Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 422-436, August 2015

ELA

RANK DROPS OF RECURRENCE MATRICES*

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Abstract. A recurrence matrix is a matrix whose terms are sequential members of a linear homogeneous recurrence sequence of order k and whose dimensions are both greater than or equal to k. In this paper, the ranks of recurrence matrices are determined. In particular, it is shown that the rank of such a matrix differs from the previously found upper bound of k in only two situations: When (a_j) satisfies a recurrence relation of order less than k, and when the nth powers of distinct eigenvalues of (a_j) coincide.

Key words. Linear recurrence relations, Matrix rank, Recurrence matrices.

AMS subject classifications. 15A03, 65Q30.

1. Introduction. Let (a_j) be a complex-valued sequence, where j starts at 0. We define the $m \times n$ *matrix of the sequence* (a_j), written $M_{m,n}((a_j))$, to be the matrix

$$M_{m,n}((a_j)) = \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_n & a_{n+1} & \cdots & a_{2n-1} \\ \vdots & \vdots & & \vdots \\ a_{(m-1)n} & a_{(m-1)n+2} & \cdots & a_{mn-1} \end{bmatrix}.$$

Consider the $m \times n$ matrix of the sequence $(j+1) = (1, 2, 3, \ldots)$ Since $(j+1)$ is such a simple sequence, we might ask what the rank of $M_{m,n}((j+1))$ is. The answer is tantalizingly trivial:

$$\operatorname{rank} M_{m,n}((j+1)) = \begin{cases} 1 & m = 1 \text{ or } n = 1, \\ 2 & m, n \geq 2. \end{cases}$$

Not only is the rank bounded, but the size of the matrix hardly matters. To see this, note the rows of $M_{m,n}((j+1))$ are linear combinations of $\begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & \cdots & n-1 \end{bmatrix}$, since each row is of the form $\begin{bmatrix} a & a+1 & \cdots & a+n-1 \end{bmatrix}$, for some a.

*Received by the editors on July 1, 2014. Accepted for publication on June 28, 2015. Handling Editor: Bryan L. Shader.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 422-436, August 2015

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Noting that $(j+1)$ is a recurrence sequence, we turn to recurrence sequences to explore this behavior in a more general setting. A *linear homogeneous recurrence relation* of order k (hereafter, a *recurrence relation of order k*) is an equation of the form

$$a_j = c_1 a_{j-1} + c_2 a_{j-2} + \cdots + c_k a_{j-k}, \tag{1.1}$$

where $c_1, c_2, \ldots, c_k$ are complex numbers and $c_k \neq 0$. A *solution* to a recurrence relation is a complex-valued sequence $(a_1, a_2, \ldots)$ such that (1.1) holds for each $j \geq k$. Such a sequence is called a *recurrence sequence of order k*. Familiar examples of recurrence sequences include geometric sequences and the Fibonacci numbers.

When (a_j) is a recurrence sequence of order k and $m, n \geq k$, we call $M_{m,n}((a_j))$ a *kth-order recurrence matrix.* Our goal in this work is to investigate the ranks of recurrence matrices. Although we will restrict our attention to homogeneous recurrence sequences, there is a reduction (discussed in Section 2.1) that allows one to apply our results to a large class of inhomogeneous recurrence sequences.

The following upper bound on the rank of a recurrence matrix was determined previously by Lee and Peterson in [4, Theorem 1].

Theorem 1.1. (Lee and Peterson [4]) *The rank of a kth-order recurrence matrix is less than or equal to k.*

Let us return to our example, $M_{m,n}((j+1))$, this time assuming that $m, n \geq 2$. The sequence $(a_j) = (j+1)$ satisfies the second-order recurrence $a_j = 2a_{j-1} - a_{j-2}$, since

$$2a_{j-1} - a_{j-2} = 2(j) - (j-1) = 2j - j + 1 = j + 1 = a_j.$$

Hence, Theorem 1.1 applies. It follows that $M_{m,n}((j+1)) \leq 2$, in agreement with our earlier calculation that $M_{m,n}((j+1)) = 2$.

However, Theorem 1.1 only determines an upper bound on the rank of recurrence matrices, leaving open the problem of determining the exact rank. As we just saw, the bound is attained for some matrices, but it is possible that the rank of a kth-order recurrence matrix is strictly less than k. When this occurs, we say that the matrix has a *rank drop.* We will prove via an exact calculation of the rank that rank drops occur in only two ways, each reflecting a kind of degeneracy of the recurrence sequence. We begin with an example of each.

Example 1.2. Consider the recurrence relation $a_j = 3a_{j-1} - 2a_{j-2}$ with initial values (or *seeds*) $a_0 = 1, a_1 = 2$. Then $a_j = 2^j$ and

$$M_{3,3}((a_j)) = \begin{bmatrix} 1 & 2 & 4 \\ 8 & 16 & 32 \\ 64 & 128 & 256 \end{bmatrix}.$$

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Since each row is a multiple of the first, this matrix has rank 1, although it is a 2nd-order recurrence matrix.

