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INVERSION OF CENTROSKEWSYMMETRIC TOEPLITZ-PLUS-HANKEL BEZOUTIANS*

TORSTEN EHRHARDT† AND KARLA ROST‡

Abstract. In this paper, the inverse of a nonsingular, centroskewsymmetric Toeplitz-plus-Hankel Bezoutian B of (even) order n are computed, and a representation of B^{-1} as a sum of a Toeplitz and a Hankel matrix is found. Two possibilities are discussed. In the first one, the problem is reduced to the inversion of two skewsymmetric Toeplitz Bezoutians of order n. In the second one, the problem is tackled via the inversion of two Hankel Bezoutians of half the order $\frac{n}{2}$. The inversion of Toeplitz or Hankel Bezoutians is the subject of a previous paper [T. Ehrhardt and K. Rost. Resultant matrices and inversion of Bezoutians. *Linear Algebra Appl.*, 439:621–639, 2013.]. Both approaches lead to fast $O(n^2)$ inversion algorithms.

Key words. Bezoutian matrix, Toeplitz matrix, Hankel matrix, Toeplitz-plus-Hankel matrix, matrix inversion.

AMS subject classifications. 15A09, 15B05, 65F05.

1. Introduction. The present paper is devoted to the inversion of special types of structured matrices, so-called Toeplitz-plus-Hankel Bezoutians (shortly, $T+H$-*Bezoutians*). We assume that the matrix entries are taken from a field $\mathbb{F}$ with characteristic not equal to 2. In a previous paper [5], we investigated centrosymmetric $T+H$-Bezoutians. The focus of this paper are centroskewsymmetric (briefly, centroskew) $T+H$-Bezoutians. Recall that an $n \times n$ matrix A is called *centrosymmetric* or *centroskew*, if $J_n A J_n = A$ or $J_n A J_n = -A$, respectively, where J_n denotes the flip matrix of order n,

$$J_n := \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}. \tag{1.1}$$

Before we start to explain the content of the paper in more detail, let us give a very short historical account on Bezoutians. Bezoutians were introduced in connection with elimination theory (see [21]). Their importance for the inversion of Hankel and

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†Department of Mathematics, University of California at Santa Cruz, Santa Cruz, CA-95064, USA (tehrhard@ucsc.edu).

‡Fakultät für Mathematik, Technische Universität Chemnitz, Reichenhainer Straße 39, D-09126 Chemnitz, Germany (karla.rost@mathematik.tu-chemnitz.de).

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Toeplitz matrices was discovered by Lander [17] much later in 1974. In particular, he observed that the inverse of a nonsingular Hankel (Toeplitz) matrix is a Hankel (Toeplitz) Bezoutian and vice versa.

The inversion of Toeplitz and Hankel matrices has been the subject of a large amount of literature. The starting point were the papers of Trench [20] and Gohberg/Semencul [7]. Later, in [10], it was discovered that the inverse of a nonsingular matrix which is the sum of a Toeplitz and a Hankel matrix ($T+H$ matrix) possesses a generalized Bezoutian structure. These especially structured matrices $B = [b_{ij}]_{i,j=0}^{n-1}$ were called *Toeplitz-plus-Hankel Bezoutians* and are characterized by the property that there exists eight polynomials $\mathbf{u}_i(t), \mathbf{v}_i(t)$ $(i = 1, 2, 3, 4)$ with coefficients in $\mathbb{F}$ and of degree at most $n+1$ such that

$$\sum_{i,j=0}^{n-1} b_{ij} s^i t^j = \frac{\sum_{i=1}^{4} \mathbf{u}_i(t)\mathbf{v}_i(s)}{(t-s)(1-ts)}.$$

Again, there is a large number of papers dealing with the inversion of $T+H$ matrices (see e.g. [6], [11], [12], [15], [18], [19], and the references therein).

Up to now little attention has been devoted to the converse problem, the inversion of Bezoutians (see [8], [9]). A general approach to the inversion problem for Hankel and Toeplitz Bezoutians was given in [4]. As far as we know, the only paper dedicated to the inversion of $T+H$-Bezoutians is our paper [5]. In that paper, using results of [15] and [2], centrosymmetric $T+H$-Bezoutians were considered. Fast inversion algorithms as well as matrix representations of their inverses (which are centrosymmetric $T+H$ matrices) were presented.

In the present paper, we discuss two possibilities for how to compute the inverse of a centroskew $T+H$-Bezoutian B and how to represent the inverse as a $T+H$ matrix. Both possibilities are based on a splitting property, which was discovered in Section 8 of [13] (see also [15]), and holds for both centrosymmetric and centroskew $T+H$-Bezoutians. If B is a nonsingular, centroskew $T+H$-Bezoutian, necessarily of even size $n = 2\ell$, then B can be represented in the form $B = B_{+-} + B_{-+}$, where $B_{\pm\mp}$ have additional symmetries and, more importantly, a particular and simpler Bezoutian structure. These matrices are called *split-Bezoutians.*

In our first approach, it is proved that both splitting parts of B are directly related to nonsingular skewsymmetric Toeplitz Bezoutians. It remains to use the results of [4] to compute the inverses of these Toeplitz Bezoutians and to represent them as Toeplitz matrices. From there, the representation of B^{-1} as a $T+H$ matrix is obtained.

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The second approach is analogous to the method of inversion used in [5] for inverting centrosymmetric $T+H$-Bezoutains. Starting again with the splitting, we use now a result of [15] to transform B_{+-} and B_{-+} into nonsingular Hankel Bezoutians of half the order $\ell = \frac{n}{2}$. Then we take advantage of formulas and algorithms established in [4] in order to compute the inverses of these Hankel Bezoutians, which are Hankel matrices H_1 and H_2 the parameters of which are given by the solutions of corresponding Bezout equations (as described in [4]). At this point the formula for the inverse of the $T+H$-Bezoutian B is of the form

$$B^{-1} = W^{-T} \begin{bmatrix} \mathbf{0} & H_2 \\ H_1 & \mathbf{0} \end{bmatrix} W^{-1} ,$$

where W is a certain explicit transformation (involving triangular matrices). It remains to discover the Toeplitz-plus-Hankel structure behind this representation, i.e., we want to find a Toeplitz matrix T and a Hankel matrix H such that

$$B^{-1} = T + H .$$

This goal can be achieved utilizing finite versions of results given in [2].

The paper is structured as follows. After some preliminaries in Section 2, we recall, in Section 3, some basic facts on centroskew Toeplitz-plus-Hankel matrices. In Section 4, the inversion of Toeplitz Bezoutians and of Hankel Bezoutians is discussed, and the relevant results from [4] are recalled. Section 5 is dedicated to the splitting of centroskew $T+H$-Bezoutians. Moreover, an algorithm is discussed to decide whether a centroskew matrix B is a nonsingular $T+H$-Bezoutian. In Sections 6 and 7, the two possibilities for the inversion of centroskew $T+H$-Bezoutians are deduced. At the end of both sections, a corresponding fast algorithm is presented. Here fast means $O(n^2)$ complexity, where n is the order of B. In Section 8, we discuss the connections, the advantages, and disadvantages of both approaches.

If the reader wants to invert centroskew $T+H$-Bezoutians or test our algorithms, we refer him to the software on one of the authors' website:

http://www.tu-chemnitz.de/mathematik/ang_funktionalanalysis/rost/software/

2. Preliminaries. In what follows, we consider vectors or matrices whose entries are taken from a field $\mathbb{F}$ with a characteristic not equal to 2. By $\mathbb{F}^n$ we denote the linear space of all vectors of length n, by $\mathbb{F}^{m\times n}$ the linear space of all $m \times n$ matrices, and by I_n the identity matrix in $\mathbb{F}^{n\times n}$.

It will often be convenient to use polynomial language. Let $\mathbb{F}^n[t]$ denote the linear space of all polynomials in t of degree less than n with coefficients in $\mathbb{F}$. To

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each $\mathbf{x} = (x_j)_{j=0}^{n-1} \in \mathbb{F}^n$, we associate the polynomial

$$\mathbf{x}(t) := \sum_{j=0}^{n-1} x_j t^j \in \mathbb{F}^n[t]. \tag{2.1}$$

Occasionally, when using a different indexing, $\mathbf{x} = (x_j)_{j=-n+1}^{n-1} \in \mathbb{F}^{2n-1}$, we associate the polynomial

$$\mathbf{x}(t) := t^{n-1} \sum_{j=-n+1}^{n-1} x_j t^j \in \mathbb{F}^{2n-1}[t]. \tag{2.2}$$

Moreover, we associate to a matrix $A = [\, a_{ij} \,]_{i,j=0}^{n-1}$ the bivariate polynomial

$$A(t,s) := \sum_{i,j=0}^{n-1} a_{ij}\, t^i s^j$$

and call it the *generating polynomial of* A.

