + 2n))x - 2n(r(p +$

$q + n) + 2pq)$.

COROLLARY 2.6. *Let $K_{m,n}$ and $K_{p,q}$ be two complete bipartite graphs. Then*

$$\phi(Q(K_{m,n} \vee K_{p,q}); x) = \frac{\phi(K_{m,n}; x - p - q)\phi(Q(K_{p,q}); x - m - n)}{(x - p - q)(x - m - n)(x - m - n - p - q)^2} h(x),$$

where $h(x) = (x - p - q)(x - m - n)(x - m - n - p - q)^2 - ((m + n)(x - p - q) - (m - n)^2) \times ((p + q)(x - m - n) - (p - q)^2)$.

3. A lower bound for the i th largest signless Laplacian eigenvalue. In this section, we will give a lower bound for the i th largest signless Laplacian eigenvalue of a graph. Before proceeding, we need to mention two well known results from matrix theory. Recall that $\theta_1(M) \geq \theta_2(M) \geq \dots \geq \theta_n(M)$ denote the eigenvalues of an $n \times n$ symmetric matrix M .

LEMMA 3.1. (See [14, Cauchy interlacing theorem]) *Let H be a Hermitian matrix of order n . If H_m is a principal submatrix of H of order m with $1 \leq m \leq n$, then for $1 \leq i \leq m$,*

$$\theta_{i+n-m}(H) \leq \theta_i(H_m) \leq \theta_i(H).$$

LEMMA 3.2. (See [14, Weyl's inequality]) *Let $H = M + P$, where M and P are Hermitian matrices of order n . Then for each $i = 1, 2, \dots, n$,*

$$\theta_i(M) + \theta_n(P) \leq \theta_i(H) \leq \theta_i(M) + \theta_1(P).$$

THEOREM 3.3. *Let G be a graph of order n with non-increasing degree sequence $\deg(G) = (d_1, d_2, \dots, d_n)$. Let G_i be the graph induced by vertices $v_1, v_2, \dots, v_i$, where $2 \leq i \leq n$. Then*

$$\nu_i(G) \geq d_i + \lambda_i(G_i).$$

Proof. Index the rows and columns of $Q(G)$ by $v_1, v_2, \dots, v_n$. Let M be the top-left principal $i \times i$ submatrix of $Q(G)$. By Lemma 3.1, $\nu_i(G) \geq \theta_i(M)$. Note that $M = D_i + (d_i I_i + A(G_i))$, where D_i is the $i \times i$ diagonal matrix with $d_1 - d_i, d_2 - d_i, \dots, d_{i-1} - d_i, 0$ as diagonal entries. Thus, by Lemma 3.2, $\theta_i(M) \geq \theta_i(D_i) + \theta_i(d_i I_i + A(G_i)) = d_i + \lambda_i(G_i)$. $\square$

Let P_n denotes a *path* on n vertices. Note that $\lambda_2(2P_1) = 0$ and $\lambda_2(P_2) = -1$. By Theorem 3.3, we obtain the following result immediately.

COROLLARY 3.4. (See [11, Corollary 3.2]) *Let G be a graph of order n with non-increasing degree sequence $\deg(G) = (d_1, d_2, \dots, d_n)$. Then*

- (a) $\nu_2(G) \geq d_2$, if v_1 and v_2 are not adjacent;
- (b) $\nu_2(G) \geq d_2 - 1$, if v_1 and v_2 are adjacent.

Since $\lambda_3(3P_1) = 0$, $\lambda_3(P_2 \cup P_1) = \lambda_3(K_3) = -1$ and $\lambda_3(P_3) = -\sqrt{2}$, Theorem 3.3 implies the following result.

COROLLARY 3.5. (See [21, Theorem 3.1]) *Let G be a graph of order n with non-increasing degree sequence $\deg(G) = (d_1, d_2, \dots, d_n)$. Then*

- (a) $\nu_3(G) \geq d_3$, if v_1, v_2 and v_3 induce $3P_1$;
- (b) $\nu_3(G) \geq d_3 - 1$, if v_1, v_2 and v_3 induce $P_2 \cup P_1$ or K_3 ;
- (c) $\nu_3(G) \geq d_3 - \sqrt{2}$, if v_1, v_2 and v_3 induce P_3 .