This can be explained by noting that $a_j = 2^j$ also satisfies the first order recurrence $a_j = 2a_{j-1}$. Hence, by Theorem 1.1, $\operatorname{rank} M_{3,3}((a_j))$ is bounded above by 1, rather than 2 as initially predicted. We will call this an *order rank drop.* This pattern was previously observed and characterized in the order 2 case in Theorem 2 of [4]. Order rank drops will be characterized in Section 3.

EXAMPLE 1.3. Consider the recurrence relation $a_j = a_{j-2}$. The effect of this recurrence relation is to periodically repeat the seed. In particular, let $a_0 = 2, a_1 = 0$. Then $a_j = 2$ for even j and $a_j = 0$ for odd j. If we construct a 3×3 matrix from this sequence,

$$M_{3,3}((a_j)) = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix},$$

we have a rank 2 matrix. However, if we construct a 4×4 matrix from the sequence,

$$M_{4,4}((a_j)) = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix},$$

the result has rank 1.

Here the rank of the matrix depends on its width. When this happens, we say that the recurrence sequence has a *width rank drop.* These will be investigated in Section 4.

2. Solution sets of recurrence relations. It will be convenient to develop some basic facts about the solution sets of recurrence relations. Readers who are already familiar with solutions to linear homogeneous recurrence relations may wish to skip to Corollary 2.4.

To each recurrence relation of order k,

$$a_j = c_1 a_{j-1} + c_2 a_{j-2} + \cdots + c_k a_{j-k}, \tag{2.1}$$

there is associated a *characteristic polynomial* of degree k,

$$f(\lambda) = \lambda^k - c_1\lambda^{k-1} - c_2\lambda^{k-2} - \cdots - c_{k-1}\lambda - c_k.$$

The roots of the characteristic polynomial are called the *eigenvalues* of the recurrence relation. (Note that since we have assumed that $c_k \neq 0$, eigenvalues are always

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A publication of the International Linear Algebra Society
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nonzero.) These definitions are justified by the fact that we may use the characteristic polynomial $f(\lambda)$ to create a linear operator that vanishes on solutions of (2.1).

Let Λ be the linear operator on sequences defined by $\Lambda(a_j) = (a_{j+1})$. We may define an operator $f(\Lambda)$ by

$$f(\Lambda) = \Lambda^k - c_1\Lambda^{k-1} - c_2\Lambda^{k-2} - \cdots - c_{k-1}\Lambda - c_k I,$$

where I is the identity operator. Multiplication of two such operators $f(\Lambda), g(\Lambda)$ is taken to be their composition, which coincides with multiplication of ordinary polynomials in the sense that $f(\Lambda)g(\Lambda) = (fg)(\Lambda)$. Given these definitions, for an arbitrary sequence (a_j),

$$\begin{aligned} f(\Lambda)(a_j) &= (\Lambda^k - c_1\Lambda^{k-1} - c_2\Lambda^{k-2} - \cdots - c_{k-1}\Lambda - c_k I)(a_j) \\ &= (a_{j+k} - c_1 a_{j+k-1} - c_2 a_{j+k-2} - \cdots - c_{k-1}a_{j+1} - c_k a_j). \end{aligned}$$

So, $f(\Lambda)(a_j) = (0)$ if and only if (a_j) satisfies the recurrence relation (2.1). That is, $\ker f(\Lambda)$ is the solution set of the recurrence relation.

For a concrete example, consider the Fibonacci sequence $(a_j) = (1, 1, 2, 3, 5, \ldots)$ It satisfies the recurrence $a_j = a_{j-1} + a_{j-2}$, which has the characteristic polynomial $f(\lambda) = \lambda^2 - \lambda - 1$. The corresponding operator obtained by evaluating $f(\lambda)$ at Λ is $\Lambda^2 - \Lambda - I$. We apply this operator to the sequence and compute:

$$\begin{array}{crlccccccc} & \Lambda^2(a_j) & = 2 & 3 & 5 & 8 & 13 & 21 & 34 & \cdots \\ - & \Lambda(a_j) & = 1 & 2 & 3 & 5 & 8 & 13 & 21 & \cdots \\ - & I(a_j) & = 1 & 1 & 2 & 3 & 5 & 8 & 13 & \cdots \\ \hline & (\Lambda^2 - \Lambda - I)(a_j) & = 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{array}$$

So, (a_j) is in the kernel of $\Lambda^2 - \Lambda - I$, as expected.

We will now derive a canonical set of basis vectors of $\ker f(\Lambda)$, which we will call *fundamental solutions*.

LEMMA 2.1. *The solution set of an order k recurrence has dimension k. Equivalently,* $\dim \ker f(\Lambda) = \deg f(\lambda)$.

