The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 112-117, June 2002

ELA

VARIATIONAL CHARACTERIZATIONS OF THE SIGN-REAL AND THE SIGN-COMPLEX SPECTRAL RADIUS*

SIEGFRIED M. RUMP†

Key words. Generalized spectral radius, sign-real spectral radius, sign-complex spectral radius, Perron-Frobenius theory.

AMS subject classifications. 15A48, 15A18

Abstract. The sign-real and the sign-complex spectral radius, also called the generalized spectral radius, proved to be an interesting generalization of the classical Perron-Frobenius theory (for nonnegative matrices) to general real and to general complex matrices, respectively. Especially the generalization of the well-known Collatz-Wielandt max-min characterization shows one of the many one-to-one correspondences to classical Perron-Frobenius theory. In this paper the corresponding inf-max characterization as well as variational characterizations of the generalized (real and complex) spectral radius are presented. Again those are almost identical to the corresponding results in classical Perron-Frobenius theory.

1. Introduction. Denote $\mathbb{R}_+ := \{x \geq 0 : \ x \in \mathbb{R}\}$, and let $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{C}\}$. The generalized spectral radius is defined [6] by

$$(1.1)\quad \rho^{\mathbb{K}}(A) := \max\{|\lambda| : \ \exists\, 0 \neq x \in \mathbb{K}^n, \ \exists\, \lambda \in \mathbb{K}, \ |Ax| = |\lambda x|\} \quad \text{for} \quad A \in M_n(\mathbb{K}).$$

Note that absolute value and comparison of matrices and vectors are always to be understood componentwise. For example, $A \leq |C|$ for $A \in M_n(\mathbb{R})$, $C \in M_n(\mathbb{C})$ is equivalent to $A_{ij} \leq |C_{ij}|$ for all i, j.

For $\mathbb{K} = \mathbb{R}_+$ the quantity in (1.1) is the classical Perron root, for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ it is the sign-real or sign-complex spectral radius, respectively. Note that the quantities are only defined for matrices over the specific set $\mathbb{K}$, and also note that for $\rho^{\mathbb{R}}$ the maximum $|\lambda|$ is only taken over real λ and real x. Vectors $0 \neq x \in \mathbb{K}^n$ and scalars $\lambda \in \mathbb{K}$ satisfying the nonlinear eigenequation $|Ax| = |\lambda x|$ are also called generalized eigenvectors and generalized eigenvalues, respectively.

Denote the set of signature matrices over $\mathbb{K}$ by $\mathcal{S}(\mathbb{K})$, which are diagonal matrices S with $|S_{ii}| = 1$ for all i. In short notation $S \in \mathcal{S}(\mathbb{K}) :\Leftrightarrow \ S \in M_n(\mathbb{K})$ and $|S| = I$. For $\mathbb{K} = \mathbb{R}_+$ this is just the identity matrix I, for $\mathbb{K} = \mathbb{R}$ the set of $S = \operatorname{diag}(\pm 1)$ or diagonal orthogonal, and for $\mathbb{K} = \mathbb{C}$ the set of diagonal unitary matrices. Obviously, for $y \in \mathbb{K}^n$ there is $S \in \mathcal{S}(\mathbb{K})$ with $Sy \geq 0$. In case $|y| > 0$, this S is uniquely determined. Note that $S^{-1} = S^* \in \mathcal{S}(\mathbb{K})$ for all $S \in \mathcal{S}(\mathbb{K})$.

By definition (1.1) there is $y \in \mathbb{K}^n$ with $|Ay| = |ry| = r|y|$ for $r := \rho^{\mathbb{K}}(A)$, and therefore for $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{C}\}$,

$$(1.2)\qquad \exists\, S \in \mathcal{S}(\mathbb{K}), \ \ \exists\, 0 \neq y \in \mathbb{K}^n \ : \ SAy = ry$$

*Received by the editors on 20 April 2002. Final manuscript accepted on 7 June 2002. Handling editor: Ludwig Elsner.

†Institut für Informatik III, Technical University Hamburg-Harburg, Schwarzenbergstr. 95, 21071 Hamburg, Germany (rump@tu-harburg.de).

