



ON DERIVATIVES AND NORMS OF GENERALIZED MATRIX FUNCTIONS AND RESPECTIVE SYMMETRIC POWERS*

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Abstract. In recent papers, the authors obtained formulas for directional derivatives of all orders, of the immanant and of the m -th ξ -symmetric tensor power of an operator and a matrix, when ξ is a character of the full symmetric group. The operator norm of these derivatives was also calculated. In this paper, similar results are established for generalized matrix functions and for every symmetric tensor power.

Key words. Generalized matrix function, Derivative, ξ -Symmetric tensor power.

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1. Introduction. Let ξ be an irreducible character of the symmetric group. There are formulas for the higher order derivatives of the immanant and the ξ -symmetric tensor power of an operator that were proved in [6]. In this paper, we also defined the matrix $K_\xi(A)$, which generalized the already known concepts of $\vee^m A$, m -th induced power of A and the m -th compound of A , represented by $\wedge^m A$. There are also formulas for the norms of some of these derivatives, calculated in [7], following the work done in [3] and [4].

This paper follows along the lines of our previous work, but instead of considering an irreducible character of the full symmetric group we will consider a character of any subgroup G of S_m . Some of the proofs for the derivatives carry through with some adjustments, however, when considering the norm of the derivatives of these generalized functions some new questions arise, because in this case there are no relations between the character ξ and the partitions of m .

2. Definitions. We will write $M_n(\mathbb{C})$ to represent the vector space of the square matrices of order n with complex entries. Let G be a subgroup of the permutation group of order n , S_n . Let $A \in M_n(\mathbb{C})$ and ξ be a character of G . We define the

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generalized matrix function determined by ξ and G as:

$$d_{\xi}^G(A) = \sum_{\sigma \in G} \xi(\sigma) \prod_{i=1}^n a_{i\sigma(i)}.$$

This is a multilinear map in the columns of the matrix A and a polynomial map in the matrix entries, hence differentiable.

It is our purpose to obtain formulas for higher order derivatives of this function. Let V_1, \dots, V_n be n vector spaces over \mathbb{C} , and let $\phi : V_1 \times \dots \times V_n \rightarrow \mathbb{C}$ be a multilinear form. For $A, X^1, \dots, X^k \in V_1 \times \dots \times V_n$, the k -th derivative of ϕ at A in the direction of (X^1, \dots, X^k) is given by the expression

$$D^k \phi(A)(X^1, \dots, X^k) := \frac{\partial^k}{\partial t_1 \dots \partial t_k} \Big|_{t_1 = \dots = t_k = 0} \phi(A + t_1 X^1 + \dots + t_k X^k).$$

This is a multilinear function defined on $M_n(\mathbb{C})^k$.

Given a matrix $A \in M_n(\mathbb{C})$, we will represent by $A_{[i]}$ the i -th column of A , $i = 1, \dots, n$.

Let $\Gamma_{m,n}$ be the set of all maps from the set $\{1, \dots, m\}$ into the set $\{1, \dots, n\}$. This set can also be identified with the collection of multiindices $\{(i_1, \dots, i_m) : i_j \leq n\}$. If $\alpha \in \Gamma_{m,n}$, this correspondence associates to α the m -tuple $(\alpha(1), \dots, \alpha(m))$. We will consider the lexicographic order in the set $\Gamma_{m,n}$.

We denote by $Q_{k,n}$ the set of strictly increasing maps in $\Gamma_{m,n}$, and by $G_{k,n}$ the set of increasing maps.

3. Derivatives of d_{ξ}^G . Let k be a natural number, $1 \leq k \leq n$, $A, X^1, \dots, X^k \in M_n(\mathbb{C})$, and t_1, \dots, t_k k variables.