Proof. Let $f(\lambda)$ be the characteristic polynomial of the recurrence. The mapping $\phi : \ker f(\Lambda) \to \mathbb{C}^k$ defined by taking a solution (a_j) to its first k values, $\begin{bmatrix} a_0 & a_1 & \cdots & a_{k-1} \end{bmatrix}$, is an isomorphism of vector spaces. Therefore, $\dim \ker f(\Lambda) = k$. $\square$

Suppose $f(\lambda) = \prod_{l=1}^{q} (\lambda - \lambda_l)^{k_l}$ is a characteristic polynomial, where the eigenvalues λ_l are distinct. Then $\ker (\Lambda - \lambda_l)^{k_l}$ is a subspace of $\ker f(\Lambda)$ for each l. As a first step toward finding a fundamental solution set for the corresponding recurrence

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 422-436, August 2015

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relation, we start by finding a basis of $\ker(\Lambda - \lambda_l)^{k_l}$ for each l.

Lemma 2.2. *Let λ be a nonzero complex number and let n be a positive integer. Then $\ker((\Lambda-\lambda)^n)$ has the basis $\{(\lambda^j), (j\lambda^j), \ldots, (j^{n-1}\lambda^j)\}$.*

Proof. We will first prove that for each n, $\{(\lambda^j), (j\lambda^j), \ldots, (j^{n-1}\lambda^j)\} \subseteq \ker((\Lambda - \lambda)^n)$. The proof is by induction.

Let $n = 1$. Then $(\Lambda - \lambda)(\lambda^j) = (\lambda^{j+1} - \lambda\lambda^j) = (0)$, so $(\lambda_j) \in \ker(\Lambda - \lambda)$.

Suppose $\{(\lambda^j), (j\lambda^j), \ldots, (j^{n-1}\lambda^j)\} \subseteq \ker(\Lambda - \lambda)^n$ for $n = k$, for some integer $k \geq 1$. Clearly, $\{(\lambda^j), (j\lambda^j), \ldots, (j^{n-1}\lambda^j)\} \subseteq \ker(\Lambda - \lambda)^{n+1}$. It remains to show that $(j^n\lambda^j) \in \ker(\Lambda - \lambda)^{n+1}$. Now,

$$\begin{aligned}(\Lambda-\lambda)^{n+1}(j^n\lambda^j) &= (\Lambda-\lambda)^n((j+1)^n\lambda^{j+1} - j^n\lambda^{j+1})\\ &= (\Lambda-\lambda)^n\left(\lambda\binom{n}{1}j^{n-1}\lambda^j + \lambda\binom{n}{2}j^{n-2}\lambda^j + \cdots + \lambda\binom{n}{n}\lambda^j\right)\\ &= (0),\end{aligned}$$

where we have used the binomial theorem on the second line and the induction hypothesis on the third. Therefore, $\{(\lambda^j), (j\lambda^j), \ldots, (j^n\lambda^j)\} \subseteq \ker(\Lambda - \lambda)^{n+1}$.

Next we will show that $(j^n\lambda^j) \notin \operatorname{Span}\{(\lambda^j), (j\lambda^j), \ldots, (j^{n-1}\lambda^j)\}$ for all n. The result is trivial for $n = 1$. For $n > 1$, suppose that $K_1, K_2, \ldots, K_n$ are complex numbers so that $(j^n\lambda^j) = K_1(\lambda^j) + K_2(j\lambda^j) + \cdots + K_n(j^{n-1}\lambda^j)$. But then

$$j^n = K_1 + K_2 j + \cdots + K_n j^{n-1}$$

for all j. This is impossible, since the left hand side is an nth degree polynomial and the right hand side is an $(n-1)$st degree polynomial. It follows that $(j^n\lambda^j) \notin \operatorname{Span}\{(\lambda^j), (j\lambda^j), \ldots, (j^{n-1}\lambda^j\}$ for all n.

Therefore, for all n, $\{(\lambda_j), (j\lambda^j), \ldots, (j^{n-1}\lambda^j)\}$ is a linearly independent subset of $\ker(\Lambda - \lambda)^n$ containing n vectors. Since $\ker(\Lambda - \lambda)^n$ has dimension n, it follows that $\{(\lambda_j), (j\lambda^j), \ldots, (j^{n-1}\lambda^j)\}$ is a basis of $\ker(\Lambda - \lambda)^n$. ◻

So far we have characterized the solution sets of recurrence relations with a single eigenvalue, possibly repeated. For the remaining recurrences it suffices to piece together the solutions corresponding to each eigenvalue.

Theorem 2.3. *Let $f(\lambda) = \prod_{l=1}^{q}(\lambda - \lambda_l)^{k_l}$ be the characteristic polynomial of a kth order recurrence relation with q distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_q$ with respective multiplicities $k_1, k_2, \ldots, k_q$. Then the solution set of the recurrence has the basis $\bigcup_{l=1}^{q}\left\{(\lambda_l^j), (j\lambda_l^j), \ldots, (j^{k_l-1}\lambda_l^j)\right\}$.*

Proof. By Corollary II in [3, p. 386], $\ker f(\Lambda) = \bigoplus_{l=1}^{q}\ker(\Lambda - \lambda_l)^{k_l}$. By the