COROLLARY 3.6. *Let G be a graph of order n with non-increasing degree sequence $\deg(G) = (d_1, d_2, \dots, d_n)$. Let G_4 be the graph induced by vertices $v_1, \dots, v_4$. Then*

- (a) $\nu_4(G) \geq d_4$, if $G_4 \cong 4P_1$;
- (b) $\nu_4(G) \geq d_4 - 1$, if $G_4 \cong 2P_1 \cup P_2, 2P_2, P_1 \cup C_3$ or K_4 ;
- (c) $\nu_4(G) \geq d_4 - \sqrt{2}$, if $G_4 \cong P_1 \cup P_3$;
- (d) $\nu_4(G) \geq d_4 + \lambda_{\min}$, if $G_4 \cong G_4^1$, where $\lambda_{\min}$ is the smallest root of $\lambda^3 - \lambda^2 - 3\lambda + 1 = 0$;
- (e) $\nu_4(G) \geq d_4 - \frac{\sqrt{17}-1}{2}$, if $G_4 \cong G_4^2$;
- (f) $\nu_4(G) \geq d_4 - \frac{1+\sqrt{5}}{2}$, if $G_4 \cong P_4$;
- (g) $\nu_4(G) \geq d_4 - \sqrt{3}$, if $G_4 \cong K_{1,3}$;
- (h) $\nu_4(G) \geq d_4 - 2$, if $G_4 \cong C_4$,

where G_4^1 and G_4^2 are shown in Figure 3.1.

Proof. The result follows from Theorem 3.3 and Table 3.1. $\square$

COROLLARY 3.7. *Let G be a graph of order n with non-increasing degree sequence $\deg(G) = (d_1, d_2, \dots, d_n)$. Then $\nu_4(G) \geq d_4 - 2$.*

Proof. Let $f(\lambda) = \lambda^3 - \lambda^2 - 3\lambda + 1$. Note that $f(-2) = -5 < 0$, $f(-1) = 2 > 0$, $f(1) = -2 < 0$ and $f(3) = 10 > 0$. This implies that the smallest root of $f(\lambda) = 0$ lies in $(-2, -1)$. Then the result follows from Corollary 3.6. $\square$

4. Some joins determined by their signless Laplacian spectra. As mentioned in Section 1 that characterizing which graphs are determined by the spectra is very challenging, and that only several joins are proved to be determined by their

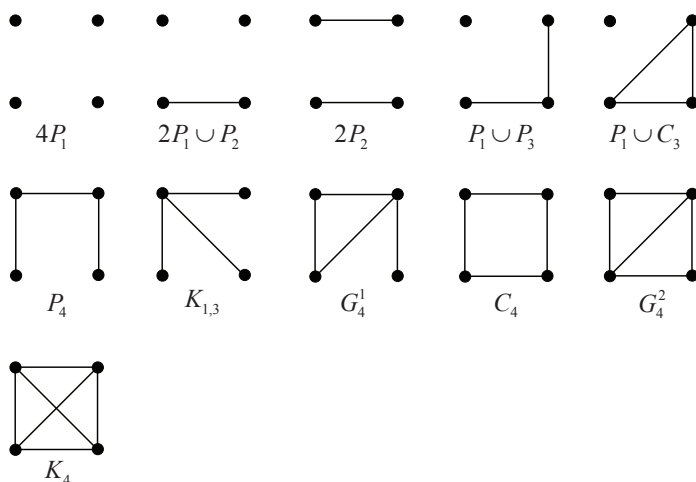


FIG. 3.1. *Graphs with 4 vertices.*

graph	characteristic polynomial	λ_4
$4P_1$	λ^4	0
$2P_1 \cup P_2$	$\lambda^2(\lambda - 1)(\lambda + 1)$	-1
$2P_2$	$(\lambda - 1)^2(\lambda + 1)^2$	-1
$P_1 \cup P_3$	$\lambda^2(\lambda^2 - 2)$	$-\sqrt{2}$
$P_1 \cup C_3$	$\lambda(\lambda - 2)(\lambda + 1)^2$	-1
P_4	$(\lambda^2 - \lambda - 1)(\lambda^2 + \lambda - 1)$	$\frac{-1-\sqrt{5}}{2}$
$K_{1,3}$	$\lambda^2(\lambda^2 - 3)$	$-\sqrt{3}$
G_4^1	$(\lambda + 1)(\lambda^3 - \lambda^2 - 3\lambda + 1)$	the smallest root of $\lambda^3 - \lambda^2 - 3\lambda + 1 = 0$
C_4	$\lambda^2(\lambda - 2)(\lambda + 2)$	-2
G_4^2	$\lambda(\lambda + 1)(\lambda^2 - \lambda - 4)$	$\frac{1-\sqrt{17}}{2}$
K_4	$(\lambda - 3)(\lambda + 1)^3$	-1

TABLE 3.1

The characteristic polynomials and the smallest eigenvalues of graphs in Figure 3.1.

signless Laplacian spectra. In this section, we will prove that some new families of joins are determined by the signless Laplacian spectra.

