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THE LAPLACIAN QUADRATIC FORM AND EDGE CONNECTIVITY OF A GRAPH*

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Abstract. Let G be a simple connected graph with associated positive semidefinite integral quadratic form $Q(x) = \sum(x(i) - x(j))^2$, where the sum is taken over all edges ij of G. It is showed that the minimum positive value of $Q(x)$ for $x \in \mathbb{Z}^n$ equals the edge connectivity of G. By restricting $Q(x)$ to $x \in \mathbb{Z}^{n-1} \times \{0\}$, the quadratic form becomes positive definite. It is also showed that the number of minimal disconnecting sets of edges of G equals twice the number of vectors $x \in \mathbb{Z}^{n-1} \times \{0\}$ for which the form Q attains its minimum positive value.

Key words. Graph, Laplacian matrix, Edge connectivity, Integral quadratic form.

AMS subject classifications. 05C50, 15A63, 15B36.

1. Statement of results. Let G be a simple connected graph (no loops or multiple edges). The vertex set for G is $V(G) = \{1, 2, \dots, n\}$ and the edge set is denoted by $E(G)$. The Laplacian quadratic form associated with G is defined by:

$$Q(x) = \sum_{ij \in E(G)} (x(i) - x(j))^2,$$

for $x = (x(1), \dots, x(n)) \in \mathbb{Z}^n$. The matrix for this quadratic form is the Laplacian matrix $L(G)$ for the graph. See [2] for a survey of results about the Laplacian matrix.

A set of t edges $E = \{i_1j_1, \dots, i_tj_t\}$ of G *disconnects* G if the graph $G' = G - E$, obtained by removing these edges from G, is not connected. And the *edge connectivity* of G is the fewest number of edges that disconnect G. We call such a set of edges a *minimal disconnecting set* of edges of G.

THEOREM 1.1. *Let G be a simple connected graph. Then the least positive value of $Q(x)$ for $x \in \mathbb{Z}^n$ equals the edge connectivity of G.*

Let k be the common value of the least positive value of $Q(x)$ and the edge connectivity of G. The next theorem compares the number of minimal disconnecting sets of edges

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of G with the number of integral vectors $x = (x(1), x(2), \ldots, x(n-1), 0)$ for which $Q(x) = k$.

THEOREM 1.2. *Let G be a simple connected graph and let k be the edge connectivity of G. Then the number of vectors $x \in \mathbb{Z}^n$ with $x(n) = 0$ such that $Q(x) = k$ is twice the number of minimal disconnecting sets of edges of G.*

The restriction of the vectors $x \in \mathbb{Z}^n$ to those with $x(n) = 0$ is necessary because $Q(x)$ is not positive definite. Indeed its null space is spanned by the all-ones vector $e = (1, 1, \ldots, 1)$ and so if $Q(x) = k$ then $Q(x + ze) = k$ for every integer z. Thus, there are infinitely many vectors y in $\mathbb{Z}^n$ for which $Q(y) = k$. But the restriction of the quadratic form to

$$\mathcal{Z} = \{x \in \mathbb{Z}^n : x(n) = 0\}$$

is positive definite, which implies that there are only finitely many vectors $y \in \mathcal{Z}$ such that $Q(y) = k$. Furthermore, the positive integers represented by Q over $\mathbb{Z}^n$ are the same as those represented by Q over $\mathcal{Z}$ because $Q(x) = Q(y)$ for $y = x - x(n)e \in \mathcal{Z}$.

Before proceeding to the proofs, we insert a few remarks about the relationship between the quadratic form Q and its restriction to $\mathcal{Z}$. If we view the restriction as a quadratic form over $(x(1), x(2), \ldots, x(n-1)) \in \mathbb{Z}^{n-1}$, then its matrix is the principal sub matrix of the Laplacian $L(G)$ in rows and columns $1, 2, \ldots, n-1$. The famous matrix tree theorem of Kirchhoff [1, 2] states that the determinant of every $(n-1) \times (n-1)$ sub matrix of $L(G)$ equals plus or minus the number of spanning trees of G. In addition, all of the $(n-1) \times (n-1)$ principal sub matrices of $L(G)$ are congruent to each other by a unimodular matrix [3, 4]. So there is nothing special about restricting Q to vectors with $x(n) = 0$. Indeed, if we restrict Q by taking $x \in \mathbb{Z}^n$ with $x(i) = 0$ for some other vertex i instead of $x(n) = 0$, all of the resulting quadratic forms are equivalent to each other.