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A publication of the International Linear Algebra Society
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previous lemma, each $\ker(\Lambda - \lambda_l)^{k_l}$ has the basis $\{(\lambda_l^j), (j\lambda_l^j), \ldots, (j^{k_l-1}\lambda_l^j)\}$. The result follows. □

An alternative proof of Theorem 2.3 using generating functions may be found in [1, pp. 70–71]. Note that if $f(\lambda), g(\lambda)$ are characteristic polynomials of recurrence sequences and $h(\lambda) = \operatorname{lcm}(f(\lambda), g(\lambda))$, then $\ker f(\Lambda) + \ker g(\Lambda) = \ker h(\Lambda)$. In particular, $\ker f(\Lambda) \subseteq \ker g(\Lambda)$ only if $f(\lambda) \mid g(\lambda)$. Combining this result with the fact that $\ker f(\Lambda) \subseteq \ker g(\Lambda)$ if $f(\lambda) \mid g(\lambda)$, we obtain the following corollary (stated without proof in [2, p. 13]).

COROLLARY 2.4. *If $f(\lambda)$ and $g(\lambda)$ are characteristic polynomials of recurrence sequences,* $\ker f(\Lambda) \subseteq \ker g(\Lambda)$ *if and only if* $f(\lambda) \mid g(\lambda)$.

2.1. Extension to certain inhomogeneous recurrences. We have so far assumed (and will continue to assume) that our recurrence sequences are homogeneous. This is not a great restriction, since many linear inhomogeneous recurrence sequences may be transformed into homogeneous recurrence sequences. Suppose (a_j) is a sequence satisfying the kth-order inhomogeneous recurrence relation

$$a_j = c_1 a_{j-1} + \cdots + c_k a_{j-k} + b_j, \tag{2.2}$$

where $b_j = \sum_{i=1}^{q} p_i(j)\lambda_i^j$ and each p_i is a polynomial of degree k_i. Let $f(\lambda) = \lambda^k - c_1\lambda^{k-1} - \cdots - c_k$. Then equation (2.2) may also be written as an equation on sequences:

$$f(\Lambda)(a_j) = (b_j). \tag{2.3}$$

Note that (b_j) is in the solution set of the recurrence with the characteristic polynomial

$$g(\lambda) = \prod_{i=1}^{q} (\lambda - \lambda_i)^{k_i}.$$

Applying $g(\Lambda)$ to both sides of (2.3), the (b_j) term disappears, leaving

$$g(\Lambda)f(\Lambda)(a_j) = (0).$$

So (a_j) satisfies the homogeneous recurrence with characteristic polynomial $g(\lambda)f(\lambda)$. We may then use our results for homogeneous recurrences on (a_j).

3. Order rank drops. Let (a_j) be a sequence satisfying a recurrence relation of order k. As stated earlier, it is possible that (a_j) satisfies a recurrence sequence of order less than k, and if so, then $M_{m,n}((a_j))$ will have an order rank drop. We now have all of the tools in place to identify the least order recurrence relation satisfied by (a_j) (hereafter the *minimal order* of (a_j)), and therefore when this occurs.

THEOREM 3.1. *Let (a_j) be a sequence satisfying a recurrence relation of order k with q distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_q$ with respective multiplicities $k_1, k_2, \ldots, k_q$.*

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Let $K_{l,i}$ be the unique constants so that

$$(a_j) = \sum_{l=1}^{q} \sum_{i=1}^{k_l} K_{l,i} \left(j^{i-1} \lambda_l^j \right).$$

Let M_l be the maximal value of i so that $K_{l,i}$ is nonzero, or zero if $K_{l,i}$ is zero for all i. Then the minimal order recurrence satisfied by (a_j) is the recurrence with the characteristic polynomial $f(\lambda) = \prod_{l=1}^{q} (\lambda - \lambda_l)^{M_l}$.

Proof. Since (a_j) is in the span of $\bigcup_{l=1}^{q} \left\{ (\lambda_l^j), (j\lambda_l^j), \ldots, (j^{M_l-1}\lambda_l^j) \right\}$, (a_j) is in the solution set of this recurrence, by Theorem 2.3.

By Corollary 2.4, any recurrence relation of order less than $\deg f(\lambda)$ satisfied by (a_j) must have a characteristic polynomial that divides $f(\lambda)$. To eliminate the possibility of satisfying an even lower recurrence relation, suppose (a_j) is in the solution set of a recurrence relation with characteristic polynomial $g(\lambda) = f(\lambda)/(\lambda - \lambda_r)$, for some r.