We will denote by $A(\alpha; X^1, \dots, X^k)$ the matrix of order n obtained from A by replacing the $\alpha(j)$ column of A by the $\alpha(j)$ column of X^j . There is a known formula for the k -th directional derivative, which can be easily deduced by considering the multilinearity (see [3] or [6]).

$$D^k d_{\xi}^G(A)(X^1, \dots, X^k) = \sum_{\sigma \in S_k} \sum_{\alpha \in Q_{k,n}} d_{\xi}^G A(\alpha; X^{\sigma(1)}, \dots, X^{\sigma(k)}).$$

In particular,

$$D^k d_{\xi}^G(A)(X, \dots, X) = k! \sum_{\alpha \in Q_{k,n}} d_{\xi}^G A(\alpha; X, \dots, X).$$



We can re-write the last expression using the concept of mixed generalized matrix function. The mixed immanant, defined in [6], is a particular case of this definition.

DEFINITION 3.1. Let X^1, \dots, X^n be n matrices of order n . We define the *mixed generalized matrix function* of X^1, \dots, X^n as

$$\Delta_\xi^G(X^1, \dots, X^n) := \frac{1}{n!} \sum_{\sigma \in S_n} d_\xi^G(X_{[1]}^{\sigma(1)}, \dots, X_{[n]}^{\sigma(n)}).$$

If $X^1 = \dots = X^t = A$, for some $t \leq n$ and $A \in M_n(\mathbb{C})$, we denote it by

$$\Delta_\xi^G(A; X^{t+1}, \dots, X^n).$$

We have that $\Delta_\xi^G(A, \dots, A) = d_\xi^G(A)$.

PROPOSITION 3.2. Let $A \in M_n(\mathbb{C})$. We have that

$$\Delta_\xi^G(A; X^1, \dots, X^k) := \frac{(n-k)!}{n!} \sum_{\sigma \in S_k} \sum_{\alpha \in Q_{k,n}} d_\xi^G A(\alpha; X^{\sigma(1)}, \dots, X^{\sigma(k)}).$$

Proof. We simply have to observe that each summand in $\Delta_\xi^G(A; X^1, \dots, X^k)$ appears $(n-k)!$ times: once we fix a permutation of the matrices X^1, \dots, X^k , these summands correspond to the possible permutations of the $n-k$ matrices equal to A . \square

As an immediate consequence of this result, we can re-write the formula we obtained for the derivative of d_ξ^G .

THEOREM 3.3.

$$(3.1) \quad D^k d_\xi^G(A)(X^1, \dots, X^k) = \frac{n!}{(n-k)!} \Delta_\xi^G(A; X^1, \dots, X^k).$$

4. On ξ -symmetric tensor powers. We present some classic facts and notation about ξ -symmetric powers that can be found in [10, Chapter 6]. In this section, G is a subgroup of S_m and ξ is an irreducible character of G . Define $T(G, \xi) \in \mathcal{L}(\otimes^m V)$ as

$$T(G, \xi) = \frac{\xi(\text{id})}{|G|} \sum_{\sigma \in G} \xi(\sigma) P(\sigma),$$

where id stands for the identity element of S_m and $P(\sigma)(v_1 \otimes \dots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(m)}$. The range of $T(G, \xi)$ is called the *symmetry class of tensors* associated

with the irreducible character ξ and it is represented by $V_\xi(G) = T(G, \xi)(\otimes^m V)$. We denote

$$v_1 * v_2 * \cdots * v_m = T(G, \xi)(v_1 \otimes v_2 \otimes \cdots \otimes v_m).$$

These vectors belong to $V_\xi(G)$ and are called *decomposable symmetrized tensors*.

Given $T \in \mathcal{L}(V)$, it is known that $V_\xi(G)$ is an invariant subspace for $\otimes^m T$. We define the ξ -symmetric tensor power of T as the restriction of $\otimes^m T$ to $V_\xi(G)$, and denote it by $K_\xi^G(T)$.

The group G acts on $\Gamma_{m,n}$ by the action $(\sigma, \alpha) \rightarrow \alpha \circ \sigma^{-1}$ where $\sigma \in S_m$ and $\alpha \in \Gamma_{m,n}$. The orbit of $\alpha \in \Gamma_{m,n}$ is $\{\alpha \cdot \sigma : \sigma \in G\} \subseteq \Gamma_{m,n}$ and the stabilizer of α is $G_\alpha = \{\sigma \in S_m : \alpha \cdot \sigma = \alpha\}$.