Given a vector $\mathbf{x} \in \mathbb{F}^n$ we denote

$$\mathbf{x}^J := J_n \mathbf{x}\,,$$

where J_n was introduced in (1.1). In polynomial language, this means $\mathbf{x}^J(t) = \mathbf{x}(t^{-1})t^{n-1}$. With this abbreviation, a vector $\mathbf{x} \in \mathbb{F}^n$ (or its corresponding polynomial) is said to be *symmetric* if $\mathbf{x} = \mathbf{x}^J$ and *skewsymmetric* if $\mathbf{x} = -\mathbf{x}^J$. The matrices

$$P_{\pm} := \tfrac{1}{2}(I_n \pm J_n) \tag{2.3}$$

are the projections from $\mathbb{F}^n$ onto the subspaces $\mathbb{F}^n_{\pm}$ consisting of all symmetric, respective skewsymmetric vectors, i.e.,

$$\mathbb{F}^n_{\pm} := \left\{ \, \mathbf{x} \in \mathbb{F}^n \; : \; \mathbf{x}^J = \pm \mathbf{x} \, \right\}.$$

The various spaces $\mathbb{F}^n_{\pm}$ for n even or odd are related to each other. This can be easily expressed in polynomial language as follows,

$$\begin{aligned} \mathbb{F}^{2\ell}_{+}[t] &= \left\{ \, (t+1)\,\mathbf{x}(t) \; : \; \mathbf{x}(t) \in \mathbb{F}^{2\ell-1}_{+}[t] \, \right\}, \\ \mathbb{F}^{2\ell}_{-}[t] &= \left\{ \, (t-1)\,\mathbf{x}(t) \; : \; \mathbf{x}(t) \in \mathbb{F}^{2\ell-1}_{+}[t] \, \right\}, \\ \mathbb{F}^{2\ell+1}_{-}[t] &= \left\{ \, (t^2-1)\,\mathbf{x}(t) \; : \; \mathbf{x}(t) \in \mathbb{F}^{2\ell-1}_{+}[t] \, \right\}. \end{aligned} \tag{2.4}$$

Recall that a matrix A of order n is called *centroskew* if $A = -J_n A J_n$. Since $(\det J_n)^2 = 1$, the order n of a nonsingular, centroskew matrix is even, $n = 2\ell$.

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It is easy to see that a matrix A is centroskew if and only if

$$P_-AP_- = P_+AP_+ = \mathbf{0}. \tag{2.5}$$

In particular, a centroskew matrix A maps $\mathbb{F}^n_\pm$ to $\mathbb{F}^n_\mp$, i.e., $AP_\pm = P_\mp AP_\pm$.

Let us recall the definition of Toeplitz and Hankel matrices. The $n \times n$ *Toeplitz matrix* generated by the vector $\mathbf{a} = (a_i)_{i=-n+1}^{n-1} \in \mathbb{F}^{2n-1}$ is the matrix

$$T_n(\mathbf{a}) = [\, a_{i-j} \,]_{i,j=0}^{n-1} .$$

We will use (2.2) in order to assign its (polynomial) symbol, in slight deviation from standard notation. An $n \times n$ *Hankel matrix* generated by $\mathbf{s} = (s_i)_{i=0}^{2n-2} \in \mathbb{F}^{2n-1}$ is the matrix

$$H_n(\mathbf{s}) = [\, s_{i+j} \,]_{i,j=0}^{n-1} ,$$

where (2.1) is used to denote its symbol.

For Toeplitz matrices we have

$$T_n(\mathbf{a})^T = J_n T_n(\mathbf{a}) J_n = T_n(\mathbf{a}^J). \tag{2.6}$$

In particular, a Toeplitz matrix is skewsymmetric if and only if it is centroskew, or, equivalently, if its symbol is a skewsymmetric vector.

3. Centroskew Toeplitz-plus-Hankel matrices. Toeplitz-plus-Hankel matrices (shortly, *$T + H$ matrices*) are matrices which are a sum of a Toeplitz and a Hankel matrix. Since $T_n(\mathbf{b})J_n$ is a Hankel matrix it is possible to represent any $T+H$ matrix by means of two Toeplitz matrices,

$$R_n = T_n(\mathbf{a}) + T_n(\mathbf{b})J_n \qquad (\mathbf{a}, \mathbf{b} \in \mathbb{F}^{2n-1}). \tag{3.1}$$

Related to this representation there is another one, using the projections (2.3) and the symbols $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{a} - \mathbf{b}$,

$$R_n = T_n(\mathbf{c})P_+ + T_n(\mathbf{d})P_- . \tag{3.2}$$

Restricting our attention to centroskew $T + H$ matrices, we have the following result regarding the underlying symbols (compare [15]).

Proposition 3.1. *The $T+H$ matrix R_n is centroskew if and only if the symbols $\mathbf{a}, \mathbf{b}$ as well as $\mathbf{c}, \mathbf{d}$ of the Toeplitz matrices in* (3.1) *(respectively* (3.2)*) can be chosen as skewsymmetric vectors. This choice is unique.*

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Proof. Let R_n be given by (3.1). Using (2.6), the centroskewsymmetry of R_n is equivalent to

$$T_n(\mathbf{a} + \mathbf{a}^J) + T_n(\mathbf{b} + \mathbf{b}^J)J_n = \mathbf{0}\,,$$

which implies

$$\mathbf{a} + \mathbf{a}^J = \mathbf{e}_{\alpha,\beta} := (\alpha, \beta, \alpha, \beta, \dots, \beta, \alpha)^T \in \mathbb{F}_+^{2n-1}$$

for some $\alpha, \beta \in \mathbb{F}$ and

$$\mathbf{b} + \mathbf{b}^J = \mathbf{f}_{\alpha,\beta} := \begin{cases} -\mathbf{e}_{\alpha,\beta} & \text{if } n \text{ is odd} \\ -\mathbf{e}_{\beta,\alpha} & \text{if } n \text{ is even.} \end{cases}$$

If we define $\hat{\mathbf{a}} = \mathbf{a} - \frac{1}{2}\mathbf{e}_{\alpha,\beta}$ and $\hat{\mathbf{b}} = \mathbf{b} - \frac{1}{2}\mathbf{f}_{\alpha,\beta}$, then $\hat{\mathbf{a}}, \hat{\mathbf{b}} \in \mathbb{F}_-^{2n-1}$ and

$$T_n(\mathbf{a}) + T_n(\mathbf{b})J_n = T_n(\hat{\mathbf{a}}) + T_n(\hat{\mathbf{b}})J_n.$$

Hence, we can choose skewsymmetric vectors as symbols, and it is also easy to see that these choices are unique. Obviously, the same is true for the symbols $\mathbf{c}$ and $\mathbf{d}$ of the representation (3.2). ◻

From now on we will assume that the symbols $\mathbf{a}, \mathbf{b}$ $(\mathbf{c}, \mathbf{d})$ of a centroskew $T + H$ matrix R_n are chosen as skewsymmetric vectors. Moreover, in this case we can also write

$$R_n = P_- T_n(\mathbf{c}) P_+ + P_+ T_n(\mathbf{d}) P_-$$

instead of (3.2) (see (2.5)).

Proposition 3.2. *The centroskew $T + H$ matrix R_n is nonsingular if and only if*

$$R_n^- := T_n(\mathbf{a}) - T_n(\mathbf{b})J_n = T_n(\mathbf{c})P_- + T_n(\mathbf{d})P_+$$

is nonsingular.

Proof. Using (2.6) for both $T_n(\mathbf{a})$ and $T_n(\mathbf{b})$, it is immediately clear that the transpose R_n^T is equal to $-T_n(\mathbf{a}) + T_n(\mathbf{b})J_n$. ◻

The following two facts are also known from [15], Corollary 3.7, but we present a simpler proof here.

Theorem 3.3. *The centroskew $T + H$ matrix $R_n = T_n(\mathbf{c})P_+ + T_n(\mathbf{d})P_-$ is nonsingular if and only if $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$ are both nonsingular.*

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Proof. Since the vector $\mathbf{c}$ and $\mathbf{d}$ are skewsymmetric, the Toeplitz matrices $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$ are skewsymmetric and centroskew. Now, using (2.5), it is easy to see that

$$\begin{bmatrix} R_n & \mathbf{0} \\ \mathbf{0} & R_n^- \end{bmatrix} = \begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix} \begin{bmatrix} \mathbf{0} & T_n(\mathbf{d}) \\ T_n(\mathbf{c}) & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix}, \tag{3.3}$$

where

$$\begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix} \begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix} = \begin{bmatrix} I_n & \mathbf{0} \\ \mathbf{0} & I_n \end{bmatrix}. \qquad \square$$

The following theorem gives some information about the inverse of a centroskew $T + H$ matrix.