By Theorem 2.3, the solution set $\ker g(\Lambda)$ has the basis

$$\bigcup_{l=1}^{q} \left\{ (\lambda_l^j), (j\lambda_l^j), \ldots, (j^{M_l-1}\lambda_l^j) \right\} \setminus \left\{ \left(j^{M_r-1} \lambda_r^j \right) \right\}.$$

Then $(a_j) - K_{r,M_r}(j^{M_r-1}\lambda_l^j)$ is in the solution set $\ker g(\Lambda)$, since it is in the span of this basis. Next, since $\ker g(\Lambda)$ is a vector space,

$$\frac{1}{K_{r,M_r}} \left[(a_j) - \left((a_j) - K_{r,M_r}(j^{M_r-1}\lambda_l^j) \right) \right] = (j^{M_l-1}\lambda^j)$$

is also in $\ker g(\Lambda)$. But this contradicts that $\bigcup_{l=1}^{q} \left\{ (\lambda_l^j), (j\lambda_l^j), \ldots, (j^{M_l-1}\lambda_l^j) \right\}$ is a basis, since then the basis vector $(j^{M_r-1}\lambda_r^j)$ is a linear combination of the other basis vectors. $\square$

Thus, we may obtain the minimal order recurrence of (a_j) by calculating its representation as a linear combination of fundamental solutions, then dropping those eigenvalues whose fundamental solutions are "unused." This allows us to lower the upper bound of Theorem 1.1:

Corollary 3.2. *With (a_j) as in Theorem 3.1, $m, n \geq \sum_{l=1}^{q} M_l$,*

$$\operatorname{rank} M_{m,n}((a_j)) \leq \sum_{l=1}^{q} M_l.$$

Other characterizations of the minimal order recurrence relation satisfied by a sequence exist. For example, [5, p. 204] provides a characterization in terms of the generating function of (a_j).

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A publication of the International Linear Algebra Society
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4. Width rank drops. In this section, we calculate the rank of recurrence matrices provided a recurrence sequence and its minimal order recurrence relation. We begin with a lemma.

LEMMA 4.1. *Suppose $k_1, k_2, \dots, k_q$ are positive integers with sum k and $\lambda_1, \lambda_2, \dots, \lambda_q$ are nonzero complex numbers. Let*

$$B_l = \begin{bmatrix} \lambda_l^0 & 0^1\lambda_l^0 & \cdots & 0^{k_l-1}\lambda_l^0 \\ \lambda_l^1 & 1^1\lambda_l^1 & \cdots & 1^{k_l-1}\lambda_l^1 \\ \vdots & \vdots & & \vdots \\ \lambda_l^{k-1} & (k-1)^1\lambda_l^{k-1} & \cdots & (k-1)^{k_l-1}\lambda_l^{k-1} \end{bmatrix}.$$

Then the $k \times k$ matrix

$$M = \begin{bmatrix} B_1 & B_2 & \cdots & B_q \end{bmatrix}$$

has rank equal to the number of distinct columns of M.

Proof. Clearly, the rank of M is less than or equal to the number of distinct columns, since repeated columns contribute nothing to the rank of M. To see that the distinct columns are linearly independent, let $f(\lambda) = \prod_{l=1}^{q}(\Lambda - \lambda_l)^{k_l}$. Note that then the distinct columns are the images of distinct basis vectors for $\ker f(\Lambda)$ under the isomorphism $\phi : \ker f(\Lambda) \to \mathbb{C}^k$, defined in Lemma 2.1, that takes each sequence to its initial k values. $\square$

This also proves the well-known fact that the rank of a Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & & \vdots \\ \lambda_1^n & \lambda_2^n & \cdots & \lambda_n^n \end{bmatrix}$$

is the number of distinct values taken on by $\lambda_1, \lambda_2, \dots, \lambda_n$.

We now calculate the rank of a recurrence matrix. We are motivated by the following observation. Suppose (a_j) satisfies a recurrence relation with non-repeated eigenvalues $\lambda_1, \dots, \lambda_k$. Then $(a_j) = \sum_{i=1}^{k} K_i(\lambda_i^j)$ for some constants $K_1, \dots, K_k$, and we have the factorization

$$M_{m,n}((a_j)) = \begin{bmatrix} 1 & \cdots & 1 \\ \lambda_1^n & \cdots & \lambda_k^n \\ \vdots & & \vdots \\ (\lambda_1^n)^m & \cdots & (\lambda_k^n)^m \end{bmatrix} \begin{bmatrix} K_1 & & & 0 \\ & K_2 & & \\ & & \ddots & \\ 0 & & & K_k \end{bmatrix} \begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^n \\ 1 & \lambda_2 & \cdots & \lambda_2^n \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_k & \cdots & \lambda_k^n \end{bmatrix}.$$

Each row of the rightmost matrix consists of the first n values of a fundamental solution. Similarly, each column of the leftmost matrix takes the form of the first m

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A publication of the International Linear Algebra Society
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values of a fundamental solution. Lemma 4.1 then allows the rank of each matrix to be determined. The following proof utilizes a factorization with the same properties in the general case.

THEOREM 4.2. *Let (a_j) be a recurrence sequence with minimal order k and q distinct eigenvalues $\lambda_1, \dots, \lambda_q$ with multiplicities $k_1, \dots, k_q$ respectively. Let S_n be the set of distinct values taken by $\lambda_1^n, \dots, \lambda_q^n$. Then if $m, n \geq k$,*

$$\operatorname{rank} M_{m,n}(a_j) = \sum_{s \in S_n} \max_l \{k_l : \lambda_l^n = s\}.$$

Proof. By Theorem 2.3,

$$(a_j) = \sum_{l=1}^{q} \sum_{i=1}^{k_l} K_{l,i} \left(j^{i-1} \lambda_l^j \right)$$

for some constants $K_{l,i}$. Since (a_j) has minimal order k, K_{l,k_l} is nonzero for each l.