We should also note that the Laplacian matrices $L(G_1), L(G_2)$ are congruent by a unimodular matrix if and only if the graphs G_1, G_2 are cycle isomorphic [3, 4]. Thus, every invariant for unimodular congruence is shared by all graphs in the same cycle-isomorphism class.

2. Proofs. Let G be a simple connected graph, k be the edge connectivity of G, and m be the minimum positive integer represented by Q. The general outline for the proofs is to show that $m = k$ and that if $Q(x) = m$ for $x \in \mathcal{Z}$ then all the coordinates of x are either in $\{0, 1\}$ or all are in $\{0, -1\}$. Then we establish a bijection between the minimal disconnecting sets of edges of G and the vectors $x \in \{0, 1\}^{n-1} \times \{0\}$ with $Q(x) = m$. This will prove Theorem 1.2 because if $Q(x) = m$ for some $x \in \mathcal{Z}$ then

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$Q(-x) = m$ as well. Thus, every pair of vectors $\pm x$ with $Q(x) = m$ corresponds to a minimal disconnecting set of edges of G.

2.1. A lemma from graph theory. We need the following lemma about connected graphs:

Lemma 2.1. *Let G be a simple connected graph and $E = \{i_1j_1, \ldots, i_kj_k\}$ be a minimal disconnecting set of edges of G. Then the graph $G' = G - E$ obtained by removing the edges in E has exactly two connected components.*

Proof. Since E disconnects G, G' has at least two components. Suppose it has more than two components. The vertices i_k, j_k are in just one or two of the components leaving a third component whose vertices do not include either i_k or j_k. It follows that this third component is still a component of the subgraph $G'' = G - \{i_1j_1, \ldots, i_{k-1}j_{k-1}\}$. Thus, $\{i_1j_1, \ldots, i_{k-1}j_{k-1}\}$ disconnects G, which contradicts the minimality of k. □

2.2. Notation. We use the following notation: For a positive integer l, let

$$\mathcal{X}(l) = \{x \in \{0,1\}^{n-1} \times \{0\} : Q(x) = l\},$$
$$\mathcal{E}(l) = \{E \subseteq E(G) : E \text{ disconnects } G \text{ and } |E| = l\}.$$

Of course, $\mathcal{X}(l)$ is empty if $l < m$ and $\mathcal{E}(l)$ is empty if $l < k$. Later we will show that $\mathcal{X}(m)$ is not empty. That is, there is a $(0,1)$ vector x with $Q(x) = m$.

For each $x \in \{0,1\}^{n-1} \times \{0\}$, partition the vertices of G into two sets:

$$V_0(x) = \{i \in \{1,2,\ldots,n\} : x(i) = 0\},$$
$$V_1(x) = \{i \in \{1,2,\ldots,n\} : x(i) = 1\},$$

and the edges of G into three sets:

$$E_0(x) = \{ij \in E(G) : x(i) = x(j) = 0\},$$
$$E_1(x) = \{ij \in E(G) : x(i) = x(j) = 1\},$$
$$E_{01}(x) = \{ij \in E(G) : x(i) = 0 \text{ and } x(j) = 1, \text{ or } x(i) = 1 \text{ and } x(j) = 0\}.$$

One thing is already clear: If $x \in \{0,1\}^{n-1} \times \{0\}$ then

$$|E_{01}(x)| = Q(x). \tag{2.1}$$

Since $E_0(x)$, $E_1(x)$, $E_{01}(x)$ partition the edges of G, the sum $\sum(x(i) - x(j))^2$ over all edges ij of G equals the sum of three sums: Over edges in $E_0(x)$, edges in $E_1(x)$ and edges in $E_{01}(x)$. The first and second sums are zero and the third sum equals $|E_{01}(x)|$.