LEMMA 4.1. (See [11, Theorem 4.1]) *Let G be connected graph of order n . Then $\nu_n(G) < d_n(G)$.*

Let G be a graph with n vertices. Define $T_k = \sum_{i=1}^n \nu_i(G)^k$ to be the k th Q -

spectral moment of G , $k = 0, 1, 2, \dots$

LEMMA 4.2. (See [6, Corollary 4.3]) *Let G be a graph with n vertices, m edges and $n_3(G)$ triangles. Then*

$$T_0 = n, \quad T_1 = \sum_{i=1}^n d_i = 2m, \quad T_2 = 2m + \sum_{i=1}^n d_i^2, \quad T_3 = 6n_3(G) + 3 \sum_{i=1}^n d_i^2 + \sum_{i=1}^n d_i^3.$$

LEMMA 4.3. *Let G be an r -regular graph on n vertices and G is determined by its signless Laplacian spectrum. Let H be a graph Q -cospectral with $G \vee K_m$. If $d_1(H) = d_2(H) = \dots = d_m(H) = n + m - 1$, then $H \cong G \vee K_m$.*

Proof. The following equations follow from Lemma 4.2:

$$(4.1) \quad \sum_{i=1}^{m+n} d_i(H) = m(n + m - 1) + n(r + m),$$

$$(4.2) \quad \sum_{i=1}^{m+n} d_i(H)^2 = m(n + m - 1)^2 + n(r + m)^2.$$

Plugging $d_1(H) = d_2(H) = \dots = d_m(H) = n + m - 1$ into equations (4.1) and (4.2), we have

$$\sum_{i=m+1}^{m+n} d_i(H) = n(r + m), \quad \sum_{i=m+1}^{m+n} d_i(H)^2 = n(r + m)^2.$$

Then

$$\sum_{i=m+1}^{m+n} (d_i(H) - r - m)^2 = \sum_{i=m+1}^{m+n} d_i(H)^2 - 2(r + m) \sum_{i=m+1}^{m+n} d_i(H) + \sum_{i=m+1}^{m+n} (r + m)^2 = 0.$$

This implies that $d_{m+1}(H) = d_{m+2}(H) = \dots = d_{m+n}(H) = r + m$. Then $H \cong G_1 \vee K_m$, where G_1 is an r -regular graph. Corollary 2.4 implies that G and G_1 are Q -cospectral graph. Therefore, $H \cong G \vee K_m$ comes from the assumption that G is determined by its signless Laplacian spectrum. $\square$

REMARK 4.4. It is known [9, Proposition 3] that a regular graph G is determined by its signless Laplacian spectrum if and only if G is determined by its adjacency spectrum. So, the assumption that G is determined by its signless Laplacian spectrum in Lemma 4.3 can be replaced by that G is determined by its adjacency spectrum.

THEOREM 4.5. *Let G be an $(n - 2)$ -regular graph on n vertices. Then $G \vee K_m$ is determined by its signless Laplacian spectrum.*

Proof. Let H be a graph Q -cospectral with $G \vee K_m$. Then Lemma 4.2 implies that

$$(4.3) \quad \sum_{i=1}^{m+n} d_i(H) = m(m+n-1) + n(m+n-2) = (m+n)^2 - (m+n) - n.$$

Note that $d_i(H) \leq m+n-1$ for $i = 1, 2, \dots, m+n$, since H has $m+n$ vertices. Consider the following cases.

If $d_m(H) = m+n-1$, then $d_1(H) = d_2(H) = \dots = d_m(H) = m+n-1$. Note that G is determined by its signless Laplacian spectrum [8]. By Lemma 4.3, $H \cong G \vee K_m$.

If $d_m(H) \leq m+n-2$, then

$$\sum_{i=1}^{m+n} d_i(H) \leq (m-1)(m+n-1) + (n+1)(m+n-2) = (m+n)^2 - (m+n) - (n+1),$$

a contradiction to equation (4.3). $\square$

REMARK 4.6. The special cases of Theorem 4.5 when $m = 1$ and $m = 2$ were considered in [2] and [20], respectively.

Note that $\phi(Q(\overline{K}_n); x) = x^n$ and $\phi(Q(K_m); x) = (x-2m+2)(x-m+2)^{m-1}$. The following result follows from Corollary 2.4 by letting $G_1 = \overline{K}_n$ and $G_2 = K_m$.