Let $A_{l,i} = M_{m,n}\left((j^{i-1}\lambda_l^j) \right)$. Then

$$M_{m,n}((a_j)) = \sum_{l=1}^{q} \sum_{i=1}^{k_l} K_{l,i} A_{l,i}.$$

Let $\mathbf{a}_{l,i}$ be the first row of $A_{l,i}$,

$$\mathbf{a}_{l,i} = \begin{bmatrix} 0^{i-1}\lambda_l^0 & 1^{i-1}\lambda_l^1 & \cdots & (n-1)^{i-1}\lambda_l^{n-1} \end{bmatrix}.$$

The cth row of $A_{l,i}$ (with c starting at 0) is

$$\begin{bmatrix} (cn)^{i-1}\lambda_l^{cn} & (cn+1)^{i-1}\lambda_l^{cn+1} & \cdots & (cn+n-1)^{i-1}\lambda_l^{cn+n-1} \end{bmatrix}.$$

We would like to rewrite this as linear combinations of $\mathbf{a}_{l,1}, \dots, \mathbf{a}_{l,i}$, in order to factor $M_{m,n}((a_j))$. By the binomial theorem, the element in the cth row and jth column of $A_{l,i}$ is

$$\begin{aligned} (j+cn)^{i-1}\lambda_l^{j+cn} &= \sum_{r=0}^{i-1} \binom{i-1}{r} j^r (cn)^{(i-1)-r} \lambda_l^{j+cn} \\ &= \sum_{r=0}^{i-1} (cn)^{(i-1)-r} (\lambda_l^n)^c \binom{i-1}{r} j^r \lambda_l^j. \end{aligned}$$

So, the cth row of $A_{l,i}$ may be expressed as the product

$$\begin{bmatrix} (cn)^{i-1}(\lambda_l^n)^c & \cdots & cn(\lambda_l^n)^c & (\lambda_l^n)^c \end{bmatrix} \begin{bmatrix} \binom{i-1}{0} & & & 0 \\ & \binom{i-1}{1} & & \\ & & \ddots & \\ 0 & & & \binom{i-1}{i-1} \end{bmatrix} \begin{bmatrix} \mathbf{a}_{l,1} \\ \mathbf{a}_{l,2} \\ \vdots \\ \mathbf{a}_{l,i} \end{bmatrix}.$$

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Let

$$\mathbf{b}_{l,i} = \begin{bmatrix} 0^{i-1}(\lambda_l^n)^0 \\ n^{i-1}(\lambda_l^n)^1 \\ \vdots \\ ((m-2)n)^{i-1}(\lambda_l^n)^{m-2} \\ ((m-1)n)^{i-1}(\lambda_l^n)^{m-1} \end{bmatrix}.$$

Then we may factor $A_{l,i}$ as

$$A_{l,i} = B_{l,i}C_{l,i}D_{l,i},$$

where $B_{l,i}$ is the $m \times i$ matrix

$$B_{l,i} = \begin{bmatrix} \mathbf{b}_{l,i} & \mathbf{b}_{l,i-1} & \cdots & \mathbf{b}_{l,1} \end{bmatrix},$$

$C_{l,i}$ is the $i \times i$ diagonal matrix

$$C_{l,i} = \begin{bmatrix} \binom{i-1}{0} & & & 0 \\ & \binom{i-1}{1} & & \\ & & \ddots & \\ 0 & & & \binom{i-1}{i-1} \end{bmatrix},$$

and finally, $D_{l,i}$ is the $i \times n$ matrix

$$D_{l,i} = \begin{bmatrix} \mathbf{a}_{l,1} \\ \mathbf{a}_{l,2} \\ \vdots \\ \mathbf{a}_{l,i} \end{bmatrix}.$$

Since for an eigenvalue λ_l the $B_{l,i}$ matrices share columns and the $D_{l,i}$ matrices share rows, we may combine the matrices $B_{l,i}, C_{l,i}$, and $D_{l,i}$ as follows:

$$\sum_{i=1}^{q_l} K_{l,i}A_{l,i} = \sum_{i=1}^{q_l} B_{l,i}(K_{l,i}C_{l,i})D_{l,i} = B_lC_lD_l,$$

where $B_l = B_{l,q_l}$, $D_l = D_{l,q_l}$, and C_l is the $q_l \times q_l$ lower triangular matrix

$$C_l = \begin{bmatrix} K_{l,q_l}\binom{q_l-1}{0} & 0 & \cdots & 0 & 0 \\ K_{l,q_l-1}\binom{q_l-2}{0} & K_{l,q_l}\binom{q_l-1}{1} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_{l,2}\binom{1}{0} & K_{l,3}\binom{2}{1} & \cdots & K_{l,q_l}\binom{q_l-1}{q_l-2} & 0 \\ K_{l,1}\binom{0}{0} & K_{l,2}\binom{1}{1} & \cdots & K_{l,q_l-1}\binom{q_l-2}{q_i-2} & K_{l,q_l}\binom{q_l-1}{q_l-1} \end{bmatrix}.$$