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2.3. The map $\theta : \mathcal{E}(k) \to \mathcal{X}(k)$. Let k be the edge connectivity of G and let $E \in \mathcal{E}(k)$ be a minimal disconnecting set of edges of G. By Lemma 2.1, the subgraph $G' = G - E$ has two connected components, H_0, H_1. To be definite we take H_0 to be the component containing vertex n. Define $x_E \in \{0,1\}^{n-1} \times \{0\}$ by

$$x_E(i) = \begin{cases} 0, & \text{if } i \text{ is a vertex of } H_0, \\ 1, & \text{if } i \text{ is a vertex of } H_1. \end{cases}$$

The edges of G are partitioned by the edges of H_0, the edges of H_1, and E. Thus, $Q(x_E) = |E| = k$. So, $x_E \in \mathcal{X}(k)$ and the function $E \to x_E$ maps $\mathcal{E}(k)$ into $\mathcal{X}(k)$. It follows from the minimality of m that $m \leq k$.

2.4. $\mathcal{X}(m)$ is not empty. Again let m be the minimum positive integer represented by Q, say $Q(x) = m$ for some $x \in \mathcal{Z}$. Define a zero-one vector y by $y(i) = 0$ whenever $x(i)$ is even and $y(i) = 1$ whenever $x(i)$ is odd. Since $x(n) = 0$ is even, $y(n) = 0$. Now $y \neq 0$ because if all the coordinates of x are even, then $x/2 \in \mathcal{Z}$ and $Q(x/2) = m/4$, which contradicts the minimality of m. Clearly, $Q(y) \leq Q(x) = m$. Since $y \neq 0$ and m is minimal we have $Q(y) = m$. That is $y \in \mathcal{X}(m)$, which shows that $\mathcal{X}(m)$ is not empty.

2.5. $m = k$. Let y be any vector in $\mathcal{X}(m)$. Then $E_{01}(y)$ is a disconnecting set of edges of G and (by Equation (2.1)) $|E_{01}(y)| = Q(y) = m$. From the minimality of k, we have $k \leq m$. Therefore, $k = m$ and Theorem 1.1 is proved.

From here on we use k to denote both the minimum positive value of $Q(x)$ and the edge connectivity of G.

2.6. $\theta : \mathcal{E}(k) \to \mathcal{X}(k), x \to x_E$ is one-to-one. Let E, F be disconnecting sets of edges in $\mathcal{E}(k)$ with $x_E = x_F$. Then

$$\begin{aligned} G - E &= H_0 + H_1, \\ G - F &= K_0 + K_1, \end{aligned}$$

where H_0, H_1 are the components of $G - E$, K_0, K_1 are the components of $G - F$, and n is a vertex in H_0 and K_0. Since $x_E = x_F$ we have $i \in V(H_0)$ if and only if $i \in V(K_0)$. Thus, $V(H_0) = V(K_0)$. The edges of H_0 are just the edges ij of G with $i, j \in V(H_0)$. It follows that $E(H_0) = E(K_0)$. Similarly $E(H_1) = E(K_1)$. The edges of G are partitioned in two ways

$$\begin{aligned} E(G) &= E(H_0) \uplus E(H_1) \uplus E, \\ E(G) &= E(K_0) \uplus E(K_1) \uplus F. \end{aligned}$$

Thus, $E = F$.

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2.7. $\theta : \mathcal{E}(k) \to \mathcal{X}(k), x \to x_E$ **is onto.** Let $x \in \mathcal{X}(k)$. We must show that there exists $E \in \mathcal{E}(k)$ such that $x = x_E$. The obvious, and correct, candidate is $E = E_{01}(x)$.

Let $H_i(x)$ be the subgraph of G with vertices $V_i(x)$ and edges $E_i(x)$ for $i = 1, 2$. Clearly, $H_0(x), H_1(x)$ are the components of $G' = G - E$. So $x_E(i) = 0$ if and only if i is a vertex of $H_0(x)$. Also $x(i) = 0$ if and only if $i \in V_0(x) = V(H_0)$. So $x_E = x$ and θ maps $\mathcal{E}(k)$ onto $\mathcal{X}(k)$.

We have proved that $|\mathcal{E}(k)| = |\mathcal{X}(k)|$.

2.8. If $x \in \mathcal{Z}$ **and** $Q(x) = k$ **then** $x \in \mathcal{X}(k)$ **or** $-x \in \mathcal{X}(k)$**.** In this section, we show that the only vectors $x \in \mathcal{Z}$ for which Q achieves the minimum positive value k are those all of whose coordinates are in $\{0, 1\}$ or all are in $\{0, -1\}$.