If V is a Hilbert space and $E = \{e_1, \dots, e_n\}$ is an orthonormal basis of V , then the set

$$\{e_\alpha^\otimes = e_{\alpha(1)} \otimes e_{\alpha(2)} \otimes \cdots \otimes e_{\alpha(m)} : \alpha \in \Gamma_{m,n}\}$$

is an orthonormal basis of the m -th tensor power of V . By the definition of $V_\xi(G)$, we have

$$V_\xi(G) = \langle \{e_\alpha^* = T(G, \xi)(e_\alpha^\otimes) : \alpha \in \Gamma_{m,n}\} \rangle.$$

In general, this is not a basis of $V_\xi(G)$. Let

$$\Omega = \Omega_\xi = \{\alpha \in \Gamma_{m,n} : e_\alpha^* \neq 0\}.$$

For a characterization of the nonzero decomposable symmetrized tensors, we have the formula

$$(4.1) \quad \|e_\alpha^*\|^2 = \frac{\xi(\text{id})}{|G|} \sum_{\sigma \in G_\alpha} \xi(\sigma).$$

So the nonzero decomposable symmetrized tensors are $\{e_\alpha^* : \alpha \in \Omega\}$. Let Δ be the system of distinct representatives for the quotient set $\Gamma_{m,n}/G$, constructed by taking the first element in each orbit, for the lexicographic order of indices. It is easy to check that $\Delta \subseteq G_{m,n}$, where $G_{m,n}$ is the set of all increasing sequences of $\Gamma_{m,n}$.

It can be proved that the set $\{e_\alpha^* : \alpha \in \Delta \cap \Omega\}$ is linearly independent. So there is a set $\widehat{\Delta}$, such that $\Delta \cap \Omega \subseteq \widehat{\Delta} \subseteq \Omega$ such that

$$\mathcal{E}' := \{e_\alpha^* : \alpha \in \widehat{\Delta}\}$$

is a basis for $V_\xi(G)$. We will consider this basis fixed.

5. Derivatives of the ξ -symmetric tensor power of an operator. In this section, we present a formula for higher order derivatives that generalizes formulas in [2] and [3]. It is known that, given $X^1, \dots, X^m \in \mathcal{L}(V)$, the space $V_\xi(G)$ is invariant for the map defined as

$$X^1 \tilde{\otimes} X^2 \tilde{\otimes} \dots \tilde{\otimes} X^m := \frac{1}{m!} \sum_{\sigma \in S_m} X^{\sigma(1)} \otimes \dots \otimes X^{\sigma(m)}.$$

See for instance [10, p. 184]. We will denote the restriction of this map to $V_\xi(G)$ by $X^1 * \dots * X^m$ and call it the *symmetrized ξ -symmetric tensor product* of the operators X^1, \dots, X^m . We remark that this notation does not convey the fact that the product depends on the character ξ and the subgroup G . In [3], the following formula is deduced:

$$(5.1) \quad D^k(\otimes^m T)(X^1, \dots, X^k) = \frac{m!}{(m-k)!} \underbrace{T \tilde{\otimes} \dots \tilde{\otimes} T}_{m-k \text{ copies}} \tilde{\otimes} X^1 \tilde{\otimes} \dots \tilde{\otimes} X^k.$$

If $k > m$ all derivatives are zero. From this we can deduce a formula for the derivative $D^k K_\xi^G(T)(X^1, \dots, X^k)$, using the same techniques. We need the following formulas, also from [3]: for L linear and f a k times differentiable function, we have:

$$(5.2) \quad D^k(L \circ f)(a)(x^1, \dots, x^k) = L \circ D^k f(a)(x^1, \dots, x^k).$$

$$(5.3) \quad D^k(f \circ L)(A)(x^1, \dots, x^k) = D^k f(L(A))(L(x^1), \dots, L(x^k)).$$

THEOREM 5.1. *Using the notation we have established, we have*

$$D^k K_\xi^G(T)(X^1, \dots, X^k) = \frac{m!}{(m-k)!} T * \dots * T * X^1 * \dots * X^k.$$

If $m = k$ this formula does not depend on T , and if $k > m$ all derivatives are zero.