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A publication of the International Linear Algebra Society
Volume 30, pp. 422-436, August 2015

Now we may write $M_{m,n}((a_j)) = \sum_{l=1}^{q} B_l C_l D_l$ as BCD, where B is the block matrix

$$\begin{bmatrix} B_1 & B_2 & \cdots & B_q \end{bmatrix},$$

C is the block diagonal matrix

$$\begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_q \end{bmatrix},$$

and D is the block matrix

$$\begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_q \end{bmatrix}.$$

Since the K_{l,q_l}s are nonzero and the corresponding binomial coefficients are nonzero, C is a triangular matrix with a nonzero diagonal, and therefore, C has full rank. Also, $\operatorname{rank} D = \operatorname{rank} D^T = k$ by Lemma 4.1.

Since C is a $k \times k$ matrix of rank k and D is a $k \times m$ matrix of rank k, CD is a $k \times m$ matrix of rank k, and it follows $\operatorname{rank} M_{m,n}((a_j)) = \operatorname{rank} BCD = \operatorname{rank} B$. The columns of B have the same form as the canonical fundamental solutions of recurrence relations. By Lemma 4.1, the rank of B is the number of distinct columns of B. Thus,

$$\operatorname{rank} M_{m,n}((a_j)) = \operatorname{rank} B = \sum_{s \in S_n} \max_{l} \{q_l : \lambda_l^n = s\},$$

as desired. □

Corollary 4.3. *Let (a_j) be a recurrence sequence with minimal order k and non-repeated eigenvalues $\lambda_1, \ldots, \lambda_k$. Then $M_{m,n}((a_j))$, where $m, n \geq k$, has rank equal to the number of distinct values taken on by $\lambda_1^n, \ldots, \lambda_k^n$.*

For a given recurrence sequence (a_j), calculating the rank of $M_{m,n}((a_j))$ proceeds in two steps. First, one uses Theorem 3.1 to calculate the minimal order recurrence relation satisfied by (a_j). Then one uses Theorem 4.2 to obtain the actual rank. The same procedure may be followed for an inhomogeneous recurrence sequence after first applying the reduction of Section 2.1.

Corollary 4.4. *If the rank of a recurrence matrix drops as in Theorem 4.2 for a matrix with n columns, then it also drops for a matrix with kn columns, for any natural number k.*

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A publication of the International Linear Algebra Society
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Proof. By Theorem 4.2, the rank drops whenever $\lambda_i^n = \lambda_j^n$ for distinct i, j. Then for any natural number k, $\lambda_i^{kn} = \lambda_j^{kn}$. $\square$

Thus, if there are rank drops associated with the width of the recurrence matrix, those rank drops are periodic in n. Moreover, if $\lambda_i^n = \lambda_j^n$, then λ_i differs from λ_j by a factor of an nth root of unity. Accordingly, we say a recurrence relation with eigenvalues $\lambda_1, \ldots, \lambda_q$ has a *width rank drop of periodicity* p if for some $i \neq j$, $\lambda_i = \omega\lambda_j$, where ω is a primitive pth root of unity.

Width rank drops of any periodicity p are possible, even for recurrences of order two. To see this, consider the recurrence with characteristic polynomial $(\lambda-1)(\lambda-\zeta)$, where ζ is a primitive pth root of unity.

REMARK 4.5. Recurrence relations whose eigenvalues differ by a factor of a root of unity are called *degenerate* (See, for example [2, p. 16]). By our previous discussion, the recurrence relations that result in matrices with periodic rank drops coincide with the degenerate recurrence sequences.

To see concretely what Theorem 4.2 says about width dependence, let us return to Example 1.3.

EXAMPLE 4.6. The characteristic polynomial of the recurrence $a_j = a_{j-2}$ is $\lambda^2 - 1$, which has the roots $\lambda_1 = 1, \lambda_2 = -1$. Therefore, the recurrence relation has fundamental solutions (1^j) and $((-1)^j)$. In particular, the sequence (a_j) from Example 1.3 may be expressed as

$$a_j = 1^j + (-1)^j = \begin{cases} 2 & \text{if } n \text{ even,} \\ 0 & \text{if } n \text{ odd.} \end{cases}$$

Consider the 3×3 matrix

$$M_{5,5}((a_j)) = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}.$$

We may apply the factorization from the proof to obtain

$$M_{3,3}((a_j)) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}.$$

In this case, B, C, and D have rank 2, so $M_{3,3}((a_j))$ has rank 2. Since $\lambda_2^3 = (-1)^3 = -1$ while $\lambda_1^3 = 1^3 = 1$, the number of distinct values taken by λ_i^n is also 2.