The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 112-117, June 2002

and

$$\exists\, S_1, S_2 \in \mathcal{S}(\mathbb{K}),\;\; \exists\, x \geq 0,\; x \neq 0 \;:\; S_1 A S_2 x = rx. \tag{1.3}$$

Among the variational characterizations of the Perron root are

$$\max_{x\geq 0} \min_{x_i \neq 0} \frac{(Ax)_i}{x_i} = \rho^{\mathbb{R}_+}(A) = \rho(A) = \inf_{x>0} \max_i \frac{(Ax)_i}{x_i} \quad \text{for} \quad A \geq 0 \tag{1.4}$$

and

$$\max_{x\geq 0} \min_{\substack{y\geq 0\\ y^T x\neq 0}} \frac{y^T A x}{y^T x} = \rho(A) = \min_{y\geq 0} \max_{\substack{x\geq 0\\ y^T x\neq 0}} \frac{y^T A x}{y^T x} \quad \text{for} \quad A \geq 0.$$

The purpose of this paper is to prove a generalization of both characterizations for the generalized spectral radius.

We note that the only non-obvious property of the generalized spectral radius we use is [6, Corollary 2.4]

$$\rho^{\mathbb{K}}(A[\mu]) \leq \rho^{\mathbb{K}}(A) \quad \text{for } \mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{C}\},\; A \in M_n(\mathbb{K}) \text{ and } \mu \subseteq \{1, \ldots, n\}. \tag{1.5}$$

2. Variational characterizations. For the following results we need three preparatory lemmata, the first showing that for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ there exists a generalized eigenvector in every orthant.

LEMMA 2.1. *Let* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ *and* $A \in M_n(\mathbb{K})$ *be given. Then*

$$\forall\, S \in \mathcal{S}(\mathbb{K}),\;\; \exists\, 0 \neq z \in \mathbb{K}^n,\;\; \exists\, \lambda \in \mathbb{R}_+ \;:\quad Sz \geq 0,\; |Az| = \lambda |z|.$$

REMARK 2.2. The condition $Sz \geq 0$ for $z \in \mathbb{K}^n$ means $Sz \in \mathbb{R}^n$ and $Sz \geq 0$, or shortly $Sz \in \mathbb{R}^n_+$. Note that Lemma 2.1 is also true for $\mathbb{K} = \mathbb{R}_+$, in which case $S \in \mathcal{S}(\mathbb{K})$ implies $S = I$.

Proof of Lemma 2.1. Let fixed $S \in \mathcal{S}(\mathbb{K})$ be given and define $\mathcal{O} := \{z \in \mathbb{K}^n : \|z\|_1 = 1,\; Sz \geq 0\}$. The set $\mathcal{O}$ is nonempty, compact and convex. If there exists some $z \in \mathcal{O}$ with $Az = 0$ we are finished with $\lambda = 0$. Suppose $Az \neq 0$ for all $z \in \mathcal{O}$ and define $\varphi(x) := \|Ax\|_1^{-1} \cdot S^*|Ax|$. Then φ is well-defined and continuous on $\mathcal{O}$, and $\varphi :\; \mathcal{O} \to \mathcal{O}$, such that by Brouwer's theorem there exists a fixed point $z \in \mathcal{O}$ with $\varphi(z) = \|Az\|_1^{-1} \cdot S^*|Az| = z$. Then $|Az| = \lambda S z = \lambda |z|$ with $\lambda = \|Az\|_1$. ☐

The next lemma states a property of vectors out of the interior of a certain orthant.

LEMMA 2.3. *Let* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $A \in M_n(\mathbb{K})$ *and define* $r := \rho^{\mathbb{K}}(A)$. *Then*

$$\forall\, S \in \mathcal{S}(\mathbb{K}),\;\; \forall\, \varepsilon > 0,\;\; \exists\, z \in \mathbb{K}^n \;:\quad Sz > 0,\; |Az| \leq (r + \varepsilon) \cdot |z|.$$

Proof. We proceed by induction. For $n = 1$, it is $r = |A_{11}| \in \mathbb{R}_+$, and $z := \operatorname{sign}(S_{11}) \in \mathbb{K}$ does the job. Suppose the lemma is proved for dimension less than n. For given $S \in \mathcal{S}(\mathbb{K})$ there exists by Lemma 2.1 some $0 \neq z \in \mathbb{K}^n$ and $\lambda \in \mathbb{R}_+$ with $Sz \geq 0$

The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 112-117, June 2002

ELA

and $|Az| = \lambda|z|$. Then $\lambda \le r$ by definition (1.1). If $Sz > 0$ we are finished. Let $\mu := \{j : z_j \neq 0\}$ and let $\overline{\mu} := \{1,\dots,n\}\backslash\mu$ such that

$$\left|\begin{bmatrix} T & U \\ V & W \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix}\right| = \lambda\left|\begin{bmatrix} x \\ 0 \end{bmatrix}\right| \quad \text{with}$$

$$T = A[\mu],\ U = A[\mu,\overline{\mu}],\ V = A[\overline{\mu},\mu],\ W = A[\overline{\mu}],\ z[\mu] = x \quad \text{and} \quad z[\overline{\mu}] = 0.$$

Then $|Tx| = \lambda|x|$, $Vx = 0$ and $|x| > 0$.