THEOREM 3.4. *Let the centroskew $T + H$ matrix $R_n = T_n(\mathbf{c})P_+ + T_n(\mathbf{d})P_-$ be nonsingular. Then its inverse is given by*

$$R_n^{-1} = T_n(\mathbf{c})^{-1}P_- + T_n(\mathbf{d})^{-1}P_+.$$

Proof. We can use (3.3) and pass to the inverse,

$$\begin{bmatrix} R_n^{-1} & \mathbf{0} \\ \mathbf{0} & (R_n^-)^{-1} \end{bmatrix} = \begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix} \begin{bmatrix} \mathbf{0} & T_n(\mathbf{c})^{-1} \\ T_n(\mathbf{d})^{-1} & \mathbf{0} \end{bmatrix} \begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix}.$$

Noting that $T_n(\mathbf{c})^{-1}$ and $T_n(\mathbf{d})^{-1}$ are centroskew, the proof is easy to complete by using (2.5). $\square$

The inverses $T_n(\mathbf{c})^{-1}$ and $T_n(\mathbf{d})^{-1}$ of the Toeplitz matrices are so-called Toeplitz Bezoutians, which together with their Hankel counterparts are analyzed next.

4. Toeplitz and Hankel Bezoutians. For later use, we are going to introduce the notions of Toeplitz Bezoutians (shortly, *T-Bezoutians*) and Hankel Bezoutians (shortly, *H-Bezoutians*).

A matrix $B \in \mathbb{F}^{n\times n}$ is called a *T-Bezoutian* if there exists vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}^{n+1}$ such that, in polynomial language,

$$B(t,s) = \frac{\mathbf{u}(t)\mathbf{v}^J(s) - \mathbf{v}(t)\mathbf{u}^J(s)}{1 - ts}.$$

In this case, we write $B = \mathrm{Bez}_T(\mathbf{u}, \mathbf{v})$. Analogously, a matrix $B \in \mathbb{F}^{n\times n}$ is called an *H-Bezoutian* if there exists vectors $\mathbf{u}, \mathbf{v} \in \mathbb{F}^{n+1}$ such that

$$B(t,s) = \frac{\mathbf{u}(t)\mathbf{v}(s) - \mathbf{v}(t)\mathbf{u}(s)}{t - s}.$$

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Then we write $B = \mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$. It is also possible to define T- and H-Bezoutians via suitable displacement transformations. However, we will not make use of it.

H-Bezoutians are always symmetric, while T-Bezoutians B are always persymmetric, i.e., $J_n B J_n = B^T$. The two kinds of Bezoutians are related to each other by $\mathrm{Bez}_H(\mathbf{u}, \mathbf{v}) = -\mathrm{Bez}_T(\mathbf{u}, \mathbf{v}) J_n$.

It is well known (see, e.g., [9]) that $\mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$ (or $\mathrm{Bez}_T(\mathbf{u}, \mathbf{v})$, respectively) is nonsingular if and only if $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are *generalized coprime*, which means that the polynomials $\mathbf{u}(t)$ and $\mathbf{v}(t)$ are coprime in the usual sense and that $\deg \mathbf{u}(t) = n$ or $\deg \mathbf{v}(t) = n$.

The following connection between Toeplitz matrices (Hankel matrices) and T-Bezoutians (H-Bezoutians) is a classical result discovered by Lander in 1974 [17].

THEOREM 4.1. *A nonsingular matrix is a T-Bezoutian (H-Bezoutian) if and only if its inverse is a Toeplitz matrix (Hankel matrix).*

Let us consider for a moment the Hankel case and discuss the question: Given the H-Bezoutian $B = \mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$ with generalized coprime polynomials $\mathbf{u}(t), \mathbf{v}(t)$, how can we compute the symbol $\mathbf{s}$ of its inverse, a Hankel matrix $H_n(\mathbf{s}) = B^{-1}$? The answer was given in [4].

THEOREM 4.2. *Assume $\mathbf{u}(t), \mathbf{v}(t) \in \mathbb{F}^{n+1}[t]$ to be generalized coprime polynomials, and let $B = \mathrm{Bez}_H(\mathbf{u}, \mathbf{v})$. Then B is nonsingular, the Bezout equations*

$$\mathbf{u}(t)\boldsymbol{\alpha}(t) + \mathbf{v}(t)\boldsymbol{\beta}(t) = 1\,, \tag{4.1}$$

$$\mathbf{u}^J(t)\boldsymbol{\gamma}^J(t) + \mathbf{v}^J(t)\boldsymbol{\delta}^J(t) = 1 \tag{4.2}$$

have unique solutions $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t), \boldsymbol{\gamma}(t), \boldsymbol{\delta}(t) \in \mathbb{F}^n[t]$, and $\mathbf{s} = (s_i)_{i=0}^{2n-2} \in \mathbb{F}^{2n-1}$ given by

$$\mathbf{s}^J(t) = -\boldsymbol{\alpha}(t)\boldsymbol{\delta}(t) + \boldsymbol{\beta}(t)\boldsymbol{\gamma}(t)$$

is the symbol of the inverse of B, $B^{-1} = H_n(\mathbf{s}) = [\, s_{i+j} \,]_{i,j=0}^{n-1}$.

For T-Bezoutians, the analogous result reads as follows [4].

THEOREM 4.3. *Assume $\mathbf{u}(t), \mathbf{v}(t) \in \mathbb{F}^{n+1}[t]$ to be generalized coprime polynomials, and let $B = \mathrm{Bez}_T(\mathbf{u}, \mathbf{v})$. Then B is nonsingular, the Bezout equations* (4.1) *and* (4.2) *have unique solutions $\boldsymbol{\alpha}(t), \boldsymbol{\beta}(t), \boldsymbol{\gamma}(t), \boldsymbol{\delta}(t) \in \mathbb{F}^n[t]$, and $\mathbf{c} = (c_i)_{i=-n+1}^{n-1} \in \mathbb{F}^{2n-1}$ given by*

$$\mathbf{c}(t) = t^{n-1} \sum_{i=-n+1}^{n-1} c_i t^i = \boldsymbol{\alpha}(t)\boldsymbol{\delta}(t) - \boldsymbol{\beta}(t)\boldsymbol{\gamma}(t)$$

is the symbol of the inverse of B, $B^{-1} = T_n(\mathbf{c}) = [\, c_{i-j} \,]_{i,j=0}^{n-1}$.

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For our purposes, it is important to specialize the previous result to the case of centroskew T-Bezoutians. As shown in [14], Section 5, if the T-Bezoutian $\mathrm{Bez}_T(\mathbf{u},\mathbf{v})$ is nonsingular and centroskew, then $\mathbf{u},\mathbf{v}$ are symmetric vectors, i.e., $\mathbf{u},\mathbf{v}\in\mathbb{F}_+^{n+1}$ (with n, of course, being even). Thus, we have $\boldsymbol{\alpha}=\boldsymbol{\gamma}^J$ and $\boldsymbol{\beta}=\boldsymbol{\delta}^J$ for the (unique) solutions of (4.1) and (4.2). This implies that

$$\mathbf{c}(t)=t^{n-1}\sum_{i=-n+1}^{n-1}c_it^i=\boldsymbol{\alpha}(t)\boldsymbol{\beta}^J(t)-\boldsymbol{\beta}(t)\boldsymbol{\alpha}^J(t). \tag{4.3}$$

Remark that $\mathbf{c}\in\mathbb{F}_-^{2n-1}$ is a skewsymmetric vector and that $T_n(\mathbf{c})=B^{-1}$ is a skewsymmetric and centroskew matrix.

5. Splitting of centroskew $\boldsymbol{T+H}$-Bezoutians.

In order to define Toeplitz-plus-Hankel Bezoutians ($T+H$-Bezoutians), let us consider the following transformation

$$\nabla_{T+H}:\mathbb{F}^{n\times n}\to\mathbb{F}^{(n+2)\times(n+2)}$$

defined by

$$\nabla_{T+H}(B)=[b_{i-1,j}+b_{i-1,j-2}-b_{i,j-1}-b_{i-2,j-1}]_{i,j=0}^{n+1},$$

where $B=[b_{ij}]_{i,j=0}^{n-1}$ stipulating $b_{ij}=0$ whenever i or j is not in the set $\{0,\dots,n-1\}$. Equivalently, in polynomial language,

$$\left(\nabla_{T+H}(B)\right)(t,s)=(t-s)(1-ts)B(t,s)\,.$$

A matrix $B\in\mathbb{F}^{n\times n}$ is called a *$T+H$-Bezoutian* if

$$\operatorname{rank}\nabla_{T+H}(B)\le 4\,.$$

This condition is equivalent to the existence of eight vectors $\mathbf{u}_i,\mathbf{v}_i$ $(i=1,2,3,4)$ in $\mathbb{F}^{n+2}$ such that

$$(t-s)(1-ts)B(t,s)=\sum_{i=1}^{4}\mathbf{u}_i(t)\mathbf{v}_i(s)\,.$$

For the $T+H$ case, we know from [10] the following important fact.