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Meanwhile, the 4×4 matrix factors as

$$M_{4,4}((a_j)) = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}.$$

We see C and D have rank 2, but B has rank 1, so $\operatorname{rank} M_{4,4}((a_j)) = 1$. Since $\lambda_2^4 = (-1)^4 = 1 = 1^4 = \lambda_1^4$, the numbers λ_i^n take only 1 value, in accordance with the rank of $M_{4,4}((a_j))$.

We might appear to have a counterexample if we seed the sequence with $a_1 = 1, a_2 = 1$, since then

$$M_{3,3}((a_j)) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

which has rank 1 although $(-1)^3 \neq 1^3$. However, this particular solution is given by $(a_j) = 1 \times (1^j) + 0 \times ((-1)^j)$. Since one of the coefficients is zero, $a_j = a_{j-2}$ is the not the minimal order recurrence of the sequence, and therefore, the hypotheses of the theorem are not satisfied.

5. Width rank drops in the order two case. As an application of the theory we have developed, we now calculate the rank of order 2 recurrence matrices in terms of their seeds. This completes Theorem 2 of [4].

THEOREM 5.1. *Suppose $m, n \geq 2$ and (a_j) satisfies the second order recurrence relation $a_j = c_1 a_{j-1} + c_2 a_{j-2}$. Then*

$$\operatorname{rank} M_{m,n}((a_j)) = \begin{cases} 0 & \text{if } a_0 = a_1 = 0, \\ 1 & \text{if } a_1^2 - c_1 a_1 a_0 - c_2 a_0^2 = 0, \\ 1 & \text{if } c_1^2 + 4c_2 \neq 0 \text{ and } \left(\frac{c_1 + \sqrt{c_1^2 + 4c_2}}{c_1 - \sqrt{c_1^2 + 4c_2}}\right)^n = 1, \\ 2 & \text{else.} \end{cases}$$

Proof. If $a_0 = a_1 = 0$, then $(a_j) = (0)$, yielding the result. For the remainder of the proof, we assume that $a_0 \neq 0$ or $a_1 \neq 0$.

Let λ_1, λ_2 be the eigenvalues of the recurrence relation. We first calculate the expression of (a_j) in terms of fundamental solutions to find its minimal order. Suppose first that $\lambda_1 \neq \lambda_2$. Then $(a_j) = K_1(\lambda_1^j) + K_2(\lambda_2^j)$. The initial values determine K_1, K_2

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A publication of the International Linear Algebra Society
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by the formula

$$\begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}.$$

Multiplying by the inverse,

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{\lambda_2 - \lambda_1} \begin{bmatrix} \lambda_2 a_0 - a_1 \\ -\lambda_1 a_0 + a_1 \end{bmatrix}$$

By Theorem 3.1, the minimal order of (a_j) is 1 if and only if K_1 or K_2 is zero. This happens if and only if

$$\begin{aligned} 0 &= (\lambda_2 a_0 - a_1)(\lambda_1 a_0 - a_1) \\ &= \lambda_1\lambda_2 - (\lambda_1 - \lambda_2)a_0 a_1 + a_1^2 \\ &= a_1^2 - c_1 a_1 a_0 - c_2 a_0^2, \end{aligned}$$

since λ_1, λ_2 are roots of the characteristic polynomial.

Next suppose $\lambda_1 = \lambda_2$. Then $(a_j) = K_1(\lambda_1^j) + K_2(j\lambda_1^j)$. K_1, K_2 are determined by

$$\begin{bmatrix} 1 & 0 \\ \lambda_1 & \lambda_1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}.$$

Multiplying by the inverse,

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \frac{1}{\lambda_1} \begin{bmatrix} \lambda_1 & 0 \\ -\lambda_1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{\lambda_1} \begin{bmatrix} \lambda_1 a_0 \\ -\lambda_1 a_0 + a_1 \end{bmatrix}$$

K_1 is necessarily nonzero. However, K_2 is zero if and only if

$$0 = (\lambda_1 a_0 - a_1)^2 = a_1^2 - c_1 a_1 a_0 - c_2 a_0^2.$$

If the minimal order is 1, then, since (a_j) is not identically 0, $M_{m,n}((a_j)) = 1$. This proves the second case of the theorem.

It remains to apply Theorem 4.2 to the case that the minimal order of (a_j) is 2. If $\lambda_1 = \lambda_2$, there are no width rank drops. Thus, if the discriminant $c_1^2 + 4c_2 = 0$, the rank is two. If $\lambda_1 \neq \lambda_2$, then it remains to check whether $\lambda_1^n = \lambda_2^n$. By the quadratic formula, this occurs when $(c_1 + \sqrt{c_1^2 + 4c_2})^n = (c_1 - \sqrt{c_1^2 + 4c_2})^n$. □

In the theorem above, the lowered rank in the first two cases is due to an order rank drop, while the reduced rank in the third case is due to a width rank drop.

Acknowledgment. The author would like to thank V. Peterson and C. Lee for their thorough reading of early drafts and for their assistance in improving the presentation of this paper.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 422-436, August 2015

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