By the induction hypothesis there exists $y' \in \mathbb{K}^{|\overline{\mu}|}$ with $S[\overline{\mu}]y' > 0$ and

$$|Wy'| \ \le\ (\rho^{\mathbb{K}}(W) + \varepsilon)|y'| \ \le\ (r+\varepsilon)|y'|,$$

where the latter inequality follows by (1.5). Define

$$\alpha := \begin{cases} \min\limits_i \left|\dfrac{x_i}{(Uy')_i}\right| & \text{for } Uy' \neq 0 \\ 1 & \text{otherwise,} \end{cases}$$

and set $y := \alpha y'$. Then $|y| > 0$ and

$$\left|A \cdot \begin{bmatrix} x \\ \varepsilon y \end{bmatrix}\right| = \left|\begin{bmatrix} Tx + \varepsilon Uy \\ \varepsilon Wy \end{bmatrix}\right| \le \begin{bmatrix} \lambda|x| + \varepsilon\alpha|Uy'| \\ \varepsilon\alpha(r+\varepsilon)|y'| \end{bmatrix} \le (r+\varepsilon)\begin{bmatrix} |x| \\ \varepsilon|y| \end{bmatrix}. \quad \square$$

The above lemma is obviously not true when replacing $r+\varepsilon$ by r, as the example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ with $\rho^{\mathbb{K}}(A) = 1$ for $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{C}\}$ shows. It is, at least for $\mathbb{K} = \mathbb{R}$, also not valid for irreducible $|A|$. Consider

$$A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}.$$

It has been shown in [5, Lemma 5.6] that $\rho^{\mathbb{R}}(A) = 1$. We show that $|Au| \le u$ is not possible for $u > 0$. Set $u := (x,y,z)^T$, then $|Au| \le u$ is equivalent to

$$\begin{array}{ccccc} -x & \le & y+z & \le & x \\ -y & \le & -x+z & \le & y \\ -z & \le & -x-y & \le & z. \end{array}$$

The second and third row imply that

$$x \le y+z \quad \text{and} \quad y \le -x+z,$$

and by the first and second row,

$$x = y+z \quad \text{and} \quad y = -x+z$$

so that $y = x - z = -x + z$ and therefore $y = 0$, which means u cannot be positive.

The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 112-117, June 2002

ELA

Third, we need a generalization of a theorem by Collatz [3, Section 2] to the complex case.

LEMMA 2.4. *Let* $A \in M_n(\mathbb{C})$, $A^*z = \lambda z$ *for* $0 \neq z \in \mathbb{R}^n$, $\lambda \in \mathbb{C}$. *Then for all* $x \in \mathbb{R}^n$ *with* $|x| > 0$ *and* $x_i z_i \geq 0$ *for all* i *the following estimations hold true:*

$$\begin{array}{ccccc} \min \operatorname{Re} \mu_i & \leq & \operatorname{Re} \lambda & \leq & \max \operatorname{Re} \mu_i \\ \min \operatorname{Im} \mu_i & \leq & \operatorname{Im} \lambda & \leq & \max \operatorname{Im} \mu_i, \end{array}$$

where $\mu_i := (Ax)_i/x_i$ *for* $1 \leq i \leq n$.

REMARK 2.5. Note that x and the left eigenvector z of A are assumed to be real.

Proof of Lemma 2.4. Similar to Collatz's original proof for the case $A \geq 0$ we note that

$$\sum_i (\lambda - \mu_i) x_i z_i = \sum_i x_i (A^* z)_i - \sum_i (Ax)_i z_i = x^* A^* z - z^* A x = 0,$$

the latter because x and z are real. Now $x_i z_i$ are real nonnegative for all i, and by $|x| > 0$ not all products $x_i z_i$ can be zero. The assertion follows. □

With these preparations we can prove the first two-sided characterization of $\rho^{\mathbb{K}}$.