LEMMA 4.7. *The signless Laplacian spectrum of $\overline{K}_n \vee K_m$ is*

$$\frac{(3m+n-2) \pm \sqrt{(m+n-2)^2 + 4mn}}{2}, m^{n-1}, (n+m-2)^{m-1},$$

where an exponent indicates the multiplicity of the corresponding signless Laplacian eigenvalue.

THEOREM 4.8. $\overline{K}_n \vee K_2$ is determined by its signless Laplacian spectrum except for $n = 3$.

Proof. If $n = 1$, then $\overline{K}_1 \vee K_2 \cong K_3$, which is determined by its signless Laplacian spectrum. In the following, we consider the cases of $n \geq 2$. Lemma 4.7 implies that the signless Laplacian spectrum of $\overline{K}_n \vee K_2$ is

$$\frac{(n+4) \pm \sqrt{n^2 + 8n}}{2}, n, 2^{n-1}.$$

Let H be Q -cospectral with $\overline{K}_n \vee K_2$. Lemma 4.1 and Corollary 3.5 imply respectively that

$$d_{n+2}(H) > \frac{(n+4) - \sqrt{n^2 + 8n}}{2} > 0, \quad d_3(H) \leq 2 + \sqrt{2}.$$

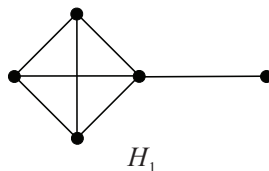


FIG. 4.1. Proof of Theorem 4.8: Possible case for H .

Then $3 \geq d_3(H) \geq d_{n+2}(H) \geq 1$. Denote by x_i , for $i = 1, 2, 3$, the number of vertices with degree i in $V(H) \setminus \{v_1(H), v_2(H)\}$, where $v_i(H)$ denotes the vertex with degree $d_i(H)$ for $i = 1, 2$. Consider the following cases.

If $d_3(H) \leq 2$, then Lemma 4.2 implies that $x_1 + x_2 = n$ and $x_1 + 2x_2 + d_2(H) + d_1(H) = 2(n+1) + 2n$. Then $x_2 = n + 2(n+1) - d_2(H) - d_1(H) \leq n$. This implies that $d_1(H) + d_2(H) \geq 2(n+1)$. Note that $d_2(H) \leq d_1(H) \leq n+1$, since H has $n+2$ vertices. Thus, we have $d_1(H) = d_2(H) = n+1$. By Lemma 4.3, $H \cong \overline{K}_n \vee K_2$.

If $d_3(H) = 3$, then by Lemma 4.2, we have

$$(4.4) \quad \begin{cases} x_1 + x_2 + x_3 = n, \\ x_1 + 2x_2 + 3x_3 + d_2(H) + d_1(H) = 2(n+1) + 2n, \\ x_1 + 4x_2 + 9x_3 + d_2(H)^2 + d_1(H)^2 = 2(n+1)^2 + 4n. \end{cases}$$

By solving the above equations, we have

$$x_1 + x_3 = (n+1 - d_1(H))(n + d_1(H) - 3) + (n+1 - d_2(H))(n + d_2(H) - 3).$$

If $n \geq d_1(H) \geq d_2(H) \geq d_3(H) = 3$, then $n \geq x_1 + x_3 \geq 2n + d_1(H) + d_2(H) - 6 > n$, a contradiction.

If $d_1(H) = n+1$ and $n \geq d_2(H) \geq d_3(H) = 3$, then $n \geq x_1 + x_3 = (n-1)^2 - (d_2(H) - 2)^2 \geq 2n - 3$. This implies that $n = 3$. Then $d_1(H) = 4$, $d_2(H) = 3$ and $x_2 = 0$. Plugging them back into (4.4), we have $x_1 = 1$ and $x_3 = 2$. So, H has 5 vertices with $\deg(H) = (4, 3, 3, 3, 1)$. There is a unique possibility for H as shown in Figure 4.1. By Maple, $\overline{K}_3 \vee K_2$ and H_1 have the same signless Laplacian spectrum: $3, (7 \pm \sqrt{33})/2, 2^2$.

If $d_1(H) = n+1$ and $d_2(H) = n+1$, then by solving (4.4), we have $x_3 = 0$, a contradiction to our assumption of $d_3(H) = 3$. $\square$

REMARK 4.9. The Q -cospectral mate $\overline{K}_3 \vee K_2$ and H_1 given in Theorem 4.8 was indeed presented in [2, Theorem 3.6], where the authors showed an infinite family of Q -cospectral mates. We also presented this result in Example 2.3.