Proof. We follow the lines of the proof in [3]. Let Q be the inclusion map defined as $Q : V_\xi(G) \rightarrow \otimes^m V$, so its adjoint operator Q^* is the projection of $\otimes^m V$ onto $V_\xi(G)$. We have

$$T_1 * \dots * T_m = Q^*(T_1 \tilde{\otimes} \dots \tilde{\otimes} T_m)Q.$$

Both maps $L \mapsto Q^*L$ and $L \mapsto LQ$ are linear, so we can apply formulas (5.1) and (5.2) and get

$$\begin{aligned} D^k K_\xi^G(T)(X^1, \dots, X^n) &= D^k(Q^*(\otimes^m T)Q)(X^1, \dots, X^k) \\ &= Q^* D^k(\otimes^m T)(X^1, \dots, X^k)Q \\ &= \frac{m!}{(m-k)!} Q^* \underbrace{(T \tilde{\otimes} \dots \tilde{\otimes} T)}_{m-k \text{ times}} \tilde{\otimes} X^1 \tilde{\otimes} \dots \tilde{\otimes} X^k Q \\ &= \frac{m!}{(m-k)!} T * \dots * T * X^1 * \dots * X^k. \quad \square \end{aligned}$$



6. Derivatives of the ξ -symmetric tensor power of a matrix. In this section, we wish to establish a formula for the k -th derivative of the ξ -symmetric tensor power of a matrix. Before we can do this, we need quite a bit of definitions, including the very definition of this matrix.

Recall \mathcal{E}' , the induced basis of $V_\xi(G)$, and let $\mathcal{E} = \{v_\alpha : \alpha \in \widehat{\Delta}\}$ be an orthonormal basis of $V_\xi(G)$.

The basis \mathcal{E} can be obtained from \mathcal{E}' via the Gram-Schmidt process, as was done in [6] and [7], but it can be any other orthonormal basis. In [9], for instance, other bases are presented, and these may eventually be more adequate.

Let B be the change of basis matrix from \mathcal{E}' to \mathcal{E} , $B = M(\text{id}_{V_\xi(G)}; \mathcal{E}', \mathcal{E})$. Since \mathcal{E} is obtained from \mathcal{E}' via the Gram-Schmidt process, this matrix *does not depend on the choice of the orthonormal basis of V* . The Gram-Schmidt process only depends on the numbers $\langle e_\alpha^*, e_\beta^* \rangle$, and, by [10, p. 163], these are given by formula

$$\langle e_\alpha^*, e_\beta^* \rangle = \frac{\xi(\text{id})}{|G|} \sum_{\sigma \in G} \xi(\sigma) \prod_{t=1}^m \langle e_{\alpha(t)}, e_{\beta\sigma(t)} \rangle.$$

Hence, they only depend on the values of $\langle e_i, e_j \rangle = \delta_{ij}$ and are independent of the vectors themselves.

We now present a technical result, from [10, p. 230] that will help us to relate the operator $K_\xi^G(T)$ to its matrix, with respect to the bases we have defined.

THEOREM 6.1. *Suppose ξ is an irreducible character of the group G . Let $E = \{e_1, \dots, e_n\}$ be an orthonormal basis of the inner product space V . Let $T \in L(V, V)$ be the unique linear operator such that $M(T, E) = A$.*

If $\alpha, \beta \in \Gamma_{m,n}$, then

$$\langle K_\xi^G(T)(e_\alpha^*), e_\beta^* \rangle = \frac{\xi(\text{id})}{|G|} d_\xi^G(A^T[\alpha|\beta]).$$

We now define $K_\xi^G(A)$, the m -th ξ -symmetric tensor power of the matrix A . As in [6], we fix an orthonormal basis E in V and consider the linear endomorphism T such that $A = M(T, E)$. Define

$$K_\xi^G(A) := M(K_\xi^G(T), \mathcal{E}).$$

The matrix $K_\xi^G(A)$ has order $t = |\widehat{\Delta}|$, with $|Q_{m,n}| \leq t$ and it *will depend on the orthonormal basis \mathcal{E}* — for different orthonormal bases, we may get different ξ -symmetric powers. This is expressed by the presence of the matrix B in several formulas.