THEOREM 2.6. *Let* $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{C}\}$ *and* $A \in M_n(\mathbb{K})$. *Then*

$$\max_{S \in \mathcal{S}(\mathbb{K})} \max_{\substack{x \in \mathbb{K}^n \\ Sx \geq 0}} \min_{x_i \neq 0} \left| \frac{(Ax)_i}{x_i} \right| = \rho^{\mathbb{K}}(A) = \max_{S \in \mathcal{S}(\mathbb{K})} \inf_{\substack{x \in \mathbb{K}^n \\ Sx > 0}} \max_i \left| \frac{(Ax)_i}{x_i} \right|. \tag{2.1}$$

REMARK 2.7. The characterization is almost identical to the classical Perron-Frobenius characterization (1.4). The difference is that for nonnegative A the nonnegative orthant is the generic one, and vectors x can be restricted to this generic orthant. For general real or complex matrices, there is no longer a generic orthant, and therefore the max-min and inf-max characterization is maximized over all orthants. Note that in the left hand side the two maximums can be replaced by $\max_{x \in \mathbb{K}^n}$, but are separated for didactic purposes.

Proof of Theorem 2.6. The result is well-known for $\mathbb{K} = \mathbb{R}_+$, and the left equality was shown in [5, Theorem 3.1] for $\mathbb{K} = \mathbb{R}$, and for $\mathbb{K} = \mathbb{C}$ it was shown in a different context in [4] and [2]; see also [6, Theorem 2.3]. We need to prove the right equality for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $S \in \mathcal{S}(\mathbb{K})$ be fixed but arbitrary and denote $r := \rho^{\mathbb{K}}(A)$. By Lemma 2.3, for every $\varepsilon > 0$ there exists some $x \in \mathbb{K}^n$ with $Sx > 0$ and $|Ax| \leq (r + \varepsilon)|x|$, so that r is larger than or equal to the r.h.s. of (2.1). We will show r is less than or equal to the r.h.s. of (2.1) to finish the proof. By (1.3) and $\rho^{\mathbb{K}}(A^*) = \rho^{\mathbb{K}}(A)$ there is $S_1, S_2 \in \mathcal{S}(\mathbb{K})$ and $0 \neq z \in \mathbb{R}^n$ with $z \geq 0$ and $S_1 A^* S_2 z = rz$. Then for any $x \in \mathbb{K}^n$ with $S_1 x > 0$, Lemma 2.4 implies that

$$\max_i \left| \frac{(Ax)_i}{x_i} \right| = \max_i \left| \frac{((S_2^* A S_1^*) \cdot S_1 x)_i}{(S_1 x)_i} \right| \geq \operatorname{Re} r = r. \qquad \square$$

Finally we give a second two-sided characterization of the generalized spectral radius.

The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 112-117, June 2002

ELA

THEOREM 2.8. *Let $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{C}\}$ and $A \in M_n(\mathbb{K})$. Then*

$$\max_{S_1,S_2\in\mathcal{S}(\mathbb{K})} \max_{\substack{x\in\mathbb{K}^n\\ S_1x\geq 0}} \min_{\substack{y\in\mathbb{K}^n\\ S_2y\geq 0\\ |y^*|\,|x|\neq 0}} \frac{|y^*Ax|}{|y^*|\,|x|} = \rho^{\mathbb{K}}(A) = \max_{S_1,S_2\in\mathcal{S}(\mathbb{K})} \min_{\substack{y\in\mathbb{K}^n\\ S_2y\geq 0}} \max_{\substack{x\in\mathbb{K}^n\\ S_1x\geq 0\\ |y^*|\,|x|\neq 0}} \frac{|y^*Ax|}{|y^*|\,|x|}. \tag{2.2}$$

Proof. Let, according to (1.2), $SAx = rx$ for $S \in \mathcal{S}(\mathbb{K})$, $0 \neq x \in \mathbb{K}^n$ and $r = \rho^{\mathbb{K}}(A)$. Define S_1 such that $S_1x \geq 0$ and set $S_2 = S_1S$. Then for every $y \in \mathbb{K}^n$ with $S_2y \geq 0$ and $|y^*|\,|x| \neq 0$, it is $S_1x = |x|$, $S_2y = |y|$, $S_2^*S_1S = I$ and

$$y^*Ax = y^*S_2^*S_1SAx = ry^*S_2^*S_1x = r|y^*|\,|x|, \quad \text{or} \quad |y^*Ax| = r|y^*|\,|x|.$$

That means for the specific choice of S_1, S_2 and x, the ratio $|y^*Ax|/(|y^*|\,|x|)$ is equal to r independent of the choice of y provided $S_2y \geq 0$. Therefore, both the left and the right hand side of (2.2) are greater than or equal to $r = \rho^{\mathbb{K}}(A)$. This proves also that the extrema are actually achieved.