THEOREM 5.1. *A nonsingular matrix is a $T+H$-Bezoutian if and only if its inverse is a $T+H$ matrix.*

The focus of this paper are centroskew $T+H$-Bezoutians B, i.e., those which satisfy $J_nBJ_n=-B$. As we will see in Theorem 5.3 below, nonsingular, centroskew

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$T+H$-Bezoutians admit a certain splitting. Let us start with the following trivial facts concerning splitting properties of an arbitrary centroskew matrix A (see [15], Section 5).

LEMMA 5.2. *Let A be a centroskew matrix of order n. Then A allows the splitting*

$$A = A_{+-} + A_{-+}\,,$$

where $A_{+-} := AP_- = P_+A$ is a matrix the columns of which are symmetric vectors and the rows are skewsymmetric, $A_{-+} := AP_+ = P_-A$ is a matrix the columns of which are skewsymmetric vectors and the rows are symmetric. Furthermore,

$$\mathit{rank}\,A = \mathit{rank}\,A_{+-} + \mathit{rank}\,A_{-+}.$$

In the case of a centroskew $T+H$-Bezoutian B, Theorem 5.3 below will tell us that the splitting parts B_{+-} and B_{-+} can be represented as a product of three matrices. The middle factor is a so-called *split-Bezoutian of* $(+)$*type*. This is a $T+H$-Bezoutian involving two symmetric vectors $\mathbf{u}_+, \mathbf{v}_+$ the generating polynomial of which is given by

$$(\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+, \mathbf{v}_+))\,(t,s) = \frac{\mathbf{u}_+(t)\mathbf{v}_+(s) - \mathbf{v}_+(t)\mathbf{u}_+(s)}{(t-s)(1-ts)}\,.$$

The matrix $\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+, \mathbf{v}_+)$ is centrosymmetric and all rows and columns are symmetric vectors. Moreover, introduce the following $n \times (n-1)$ matrices

$$M_{n-1}^{\pm} := \begin{bmatrix} \pm 1 & 0 & \cdots & 0 \\ 1 & \pm 1 & \ddots & \vdots \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \pm 1 \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

The splitting result, which was established in [15], now reads as follows.

THEOREM 5.3. *Let n be even. Then $B \in \mathbb{F}^{n\times n}$ is a nonsingular, centroskew $T+H$-Bezoutian if and only if it can be represented in the form*

$$B = M_{n-1}^{+}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{f}_+, \mathbf{g}_+)(M_{n-1}^{-})^T + M_{n-1}^{-}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{y}_+, \mathbf{z}_+)(M_{n-1}^{+})^T \tag{5.1}$$

with $\mathbf{f}_+, \mathbf{g}_+, \mathbf{y}_+, \mathbf{z}_+ \in \mathbb{F}_+^{n+1}$ such that $\{\mathbf{f}_+(t), \mathbf{g}_+(t)\}$ and $\{\mathbf{y}_+(t), \mathbf{z}_+(t)\}$ are pairs of coprime polynomials.

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Note that the terms in the sum (5.1) are equal to the splitting parts B_{+-} and B_{-+}. The split-Bezoutians occurring therein are matrices of order $n-1$. In polynomial language, this formula reads as

$$
\begin{aligned}
B(t,s) &= (t+1)\frac{\mathbf{f}_+(t)\mathbf{g}_+(s) - \mathbf{g}_+(t)\mathbf{f}_+(s)}{(t-s)(1-ts)}(s-1) \\
&\quad + (t-1)\frac{\mathbf{y}_+(t)\mathbf{z}_+(s) - \mathbf{z}_+(t)\mathbf{y}_+(s)}{(t-s)(1-ts)}(s+1). \qquad (5.2)
\end{aligned}
$$

To see this notice that $M_{n-1}^{\pm}$ is the matrix of the operator of multiplication by $t \pm 1$ in the corresponding polynomial spaces (with respect to the canonical bases).

Remark 5.4. Different pairs of linearly independent vectors $(\mathbf{u}_+,\ \mathbf{v}_+)$ and $(\hat{\mathbf{u}}_+,\ \hat{\mathbf{v}}_+)$ produce the same split-Bezoutian of $(+)$type,

$$\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+, \mathbf{v}_+) = \mathrm{Bez}_{\mathrm{sp}}(\hat{\mathbf{u}}_+, \hat{\mathbf{v}}_+)\,,$$

if and only if there is $\Phi \in \mathbb{F}^{2\times 2}$ with $\det \Phi = 1$ such that

$$[\,\hat{\mathbf{u}}_+, \hat{\mathbf{v}}_+\,] = [\,\mathbf{u}_+, \mathbf{v}_+\,]\,\Phi.$$

Remark 5.5. Given a centroskew matrix B of even order n, one can ask how to decide whether B is a nonsingular $T+H$-Bezoutian and how to determine the vectors $\mathbf{f}_+, \mathbf{g}_+, \mathbf{y}_+, \mathbf{z}_+$ occurring in (5.1). This can be done by the following procedure:

1. Compute $B_{+-} := P_+ B$ and $B_{-+} := P_- B$.
2. Verify whether $\operatorname{rank} \nabla_{T+H}(B_{+-}) = \operatorname{rank} \nabla_{T+H}(B_{-+}) = 2$.
   (If this is not fulfilled, stop: B is singular or B is not a $T+H$-Bezoutian.)
3. Determine bases $\{\mathbf{u}_\pm, \mathbf{v}_\pm\}$ in the image of $\nabla_{T+H}(B_{\pm\mp})$.
   (Due to the properties of $B_{\pm\mp}$, we have $\mathbf{u}_\pm, \mathbf{v}_\pm \in \mathbb{F}_\pm^{n+2}$.)
4. Compute
   $$\mathbf{f}_+(t) = \mathbf{u}_+(t)/(t+1), \qquad \mathbf{g}_+'(t) = \mathbf{v}_+(t)/(t+1)$$
   and
   $$\mathbf{y}_+(t) = \mathbf{u}_-(t)/(t-1), \qquad \mathbf{z}_+'(t) = \mathbf{v}_-(t)/(t-1).$$
   (Recall (2.4) and note that $\mathbf{f}_+, \mathbf{g}_+', \mathbf{y}_+, \mathbf{z}_+' \in \mathbb{F}_+^{n+1}$.)
5. Determine whether $\{\mathbf{f}_+(t), \mathbf{g}_+'(t)\}$ and $\{\mathbf{y}_+(t), \mathbf{z}_+'(t)\}$ are pairs of coprime polynomials.
   (If this is not fulfilled, stop: B is singular.)

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6. Compute the unique vectors $\mathbf{f}'_+, \mathbf{g}_+, \mathbf{y}'_+, \mathbf{z}_+ \in \mathbb{F}^{n+1}_+$ such that

$$\nabla_{T+H}(B_{+-})(t,s) = (t+1)\left(\mathbf{f}_+(t)\mathbf{g}_+(s) - \mathbf{g}'_+(t)\mathbf{f}'_+(s)\right)(s-1)$$

and

$$\nabla_{T+H}(B_{-+})(t,s) = (t-1)\left(\mathbf{y}_+(t)\mathbf{z}_+(s) - \mathbf{z}'_+(t)\mathbf{y}'_+(s)\right)(s+1).$$

Note: In fact, there exist $\lambda, \mu \in \mathbb{F} \setminus \{0\}$ such that

$$\mathbf{f}_+(t) = \lambda\mathbf{f}'_+(t),\ \mathbf{g}_+(t) = \lambda^{-1}\mathbf{g}'_+(t),\ \mathbf{y}_+(t) = \mu\mathbf{y}'_+(t),\ \mathbf{z}_+(t) = \mu^{-1}\mathbf{z}'_+(t).$$

Therefore,

$$\nabla_{T+H}(B_{+-})(t,s) = (t+1)\left(\mathbf{f}_+(t)\mathbf{g}_+(s) - \mathbf{g}_+(t)\mathbf{f}_+(s)\right)(s-1)$$

and

$$\nabla_{T+H}(B_{-+})(t,s) = (t-1)\left(\mathbf{y}_+(t)\mathbf{z}_+(s) - \mathbf{z}_+(t)\mathbf{y}_+(s)\right)(s+1).$$

Hence, to compute λ it suffices to compare a nonzero entry of $\nabla_{T+H}(B_{+-})$ with the corresponding entry in the polynomial

$$(t+1)\left(\mathbf{f}_+(t)\mathbf{g}'_+(s) - \mathbf{g}'_+(t)\mathbf{f}_+(s)\right)(s-1).$$

The same applies to μ.

7. Now, $B = B_{+-} + B_{-+}$ is a nonsingular $T+H$-Bezoutian with

$$\begin{aligned} B_{+-} &= M^+_{n-1}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{f}_+, \mathbf{g}_+)(M^-_{n-1})^T, \\ B_{-+} &= M^-_{n-1}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{y}_+, \mathbf{z}_+)(M^+_{n-1})^T, \end{aligned}$$

where the two pairs $\{\mathbf{f}_+(t), \mathbf{g}_+(t)\}$ and $\{\mathbf{y}_+(t), \mathbf{z}_+(t)\}$ are unique up to transformations discussed in Remark 5.4.