Theorem 4.8 proved that $\overline{K}_n \vee K_2$ is determined by its signless Laplacian spectrum except for $n = 3$. One may naturally ask whether $\overline{K}_n \vee K_m$ is determined by its signless Laplacian spectrum for $m \geq 3$. From our experiences, the crucial point to prove this is to determine the degree sequence of the Q -cospectral graph with $\overline{K}_n \vee K_m$ for $m \geq 3$. This will be more difficult than what we did in Theorem 4.8.

QUESTION 4.10. *Prove or disprove that $\overline{K}_n \vee K_m$ is determined by its signless Laplacian spectrum for $m \geq 3$.*

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REFERENCES

- [1] A.E. Brouwer and W.H. Haemers. *Spectra of Graphs*. Springer, 2012.
- [2] C.-J. Bu and J. Zhou. Signless Laplacian spectral characterization of the cones over some regular graphs. *Linear Algebra Appl.*, 436:3634–3641, 2012.
- [3] C.-J. Bu and J. Zhou. Starlike trees whose maximum degree exceed 4 are determined by their Q -spectra. *Linear Algebra Appl.*, 436:143–151, 2012.
- [4] S.-Y. Cui and G.-X. Tian. The spectrum and the signless Laplacian spectrum of coronae. *Linear Algebra Appl.*, 437:1692–1703, 2012.
- [5] D.M. Cvetković, M. Doob, and H. Sachs. *Spectra of Graphs - Theory and Applications*, third edition. Johann Ambrosius Barth, Heidelberg, 1995.
- [6] D.M. Cvetković, P. Rowlinson, and S.K. Simić. Signless Laplacians of finite graphs. *Linear Algebra Appl.*, 423:155–171, 2007.
- [7] D.M. Cvetković, P. Rowlinson, and S. Simić. *An Introduction to the Theory of Graph Spectra*. Cambridge University Press, Cambridge, 2010.
- [8] D.M. Cvetković and S.K. Simić. Towards a spectral theory of graphs based on the signless Laplacian, II. *Linear Algebra Appl.*, 432:2257–2272, 2010.
- [9] E.R. van Dam and W.H. Haemers. Which graphs are determined by their spectrum? *Linear Algebra Appl.*, 373:241–272, 2003.
- [10] E.R. van Dam and W.H. Haemers. Developments on spectral characterizations of graphs. *Discrete Math.*, 309:576–586, 2009.
- [11] K. Ch. Das. On conjectures involving second largest signless Laplacian eigenvalue of graphs. *Linear Algebra Appl.*, 432:3018–3029, 2010.
- [12] Hs.H. Günthard and H. Primas. Zusammenhang von Graphtheorie und Mo-Theotie von Molekeln mit Systemen konjugierter Bindungen. *Helv. Chim. Acta*, 39:1645–1653, 1956.
- [13] M.A.A. de Freitas, N.M.M. de Abreu, R.R. Del-Vecchio, and S. Jurkiewicz. Infinite families of Q -integral graphs. *Linear Algebra Appl.*, 432:2352–2360, 2010.
- [14] R.A. Horn and C.R. Johnson. *Matrix Analysis*, second edition. Cambridge University Press, 2012.
- [15] M.-H. Liu, B.-L. Liu, and F.-Y. Wei. Graphs determined by their (signless) Laplacian spectra. *Electron. J. Linear Algebra*, 22:112–124, 2011.
- [16] M.-H. Liu. Some graphs determined by their (signless) Laplacian spectra. *Czechoslovak Math. J.*, 62:1117–1134, 2012.
- [17] C. McLeman and E. McNicholas. Spectra of coronae. *Linear Algebra Appl.*, 435:998–1007, 2011.

- [18] G.R. Omid. On a signless Laplacian spectral characterization of T -shape trees. *Linear Algebra Appl.*, 431:1607–1615, 2009.
- [19] G.R. Omid and E. Vatandoost. Starlike trees with maximum degree 4 are determined by their signless Laplacian spectra. *Electron. J. Linear Algebra*, 20:274–290, 2010.
- [20] L.-Z. Xu and C.-X. He. On the signless Laplacian spectral determination of the join of regular graphs. *Discrete Math. Algorithm. Appl.*, 6(4):1450050, 2014.
- [21] J.-F. Wang and F. Belardo. A note on the signless Laplacian eigenvalues of graphs. *Linear Algebra Appl.*, 435:2585–2590, 2011.
- [22] F.-Z. Zhang. *The Schur Complement and Its Applications*. Springer, 2005.