We have that, for $\alpha, \beta \in \widehat{\Delta}$, the (α, β) entry of $K_\xi^G(A)$ is

$$\begin{aligned} \langle K_\xi^G(T)(v_\beta), v_\alpha \rangle &= \sum_{\gamma, \delta \in \widehat{\Delta}} \langle b_{\gamma\beta} K_\xi^G(T)(e_\gamma^*), b_{\delta\alpha} e_\delta^* \rangle \\ &= \sum_{\gamma, \delta \in \widehat{\Delta}} b_{\gamma\beta} \overline{b_{\delta\alpha}} \langle K_\xi^G(T)e_\gamma^*, e_\delta^* \rangle \\ &= \frac{\xi(\text{id})}{|G|} \sum_{\gamma, \delta \in \widehat{\Delta}} b_{\gamma\beta} \overline{b_{\delta\alpha}} d_\xi^G(A[\delta|\gamma]^T) \\ &= \frac{\xi(\text{id})}{|G|} \sum_{\gamma, \delta \in \widehat{\Delta}} b_{\gamma\beta} \overline{b_{\delta\alpha}} d_\xi^G(A[\delta|\gamma]). \end{aligned}$$

Denote by $\text{gmm}_\xi(A)$ the square matrix with rows and columns indexed by $\widehat{\Delta}$, whose (γ, δ) entry is $d_\xi^G(A[\gamma|\delta])$ (the letters “gmm” stand for “generalized matrix (function)” and “minors”). With this definition, we can rewrite the previous equation as

$$(6.1) \quad K_\xi^G(A) = \frac{\xi(\text{id})}{|G|} B^* \text{gmm}_\xi(A) B.$$

Finally, denote by $\text{mixgmm}_\xi(X^1, \dots, X^n)$ the square matrix having rows and columns indexed by $\widehat{\Delta}$, whose (γ, δ) entry is $\Delta_\xi^G(X^1[\gamma|\delta], \dots, X^n[\gamma|\delta])$. With this definition, $\text{mixgmm}_\xi(A, \dots, A) = \text{gmm}_\xi(A)$. We use the same shorthand as with the mixed generalized matrix function: for $k \leq n$,

$$\text{mixgmm}_\xi(A; X^1, \dots, X^k) := \text{mixgmm}_\xi(A, \dots, A, X^1, \dots, X^k).$$

Using all the notation we have so far, we have the following result.

THEOREM 6.2.

$$D^k K_\xi^G(A)(X^1, \dots, X^k) = \frac{\xi^{S_m}(\text{id})}{(m-k)!} B^* \text{mixgmm}_\xi(A; X^1, \dots, X^k) B,$$

where ξ^{S_m} is the induced character, and $\xi^{S_m}(\text{id}) = [S_m : G]\xi(\text{id}) = m!\xi(\text{id})/|G|$.

Proof. We adapt arguments from [6]. Since the map $A \mapsto A[\delta|\gamma]$ is linear, we can apply formula (5.3) to compute the derivatives of the entries of the matrix $K_\xi(A)$. By formula (6.1), the (α, β) entry of the matrix $D^k K_\xi^G(A)(X^1, \dots, X^k)$ is

$$\frac{\xi(\text{id})}{|G|} \sum_{\gamma, \delta \in \widehat{\Delta}} b_{\gamma\beta} \overline{b_{\delta\alpha}} D^k d_\xi(A[\delta|\gamma])(X^1[\delta|\gamma], \dots, X^k[\delta|\gamma]).$$

To abbreviate notation, for fixed $\gamma, \delta \in \widehat{\Delta}$, we will write $C := A[\delta|\gamma]$, and $Z^i := X^i[\delta|\gamma]$, $i = 1, \dots, k$. Using formula (3.1), we get

$$\begin{aligned} D^k d_\xi(A[\delta|\gamma])(X^1[\delta|\gamma], \dots, X^k[\delta|\gamma]) &= D^k d_\xi(C)(Z^1, \dots, Z^k) \\ &= \frac{m!}{(m-k)!} \Delta_\xi(C; Z^1, \dots, Z^k). \end{aligned}$$