This procedure has complexity $O(n^2)$. Indeed, all the steps require (at most) $O(n^2)$ operations including Step 5 using the Euclidean algorithm. In case of the field $\mathbb{F}$ being $\mathbb{R}$ or $\mathbb{C}$, a stability issue occurs in Steps 2-3 and Step 5. While it might be of interest to analyze this issue further, we refrain from discussing it in this paper.

6. Inversion of $T+H$-Bezoutians via skewsymmetric T-Bezoutians. In this section, we present our first approach to invert centroskew $T+H$-Bezoutians. It is done via reduction to certain T-Bezoutians, which can be inverted using the result of Section 4. The following key result is based on the representation obtained in Theorem 5.3.

THEOREM 6.1. *Let $B \in \mathbb{F}^{n\times n}$ be a centroskew $T+H$-Bezoutian given in the form* (5.1) *with symmetric $\mathbf{f}_+, \mathbf{g}_+, \mathbf{y}_+, \mathbf{z}_+ \in \mathbb{F}^{n+1}_+$. Then*

$$B = 2\,\mathrm{Bez}_T(\mathbf{f}_+, \mathbf{g}_+)P_- - 2\,\mathrm{Bez}_T(\mathbf{y}_+, \mathbf{z}_+)P_+\,. \tag{6.1}$$

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Proof. Recall that (5.1) reads in polynomial language as (5.2). Obviously, the generating polynomial of $2\,\mathrm{Bez}_T(\mathbf{f}_+,\mathbf{g}_+)P_-$ is equal to

$$(\mathbf{f}_+(t)\mathbf{g}_+(s)-\mathbf{g}_+(t)\mathbf{f}_+(s))\left(\frac{1}{1-ts}+\frac{1}{t-s}\right).$$

Since

$$\frac{1}{1-ts}+\frac{1}{t-s}=\frac{(1+t)(1-s)}{(1-ts)(t-s)}$$

we obtain

$$2\,\mathrm{Bez}_T(\mathbf{f}_+,\mathbf{g}_+)P_-=M^+_{n-1}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{f}_+,\mathbf{g}_+)(M^-_{n-1})^T\,.$$

Analogously, using

$$\frac{1}{1-ts}-\frac{1}{t-s}=-\frac{(1-t)(1+s)}{(1-ts)(t-s)}\,,$$

it follows that

$$-2\,\mathrm{Bez}_T(\mathbf{y}_+,\mathbf{z}_+)P_+=M^-_{n-1}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{y}_+,\mathbf{z}_+)(M^+_{n-1})^T\,.$$

This concludes the proof. ◻

It follows from the definition of T-Bezoutians that for symmetric vectors $\mathbf{f}_+,\mathbf{g}_+$, $\mathbf{y}_+,\mathbf{z}_+$

$$B_1:=\mathrm{Bez}_T(\mathbf{f}_+,\mathbf{g}_+)\quad\text{ and }\quad B_2:=\mathrm{Bez}_T(\mathbf{y}_+,\mathbf{z}_+)$$

are centroskew and skewsymmetric, i.e., $B_i^T=J_nB_iJ_n=-B_i$.

Proposition 6.2. *Let $B\in\mathbb{F}^{n\times n}$ be a nonsingular, centroskew $T+H$-Bezoutian given by* (5.1) *or* (6.1) *with pairs $\{\mathbf{f}_+(t),\mathbf{g}_+(t)\}$ and $\{\mathbf{y}_+(t),\mathbf{z}_+(t)\}$ of symmetric coprime polynomials in $\mathbb{F}^{n+1}_+[t]$. Then*

$$B_1=\mathrm{Bez}_T(\mathbf{f}_+,\mathbf{g}_+)\quad\text{ and }\quad B_2=\mathrm{Bez}_T(\mathbf{y}_+,\mathbf{z}_+)$$

are invertible, and

$$B^{-1}=\frac{1}{2}(B_1^{-1}P_+-B_2^{-1}P_-).$$

Proof. Since the polynomials are symmetric, coprimeness implies generalized coprimeness, and hence, the T-Bezoutians are invertible. We write (6.1) as

$$\tfrac{1}{2}B=B_1P_--B_2P_+\,,$$

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and take its tranpose,

$$\tfrac{1}{2}B^T = -P_- B_1 + P_+ B_2 \,.$$

Both equations can be written, in analogy to (3.3), in the following form:

$$\frac{1}{2}\begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & -B^T \end{bmatrix} = \begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix}\begin{bmatrix} \mathbf{0} & B_1 \\ -B_2 & \mathbf{0} \end{bmatrix}\begin{bmatrix} P_+ & P_- \\ P_- & P_+ \end{bmatrix}.$$

Here one has to use that B_1 and B_2 are centroskew (see (2.5)). Notice that also this identity implies the invertibility of B_1 and B_2. Now one can pass to the inverse of this equation and obtain the desired expression for B^{-1} in terms of B_1^{-1} and B_2^{-1}. □

The inverses of the above T-Bezoutians are Toeplitz matrices

$$T_n(\mathbf{c}) = B_1^{-1} \text{ and } T_n(\mathbf{d}) = B_2^{-1}\,.$$

From Theorem 4.3 and the remarks made afterwards, we know how to obtain the symbols $\mathbf{c}, \mathbf{d}$ of these (skewsymmetric) Toeplitz matrices (see also (4.1) and (4.3)). Indeed, $\mathbf{c}, \mathbf{d} \in \mathbb{F}_-^{2n-1}$ are given by

$$\mathbf{c}(t) = \boldsymbol{\alpha}(t)\boldsymbol{\beta}^J(t) - \boldsymbol{\beta}(t)\boldsymbol{\alpha}^J(t), \tag{6.2}$$

$$\mathbf{d}(t) = \boldsymbol{\gamma}(t)\boldsymbol{\delta}^J(t) - \boldsymbol{\delta}(t)\boldsymbol{\gamma}^J(t), \tag{6.3}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \boldsymbol{\delta} \in \mathbb{F}^n$ are the solutions of the Bezout equations

$$\mathbf{g}_+(t)\boldsymbol{\alpha}(t) + \mathbf{f}_+(t)\boldsymbol{\beta}(t) = 1, \tag{6.4}$$

$$\mathbf{z}_+(t)\boldsymbol{\gamma}(t) + \mathbf{y}_+(t)\boldsymbol{\delta}(t) = 1. \tag{6.5}$$

We can now summarize this as follows.

THEOREM 6.3. *Let $B \in \mathbb{F}^{n\times n}$ be a centroskew $T+H$-Bezoutian given by* (5.1) *or* (6.1) *with pairs $\{\mathbf{g}_+(t), \mathbf{f}_+(t)\}$ and $\{\mathbf{y}_+(t), \mathbf{z}_+(t)\}$ of symmetric coprime polynomials in $\mathbb{F}_+^{n+1}[t]$. Then n is even, B is nonsingular and*

$$B^{-1} = \frac{1}{2}(T_n(\mathbf{c})P_+ - T_n(\mathbf{d})P_-), \tag{6.6}$$

where $\mathbf{c}, \mathbf{d} \in \mathbb{F}_-^{2n-1}$ are given by (6.2) *and* (6.3).

Note that (6.6) reads as

$$B^{-1} = \frac{1}{4}\left[c_{j-k} + c_{j+k+1-n}\right]_{j,k=0}^{n-1} - \frac{1}{4}\left[d_{j-k} - d_{j+k+1-n}\right]_{j,k=0}^{n-1},$$

which is a (centroskew) sum of a Toeplitz and a Hankel matrix.

REMARK 6.4. Since $\mathbf{c} = (c_i)_{i=-n+1}^{n-1}$ and $\mathbf{d} = (d_i)_{i=-n+1}^{n-1}$ are skewsymmetric vectors, it suffices to compute only their last $n-1$ components $(c_i)_{i=1}^{n-1}$ and $(d_i)_{i=1}^{n-1}$.

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To that aim, introduce for a given vector $\mathbf{x} = (x_i)_{i=0}^{n-1}$ the following upper triangular Toeplitz matrix of order n,

$$U_n(\mathbf{x}) = \begin{bmatrix} x_0 & x_1 & \cdots & \cdots & x_{n-1} \\ & x_0 & x_1 & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & x_0 & x_1 \\ 0 & & & & x_0 \end{bmatrix}.$$

Now, as is easy to see and has already been stated in Section 6 of [4], equations (6.2) and (6.3) become

$$(c_i)_{i=0}^{n-1} = U_n(\boldsymbol{\beta})\boldsymbol{\alpha} - U_n(\boldsymbol{\alpha})\boldsymbol{\beta}, \quad (d_i)_{i=0}^{n-1} = U_n(\boldsymbol{\delta})\boldsymbol{\gamma} - U_n(\boldsymbol{\gamma})\boldsymbol{\delta}, \tag{6.7}$$

where $c_0 = d_0 = 0$.