On the other hand, let $S_1, S_2 \in \mathcal{S}(\mathbb{K})$ and $x \in \mathbb{K}^n$, $S_1x \geq 0$ be fixed but arbitrarily given. Denote $\mu := \{j : \ x_j \neq 0\}$, $k := |\mu|$, and $\overline{\mu} := \{1, \ldots, n\}\backslash\mu$. By Lemma 2.1, there exists $\tilde{y} \in \mathbb{K}^k$ with $\tilde{y} \neq 0$, $S_2[\mu]\,\tilde{y} \geq 0$ and $|A^*[\mu] \cdot \tilde{y}\,| = \lambda|\,\tilde{y}\,|$ for $\lambda \geq 0$. Therefore $\lambda \leq \rho^{\mathbb{K}}(A^*[\mu]) = \rho^{\mathbb{K}}(A[\mu])$. Define $y \in \mathbb{K}^n$ by $y[\mu] := \tilde{y}$ and $y[\overline{\mu}] := 0$. Then $|y^*|\,|x| = |y[\mu]^*|\,|x[\mu]| \neq 0$ and $x[\overline{\mu}] = 0$ imply that

$$|y^*Ax| = |y[\mu]^*A[\mu]x[\mu]| \leq |y[\mu]^*A[\mu]| \cdot |x[\mu]| = \lambda|y[\mu]^*|\,|x[\mu]| = \lambda|y^*|\,|x|.$$

By (1.5),

$$\frac{|y^*Ax|}{|y^*|\,|x|} \leq \lambda \leq \rho^{\mathbb{K}}(A).$$

Therefore, for that choice of y (depending on S_1, S_2 and x) the left hand side of (2.2) is less than or equal to $\rho^{\mathbb{K}}(A)$. It remains to prove that the right hand side of (2.2) is less than or equal to $\rho^{\mathbb{K}}(A)$. Let S_1, S_2 be given, fixed but arbitrary. By Lemma 2.1, there exists $0 \neq y \in \mathbb{K}^n$ with $S_2y \geq 0$ and $|A^*y| = \lambda|y|$ for $\lambda \in \mathbb{R}_+$. Then for all $x \in \mathbb{K}^n$,

$$|y^*Ax| \leq |y^*A|\,|x| = \lambda|y^*|\,|x|,$$

such that for that choice of y (depending on S_1, S_2) the ratio $|y^*Ax|/(|y^*|\,|x|)$ is less than or equal to λ for all $x \in \mathbb{K}^n$ with $|y^*|\,|x| \neq 0$. It follows that the right hand side of (2.2) is less than or equal to $\lambda \leq \rho^{\mathbb{K}}(A^*) = \rho^{\mathbb{K}}(A)$, and the proof is finished. ◻

We note that Theorem 2.8 and its proof cover the case $\mathbb{K} = \mathbb{R}_+$, where in this case $\mathcal{S}(\mathbb{R}_+)$ consists only of the identity matrix. That means for general $A \geq 0$,

$$\max_{x\geq 0} \min_{\substack{y\geq 0\\ y^Tx\neq 0}} \frac{y^TAx}{y^Tx} = \rho(A) = \min_{y\geq 0} \max_{\substack{x\geq 0\\ y^Tx\neq 0}} \frac{y^TAx}{y^Tx}\ .$$

The Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 9, pp. 112-117, June 2002

Finally we note that for the classical Perron-Frobenius theory this characterization is mentioned without proof in the classical book by Varga [7] for irreducible matrices. As in other textbooks, the result is referenced as if it were included in [1], where in turn we only found a reference to an internal report.

REFERENCES

[1] G. Birkhoff and R.S. Varga. Reactor Criticality and Nonnegative Matrices. *J. Soc. Indust. Appl. Math.*, 6:354–377, 1958.
[2] B. Cain. private communication, 1998.
[3] L. Collatz. Einschließungssatz für die charakteristischen Zahlen von Matrizen. *Math. Z.*, 48:221–226, 1942.
[4] J.C. Doyle. Analysis of Feedback Systems with Structured Uncertainties. *IEE Proceedings, Part D*, 129:242–250, 1982.
[5] S.M. Rump. Theorems of Perron-Frobenius type for matrices without sign restrictions. *Linear Algebra Appl.*, 266:1–42, 1997.
[6] S.M. Rump. Perron-Frobenius Theory for Complex Matrices, to appear in *Linear Algebra Appl.*, 2002.
[7] R.S. Varga. *Matrix Iterative Analysis.* Prentice-Hall, Englewood Cliffs, N.J., 1962. Second edition, Springer, Berlin, 2000.