Suppose $x \in \mathcal{Z}$ and $Q(x) = k$. Define a vector $y \in \{0, 1\}^{n-1} \times \{0\}$ by

$$y(i) = \begin{cases} 0 & x(i) \text{ is even,} \\ 1 & x(i) \text{ is odd.} \end{cases}$$

Arguing as in Section 2.4, we get $y \neq 0$. Now partition the edges of G into three sets, $E_0(y)$, $E_1(y)$, and $E_{01}(y)$. It is clear that $(y(i) - y(j))^2 \leq (x(i) - x(j))^2$, for all i, j. Therefore, we have the following inequalities for the sums:

$$\begin{aligned} 0 = \sum_{ij \in E_0(y)} (y(i) - y(j))^2 &\leq \sum_{ij \in E_0(y)} (x(i) - x(j))^2 \\ 0 = \sum_{ij \in E_1(y)} (y(i) - y(j))^2 &\leq \sum_{ij \in E_1(y)} (x(i) - x(j))^2 \\ k = \sum_{ij \in E_{01}(y)} (y(i) - y(j))^2 &\leq \sum_{ij \in E_{01}(y)} (x(i) - x(j))^2. \end{aligned}$$

But $Q(x)$, which is the sum of the three sums above on the right, equals k. Therefore, $Q(y) = k$ and $y \in \mathcal{X}(k)$. In addition, we have equality for each of the three inequalities. This shows that $x(i) = x(j)$ for all $ij \in E_0(y)$, $x(i) = x(j)$ for all $ij \in E_1(y)$, and $|x(i) - x(j)| = 1$ for all $ij \in E_{01}(y)$.

We now show that there is an integer a such that $x(i) = a$ for all $i \in V_0(y)$ and an integer b such that $x(i) = b$ for all $i \in V_1(y)$. The set of edges $E_{01}(y)$ disconnects G and it is a minimal disconnecting set ($|E_{01}(y)| = k$). Lemma 2.1 applies so $G' = G - E_{01}(y) = H_0 + H_1$ where H_0, H_1 are the connected components of G' and n is a vertex of H_0. It is clear that $V(H_i) = V_i(y)$ and $E(H_i) = E_i(y)$ for $i = 1, 2$.

Because H_0 is connected, there is a path joining any two vertices in H_0. But $x(i) = x(j)$ for any edge ij in $E_0(y) = E(H_0)$. It follows that there is an integer a such that $x(i) = a$ for all $i \in V(H_0) = V_0(y)$. Likewise there is an integer b such that $x(i) = b$

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for all $i \in V(H_1) = V_1(y)$. Now $x(n) = 0$ and $n \in V(H_0)$, so $a = 0$. There is at least one edge ij in $E_{01}(y)$ or else G is not connected. By adjusting the notation we may suppose that i is a vertex in H_0 and j a vertex in H_1 for this edge in $E_{01}(y)$. Therefore, $1 = |x(i) - x(j)| = |0 - b| = 1$. It follows that $b = \pm 1$ and therefore either $x \in \mathcal{X}(k)$ or $-x \in \mathcal{X}(k)$.

2.9. Conclusion. The preceding arguments show that for every $x \in \mathcal{Z}$ with $Q(x) = k$, either $x \in \mathcal{X}(k)$ or $-x \in \mathcal{X}(k)$. And that the number of minimal disconnecting sets for G equals the number of $x \in \mathcal{X}(k)$ for which $Q(x) = k$. Thus, the number of vectors x in $\mathcal{Z}$ such that $Q(x) = k$ is twice the number of minimal disconnecting sets of edges of G. The proof of Theorem 1.2 is complete.

2.10. A combinatorial observation. The author wishes to thank the referee for this observation: If the vertices of a connected graph G are colored with two colors, 0 and 1, then the number of two-colored edges is at least the edge connectivity of G with equality if and only if the set of two-colored edges is a minimal disconnecting set of edges, E. Indeed, the number of two-colored edges is just $E_{01}(x_E)$ for the $0, 1$ coloring vector x_E.

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