Let us now present the steps of a corresponding inversion algorithm.

Algorithm 6.5. *We are given a centroskew $T+H$-Bezoutian B of even order n in the form* (5.1) *with pairs $\{\mathbf{f}_+(t), \mathbf{g}_+(t)\}$ and $\{\mathbf{y}_+(t), \mathbf{z}_+(t)\}$ of symmetric, coprime polynomials in $\mathbb{F}_+^{n+1}[t]$.*

1. *Solve the Bezout equations* (6.4) *and* (6.5) *by the extended Euclidean algorithm.*
2. *Determine the (skewsymmetric) symbols* $\mathbf{c}$ *and* $\mathbf{d}$ *by either*
   (i) computing their last components according to (6.7), *or,*
   (ii) computing them from (6.2) *and* (6.3).
3. *Compute the matrices*

$$A_{\mathbf{c}} := T_n(\mathbf{c})P_+ \text{ and } A_{\mathbf{d}} := T_n(\mathbf{d})P_- .$$

4. *Then the inverse of B is given by*

$$B^{-1} = \frac{1}{2}(A_{\mathbf{c}} - A_{\mathbf{d}}).$$

The algorithm has complexity $O(n^2)$. In fact, the extended Euclidean algorithm for the solution of the Bezout equations requires $O(n^2)$ operations. In case of $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$, its complexity can be reduced to $O(n \log^2 n)$ (see [1], [3], [16]).

The computation in Step 2, discrete convolution or polynomial multiplication, can be done with $O(n(\log n) \log\log n)$ complexity (see [19], Section 2.4). In case of $\mathbb{R}$ or $\mathbb{C}$ one can speed up to $O(n \log n)$ complexity using FFT (see [3] and references therein).

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7. Inversion of $T+H$-Bezoutians via H-Bezoutians of half order. In our second approach, we start again from the representation (5.1) of a centroskew $T+H$-Bezoutian B of order $n=2\ell$, i.e.,

$$B = M_{n-1}^{+}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{f}_+,\mathbf{g}_+)(M_{n-1}^{-})^T + M_{n-1}^{-}\mathrm{Bez}_{\mathrm{sp}}(\mathbf{y}_+,\mathbf{z}_+)(M_{n-1}^{+})^T .$$

Recall that both $\mathrm{Bez}_{\mathrm{sp}}(\mathbf{f}_+,\mathbf{g}_+)$ and $\mathrm{Bez}_{\mathrm{sp}}(\mathbf{y}_+,\mathbf{z}_+)$ are split-Bezoutians of odd order $n-1$ and of $(+)$type since the vectors $\mathbf{f}_+,\mathbf{g}_+,\mathbf{y}_+,\mathbf{z}_+\in\mathbb{F}_+^{n+1}$ are symmetric. Introduce a matrix S_ℓ of size $(2\ell-1)\times\ell$ as the isomorphism defined by

$$S_\ell:\mathbb{F}^\ell\to\mathbb{F}_+^{2\ell-1},\qquad (S_\ell\mathbf{x})(t)=\mathbf{x}(t+t^{-1})t^{\ell-1},\quad \mathbf{x}\in\mathbb{F}^\ell.$$

Notice that

$$(S_\ell)^T=\begin{bmatrix} 0 & & & & \binom{0}{0} & & & & 0\\ & & & \binom{1}{0} & 0 & \binom{1}{1} & & & \\ & & \binom{2}{0} & 0 & \binom{2}{1} & 0 & \binom{2}{2} & & \\ & \iddots & & & & & & \ddots & \\ \binom{\ell-1}{0} & 0 & \binom{\ell-1}{1} & 0 & \cdots & 0 & \binom{\ell-1}{\ell-2} & 0 & \binom{\ell-1}{\ell-1}\end{bmatrix}.$$

It was established in [15] that the just mentioned split-Bezoutians can be reduced to H-Bezoutians of half the order ℓ.

THEOREM 7.1. *Let $\mathbf{u}_+,\mathbf{v}_+\in\mathbb{F}_+^{n+1}$, $n=2\ell$, and let $\mathbf{u},\mathbf{v}\in\mathbb{F}^{\ell+1}$ be such that $\mathbf{u}_+=S_{\ell+1}\mathbf{u}$, $\mathbf{v}_+=S_{\ell+1}\mathbf{v}$. Then*

$$\mathrm{Bez}_{\mathrm{sp}}(\mathbf{u}_+,\mathbf{v}_+)=-S_\ell\,\mathrm{Bez}_H(\mathbf{u},\mathbf{v})\,S_\ell^T.$$

Notice that the pair $\mathbf{u}(t)$ and $\mathbf{v}(t)$ is generalized coprime if and only if the pair $\mathbf{u}_+(t)$ and $\mathbf{v}_+(t)$ is coprime.

Combining this theorem with Theorem 5.3, we conclude the following.

THEOREM 7.2. *Let $n=2\ell$. Then $B\in\mathbb{F}^{n\times n}$ is a nonsingular, centroskew $T+H$-Bezoutian if and only if it can be represented in the form*

$$B=M_{n-1}^{+}S_\ell\mathrm{Bez}_H(\mathbf{g},\mathbf{f})S_\ell^T(M_{n-1}^{-})^T+M_{n-1}^{-}S_\ell\mathrm{Bez}_H(\mathbf{z},\mathbf{y})S_\ell^T(M_{n-1}^{+})^T \tag{7.1}$$

with generalized coprime pairs $\{\mathbf{f}(t),\mathbf{g}(t)\}$ and $\{\mathbf{z}(t),\mathbf{y}(t)\}$.

The vectors $\mathbf{f},\mathbf{g},\mathbf{z},\mathbf{y}\in\mathbb{F}^{\ell+1}$ are given by

$$\mathbf{f}_+=S_{\ell+1}\mathbf{f},\quad \mathbf{g}_+=S_{\ell+1}\mathbf{g},\quad \mathbf{y}_+=S_{\ell+1}\mathbf{y},\quad \mathbf{z}_+=S_{\ell+1}\mathbf{z}, \tag{7.2}$$

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or, equivalently, by

$$\mathbf{f}_+(t) = t^\ell \mathbf{f}(t+t^{-1}), \quad \mathbf{g}_+(t) = t^\ell \mathbf{g}(t+t^{-1}), \text{ etc.}$$

Let us introduce the shift matrix of order m,

$$V_m = \begin{bmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ 0 & & & & 0 \end{bmatrix},$$

as well as the matrices

$$T_m^\pm = I_m \pm V_m \,, \quad T_m = I_m - V_m^2. \tag{7.3}$$

Moreover, we need the following matrices of order m,

$$Q_m = \begin{bmatrix} \binom{0}{0} & 0 & \binom{2}{1} & 0 & \cdots & \\ & \binom{1}{0} & 0 & \binom{3}{1} & & \vdots \\ & & \binom{2}{0} & 0 & \ddots & 0 \\ & & & \binom{3}{0} & \ddots & \binom{m-1}{1} \\ & & & & \ddots & 0 \\ 0 & & & & & \binom{m-1}{0} \end{bmatrix},$$

i.e.,

$$Q_m := [\, q_{ij} \,]_{i,j=0}^{m-1} \quad \text{with} \quad q_{ij} = \begin{cases} \binom{j}{\frac{j-i}{2}} & \text{if } j \geq i \text{ and } j-i \text{ is even} \\ 0 & \text{otherwise,} \end{cases}$$

as well as

$$U_m := [\, u_{ij} \,]_{i,j=0}^{m-1} \quad \text{with} \quad u_{ij} = \begin{cases} \binom{-i-1}{\frac{j-i}{2}} & \text{if } j \geq i \text{ and } j-i \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

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Noting that $\binom{-i-1}{k} = (-1)^k\binom{i+k}{k}$, we see that

$$U_m = \begin{bmatrix} \binom{0}{0} & 0 & -\binom{1}{1} & 0 & \cdots & \\ & \binom{1}{0} & 0 & -\binom{2}{1} & & \vdots \\ & & \binom{2}{0} & 0 & \ddots & 0 \\ & & & \binom{3}{0} & \ddots & -\binom{m-2}{1} \\ & & & & \ddots & 0 \\ 0 & & & & & \binom{m-1}{0} \end{bmatrix}. \tag{7.4}$$

It can be proved straightforwardly (see also Lemma 5.1 in [5]) that

$$U_{\ell+1}T_{\ell+1}Q_{\ell+1} = I_{\ell+1}. \tag{7.5}$$

Observe that $Q_{\ell+1}$ is the lower part of $S_{\ell+1}$. The upper part, i.e., the first ℓ rows of $S_{\ell+1}$, is the $\ell \times (\ell+1)$ matrix $J_{\ell+1}Q_{\ell+1}$ after cancelling its last row.

Denoting the last $\ell+1$ components of $\mathbf{f}_+, \mathbf{g}_+, \dots$ by $\mathbf{f}_+^l, \mathbf{g}_+^l, \dots$, respectively, the relation (7.2) can be written as

$$\mathbf{f} = Q_{\ell+1}^{-1}\mathbf{f}_+^l, \quad \mathbf{g} = Q_{\ell+1}^{-1}\mathbf{g}_+^l\,, \quad \text{etc.}, \tag{7.6}$$

where $Q_{\ell+1}^{-1} = U_{\ell+1}T_{\ell+1}$.

Let us continue with discussing what follows from Theorem 7.2. The representation (7.1) can be written in the form

$$B = W_n \begin{bmatrix} \mathbf{0} & \mathrm{Bez}_H(\mathbf{g},\mathbf{f}) \\ \mathrm{Bez}_H(\mathbf{z},\mathbf{y}) & \mathbf{0} \end{bmatrix} W_n^T\,,$$

where $W_n := [M_{n-1}^+S_\ell \mid M_{n-1}^-S_\ell] \in \mathbb{F}^{n\times n}$. We are going to rewrite W_n in a suitable way. A straightforward computation yields

$$M_{n-1}^{\pm}S_\ell = \begin{bmatrix} \pm J_\ell T_\ell^{\pm}Q_\ell \\ T_\ell^{\pm}Q_\ell \end{bmatrix}.$$

Thus

$$W_n = \begin{bmatrix} J_\ell & -J_\ell \\ I_\ell & I_\ell \end{bmatrix} \begin{bmatrix} T_\ell^+Q_\ell & \mathbf{0} \\ \mathbf{0} & T_\ell^-Q_\ell \end{bmatrix}.$$

This matrix is invertible, and

$$W_n^{-1} = \frac{1}{2}\begin{bmatrix} (T_\ell^+Q_\ell)^{-1} & \mathbf{0} \\ \mathbf{0} & (T_\ell^-Q_\ell)^{-1} \end{bmatrix} \begin{bmatrix} J_\ell & I_\ell \\ -J_\ell & I_\ell \end{bmatrix}.$$

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From Theorem 4.2, we obtain an inversion formula of the form

$$B^{-1} = W_n^{-T} \begin{bmatrix} \mathbf{0} & H_\ell(\mathbf{s}_2) \\ H_\ell(\mathbf{s}_1) & \mathbf{0} \end{bmatrix} W_n^{-1}. \tag{7.7}$$

Here $\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{F}^{2\ell-1}$ are obtained by solving the Bezout equations

$$\begin{aligned} \mathbf{g}(t)\boldsymbol{\alpha}_1(t) + \mathbf{f}(t)\boldsymbol{\beta}_1(t) = 1, \\ \mathbf{g}^J(t)\boldsymbol{\gamma}_1^J(t) + \mathbf{f}^J(t)\boldsymbol{\delta}_1^J(t) = 1, \end{aligned} \tag{7.8}$$

and

$$\begin{aligned} \mathbf{z}(t)\boldsymbol{\alpha}_2(t) + \mathbf{y}(t)\boldsymbol{\beta}_2(t) = 1, \\ \mathbf{z}^J(t)\boldsymbol{\gamma}_2^J(t) + \mathbf{y}^J(t)\boldsymbol{\delta}_2^J(t) = 1. \end{aligned} \tag{7.9}$$

Computing for $i = 1, 2$,

$$\mathbf{s}_i^J(t) = -\boldsymbol{\alpha}_i(t)\boldsymbol{\delta}_i(t) + \boldsymbol{\beta}_i(t)\boldsymbol{\gamma}_i(t) \tag{7.10}$$

the inversion formula (7.7) can now be written as stated in the following result.

PROPOSITION 7.3. *Let $B \in \mathbb{F}^{n\times n}$, $n = 2\ell$, be a nonsingular, centroskew $T+H$-Bezoutian given in the from* (7.1). *Then*

$$B^{-1} = \frac{1}{4}\begin{bmatrix} -J_\ell \\ I_\ell \end{bmatrix} A_\ell^{(1)} \Big[J_\ell,\, I_\ell \Big] + \frac{1}{4}\begin{bmatrix} J_\ell \\ I_\ell \end{bmatrix} A_\ell^{(2)} \Big[-J_\ell,\, I_\ell \Big] \tag{7.11}$$

with

$$A_\ell^{(1)} := (T_\ell^- Q_\ell)^{-T} H_\ell(\mathbf{s}_1)(T_\ell^+ Q_\ell)^{-1}, \quad A_\ell^{(2)} := (T_\ell^+ Q_\ell)^{-T} H_\ell(\mathbf{s}_2)(T_\ell^- Q_\ell)^{-1}.$$

Here $\mathbf{s}_1$ and $\mathbf{s}_2$ are obtained from (7.8)–(7.10).

It now remains to identify $A_\ell^{(i)}$ as centroskew $T+H$ matrices, for which we need further results. We will make use of the following three kinds of $T+H$ matrices of order ℓ, which we introduce for a skewsymmetric vector $\mathbf{c} = (c_k)_{k=-2\ell+1}^{2\ell-1}$ and a symmetric vector $\mathbf{a}^\# = (a_k^\#)_{k=-2\ell+2}^{2\ell-2}$,

$$\mathrm{TH}_\ell^\pm(\mathbf{c}) = \Big(c_{i-j} \pm c_{i+j+1} \Big)_{i,j=0}^{\ell-1}$$

and

$$\mathrm{TH}_\ell^\#(\mathbf{a}^\#) = \Big(a_{i-j}^\# + a_{i+j}^\# \Big)_{i,j=0}^{\ell-1}.$$

The proof of the following lemma is straightforward.

LEMMA 7.4. *Let a skewsymmetric vector $\mathbf{c} = (c_k)_{k=-2\ell+1}^{2\ell-1} \in \mathbb{F}_-^{4\ell-1}$, and a symmetric vector $\mathbf{a}^\# = (a_k^\#)_{k=-2\ell+2}^{2\ell-2} \in \mathbb{F}_+^{4\ell-3}$ be related via*

$$a_k^\# = c_{k+1} - c_{k-1}.$$

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Then

$$D_\ell \mathrm{TH}_\ell^{\#}(\mathbf{a}^{\#}) D_\ell = -(T_\ell^+)^T \mathrm{TH}_\ell^-(\mathbf{c}) T_\ell^- = (T_\ell^-)^T \mathrm{TH}_\ell^+(\mathbf{c}) T_\ell^+$$

with $D_\ell := \mathrm{diag}(\frac{1}{2}, 1, 1, \ldots, 1)$ *and* $T_\ell^{\pm}$ *given by* (7.3).

Notice that the relationship between $\mathbf{a}^{\#}$ and $\mathbf{c}$ can be expressed by

$$\begin{bmatrix} 2 & & & & 0 \\ 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{2\ell-1} \end{bmatrix} = \begin{bmatrix} a_0^{\#} \\ a_1^{\#} \\ \vdots \\ a_{2\ell-2}^{\#} \end{bmatrix}.$$

THEOREM 7.5. *Given a skewsymmetric vector* $\mathbf{c} = (c_k)_{k=-2\ell+1}^{2\ell-1} \in \mathbb{F}_-^{4\ell-1}$, *define*

$$\mathbf{s} = Q_{2\ell-1}^T T_{2\ell-1}^T (c_k)_{k=1}^{2\ell-1} \in \mathbb{F}^{2\ell-2} \tag{7.12}$$

with $T_{2\ell-1}$ *introduced by* (7.3). *Then*

$$H_\ell(\mathbf{s}) = -Q_\ell^T (T_\ell^+)^T \mathrm{TH}_\ell^-(\mathbf{c}) T_\ell^- Q_\ell = Q_\ell^T (T_\ell^-)^T \mathrm{TH}_\ell^+(\mathbf{c}) T_\ell^+ Q_\ell.$$

Proof. Introduce $\mathbf{a}^{\#} = (a_k^{\#})_{k=-2\ell+2}^{2\ell-2} \in \mathbb{F}_+^{4\ell-3}$ as in the previous lemma, i.e.,

$$D_{2\ell-1}^{-1} T_{2\ell-1}^T (c_k)_{k=1}^{2\ell-1} = (a_k^{\#})_{k=0}^{2\ell-2}.$$

In [2], Theorem 5, it was shown that

$$H_\ell(\mathbf{s}) = Q_\ell^T D_\ell \mathrm{TH}_\ell^{\#}(\mathbf{a}^{\#}) D_\ell Q_\ell$$

if $\mathbf{a}^{\#}$ is symmetric and $\mathbf{s} = Q_{2\ell-1}^T D_{2\ell-1} (a_k^{\#})_{k=0}^{2\ell-2}$. Combining this with the lemma, we arrive at the stated formula. ∎

Notice that using (7.5), equation (7.12) can be written as

$$(c_k)_{k=1}^{2\ell-1} = U_{2\ell-1}^T \mathbf{s}$$

with U_m given in (7.4).

Applying Theorem 7.5 to $H_\ell(\mathbf{s}_i)$ with $\mathbf{s}_i$ given in (7.10), we see that

$$A_\ell^{(1)} = \mathrm{TH}_\ell^+(\mathbf{c}^{(1)}), \qquad A_\ell^{(2)} = -\mathrm{TH}_\ell^-(\mathbf{c}^{(2)}),$$

where

$$\mathbf{c}^{(i)} = \begin{bmatrix} -J_{2\ell-1} U_{2\ell-1}^T \mathbf{s}_i \\ 0 \\ U_{2\ell-1}^T \mathbf{s}_i \end{bmatrix}. \tag{7.13}$$

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Combining this with the above formula (7.11), it follows that

$$B^{-1} = \frac{1}{4}\begin{bmatrix} -J_\ell \\ I_\ell \end{bmatrix} \mathrm{TH}_\ell^+(\mathbf{c}^{(1)})\,[\,J_\ell \;\; I_\ell\,] - \frac{1}{4}\begin{bmatrix} J_\ell \\ I_\ell \end{bmatrix} \mathrm{TH}_\ell^-(\mathbf{c}^{(2)})\,[\,-J_\ell \;\; I_\ell\,]\,,$$

which equals

$$\frac{1}{4}\left[c^{(1)}_{j-k} + c^{(1)}_{j+k+1}\right]_{j,k=-\ell}^{\ell-1} - \frac{1}{4}\left[c^{(2)}_{j-k} - c^{(2)}_{j+k+1}\right]_{j,k=-\ell}^{\ell-1}\,,$$

i.e.,

$$\frac{1}{4}\left[c^{(1)}_{j-k} + c^{(1)}_{j+k+1-n}\right]_{j,k=0}^{n-1} - \frac{1}{4}\left[c^{(2)}_{j-k} - c^{(2)}_{j+k+1-n}\right]_{j,k=0}^{n-1}\,.$$

Summarizing we arrive at the following result.

THEOREM 7.6. *The inverse of a nonsingular, centroskew $T+H$-Bezoutian B of order $n=2\ell$ given by* (7.1) *admits the representation*

$$B^{-1} = \frac{1}{2}(T_n(\mathbf{c}^{(1)})P_+ - T_n(\mathbf{c}^{(2)})P_-), \tag{7.14}$$

where $\mathbf{c}^{(i)}$ is given in (7.13) *and* (7.8)–(7.10).

Finally, let us present the steps of a corresponding algorithm for the inversion of a centroskew $T+H$-Bezoutian.

ALGORITHM 7.7. *We are given a centroskew $T+H$-Bezoutian of order $n=2\ell$ in the form* (5.1) *with pairs $\{\mathbf{f}_+(t), \mathbf{g}_+(t)\}$ and $\{\mathbf{y}_+(t), \mathbf{z}_+(t)\}$ of symmetric coprime polynomials in $\mathbb{F}_+^{n+1}[t]$.*

1. *Compute the vectors $\mathbf{f}, \mathbf{g}, \mathbf{y}, \mathbf{z} \in \mathbb{F}^{\ell+1}$ according to* (7.6)*, where $Q_{\ell+1}^{-1} = U_{\ell+1}T_{\ell+1}$ with $U_{\ell+1}$ and $T_{\ell+1}$ defined in* (7.4) *and* (7.3).
2. *Solve the Bezout equations* (7.8) *and* (7.9) *by the extended Euclidean algorithm.*
3. *Compute the vectors $\mathbf{s}_i$ by polynomial multiplication according to* (7.10).
4. *Compute the symbols $\mathbf{c}^{(i)}$ as in* (7.13).
5. *Compute the matrices*

$$A_1 := T_n(\mathbf{c}^{(1)})P_+ \;\text{ and }\; A_2 := T_n(\mathbf{c}^{(2)})P_-\,.$$

6. *Then the inverse of B is given by*

$$B^{-1} = \frac{1}{2}(A_1 - A_2).$$

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Also this algorithm requires $O(n^2)$ complexity. In comparison to Algorithm 6.5, it has additional matrix-vector multiplications in Steps 1 and 4. The matrices contain binomial coefficients, which can be generated recursively with $O(n^2)$ operations. Matrix-vector multiplication can also be carried out with $O(n^2)$ complexity.

Notice also that there is a stability problem in case of fields $\mathbb{R}$ or $\mathbb{C}$ because the binomial coefficients in the matrices U_ℓ become exponentially large.

8. Final remarks. Comparing the inversion formulas (6.6) and (7.14) we obtain for the symbols of the Toeplitz matrices the following equalities

$$\mathbf{c}^{(1)} = \mathbf{c} \quad \text{and} \quad \mathbf{c}^{(2)} = \mathbf{d}\,.$$

This is a consequence of the uniqueness of the representation (see Proposition 3.1). To compute these symbols, we have discussed two different possibilities, based on representations of the splitting parts of B,

$$B_{+-} = M_{n-1}^{+}\mathrm{Bez_{sp}}(\mathbf{f}_+, \mathbf{g}_+)(M_{n-1}^{-})^T\,, \quad B_{-+} = M_{n-1}^{-}\mathrm{Bez_{sp}}(\mathbf{y}_+, \mathbf{z}_+)(M_{n-1}^{+})^T.$$

Indeed, the representations

$$B_{+-} = 2\,\mathrm{Bez}_T(\mathbf{f}_+, \mathbf{g}_+)P_- \quad \text{and} \quad B_{-+} = -2\,\mathrm{Bez}_T(\mathbf{y}_+, \mathbf{z}_+)P_+$$

led to the first algorithm, whereas

$$B_{+-} = M_{n-1}^{+} S_\ell \mathrm{Bez}_H(\mathbf{g}, \mathbf{f}) S_\ell^T (M_{n-1}^{-})^T$$

and

$$B_{-+} = M_{n-1}^{-} S_\ell \mathrm{Bez}_H(\mathbf{z}, \mathbf{y}) S_\ell^T (M_{n-1}^{+})^T$$

were the basis of the second algorithm.

The advantage of the first approach is that it is simpler and straightforward. One has to invert two T-Bezoutians of order n, but the symbols $\mathbf{c}, \mathbf{d}$ of the corresponding two skewsymmetric Toeplitz matrices are obtained directly after solving corresponding Bezout equations.

The second approach has the benefit that one has to invert two H-Bezoutians of half the order $\ell = \frac{n}{2}$, which involves solving corresponding Bezout equations of half the size. On the other hand, one has to perform additional matrix-vector multiplication before and after this step, where the matrices are

$$Q_{\ell+1}^{-1} = U_{\ell+1} T_{\ell+1} \quad \text{and} \quad U_{2\ell-1}^T\,.$$

The second approach is the analogue of the method for inverting centrosymmetric $T + H$-Bezoutians, which we discussed in a previous paper [5]. Thus, we have shown

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here that the method of [5] also works in the case of centroskew $T+H$-Bezoutians. The problem of how our first approach can be modified to be applicable to centrosymmetric $T+H$-Bezoutians is under investigation and will be the subject of a forthcoming paper. (Note that in this case, R_n given in (3.2) nonsingular has not the consequence that $T_n(\mathbf{c})$ and $T_n(\mathbf{d})$ are nonsingular.) It seems, somewhat surprisingly, as if the centroskewsymmetric case is easier to deal with than the centrosymmetric case.

Finally, let us comment on what our algorithms can accomplish regarding solving of linear systems, $B\mathbf{x} = \mathbf{y}$, with a nonsingular, centroskew $T+H$-Bezoutian B of order n. As $\mathbf{x} = B^{-1}\mathbf{y}$ and B^{-1} is a $T+H$ matrix, using the inverse becomes advantageous even if one has to solve such a system for very few linear right hand sides. Note that one does not have to compute (and store) the entire inverse B^{-1} but only the symbols of the Toeplitz and Hankel matrix. Multiplication of a Toeplitz or a Hankel matrix with a vector can be carried out with $O(n(\log n)\log\log n)$ complexity as already mentioned in the remark made after Algorithm 6.5. In case $\mathbb{F} = \mathbb{C}$, one can speed it up to $O(n \log n)$ complexity using FFT. If $\mathbb{F} = \mathbb{R}$ then the most efficient way is to use representations of the $T+H$ matrix B^{-1} involving only four real trigonometric transformations of length n and diagonal matrices (see [12] and references therein). Clearly, the complexity is again $O(n \log n)$.

Furthermore, notice that the symbols can be obtained from the knowledge of the first row and the first column of B^{-1}. This can be easily seen from Proposition 3.1. Thus, in view of the above, the computational cost of computing the symbol of the inverse and solving a linear system are of the same order of magnitude. The complexity of standard methods for computing the solution of a linear system (not using any structure) are worse than quadratic in n, while our $O(n^2)$ algorithms provide explicit formulas and display the connection to the extended Euclidean algorithm.

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