So the (α, β) entry of $D^k K_\xi(A)(X^1, \dots, X^k)$ is

$$\begin{aligned} &\frac{\xi(\text{id})}{|G|} \sum_{\gamma, \delta \in \widehat{\Delta}} b_{\gamma\beta} \overline{b_{\delta\alpha}} \frac{m!}{(m-k)!} \Delta_\xi(C; Z^1, \dots, Z^k) = \\ &\frac{\xi(\text{id})m!}{|G|(m-k)!} \sum_{\gamma, \delta \in \widehat{\Delta}} b_{\gamma\beta} \overline{b_{\delta\alpha}} \Delta_\xi(A[\delta|\gamma]; X^1[\delta|\gamma], \dots, X^k[\delta|\gamma]). \end{aligned}$$

According to the definition of $\text{mixgmm}_\xi(A; X^1, \dots, X^k)$, we have

$$D^k K_\xi(A)(X^1, \dots, X^k) = \frac{\xi(\text{id})m!}{|G|(m-k)!} B^* \text{mixgmm}_\xi(A; X^1, \dots, X^k) B.$$

Finally, following [10, p. 97], we can reinterpret the constant:

$$m! \xi(\text{id}) / |G| = [S_m : G] \xi(\text{id}) = \xi^{S_m}(\text{id}). \quad \square$$

The formula obtained for the higher order derivatives of $K_\xi^G(A)(X^1, \dots, X^k)$ generalizes the expression obtained in [6] for the case $G = S_m$ (the only difference is that the present formula has ξ^{S_m} instead of ξ).

7. Norms. We now obtain upper bounds for the norm of these derivatives. We will need some more results and definitions, which we present in this section.

A partition π of m is an m -tuple of positive integers $\pi = (\pi_1, \dots, \pi_r)$, such that

- $\pi_1 \geq \dots \geq \pi_r$,
- $\pi_1 + \dots + \pi_r = m$.

The number of nonzero entries in the partition π is called the *length* of π and is represented by $l(\pi)$.

Given an n -tuple of real numbers $x = (x_1, \dots, x_n)$ and $\alpha \in \Gamma_{m,n}$, we define the m -tuple

$$x_\alpha := (x_{\alpha(1)}, x_{\alpha(2)}, \dots, x_{\alpha(m)}).$$

For every partition $\pi = (\pi_1, \pi_2, \dots, \pi_{l(\pi)}, 0, \dots, 0)$ of m we define $\omega(\pi)$ as

$$\omega(\pi) := (\underbrace{1, \dots, 1}_{\pi_1 \text{ times}}, \underbrace{2, \dots, 2}_{\pi_2 \text{ times}}, \dots, \underbrace{l(\pi), \dots, l(\pi)}_{\pi_{l(\pi)} \text{ times}}) \in G_{m,n} \subseteq \Gamma_{m,n}.$$

For each $\alpha \in \Gamma_{m,n}$ let $\text{Im } \alpha = \{i_1, \dots, i_l\}$, suppose that $|\alpha^{-1}(i_1)| \geq \dots \geq |\alpha^{-1}(i_l)|$. The partition of m

$$\mu(\alpha) := (|\alpha^{-1}(i_1)|, \dots, |\alpha^{-1}(i_l)|, 0, \dots, 0)$$

is called the *multiplicity partition* of α . The multiplicity partition of $\omega(\pi)$ is equal to the partition π :

$$\mu(\omega(\pi)) = \pi.$$

We have that $\text{Im } \omega(\pi) = \{1, 2, \dots, l(\pi)\}$ and that $|\alpha^{-1}(i)| = \pi_i$, for every $i = 1, 2, \dots, l(\pi)$. So

$$\mu(\omega(\pi)) = (|\alpha^{-1}(1)|, |\alpha^{-1}(2)|, \dots, |\alpha^{-1}(l(\pi))|) = (\pi_1, \pi_2, \dots, \pi_{l(\pi)}) = \pi.$$

We recall a well known order defined on the set of partitions of m . A partition $\lambda = (\lambda_1, \dots, \lambda_m)$ majorizes $\pi = (\pi_1, \dots, \pi_m)$, written $\pi \preceq \lambda$, if, for all $1 \leq s \leq m$,

$$\sum_{j=1}^s \pi_j \leq \sum_{j=1}^s \lambda_j.$$

When $G = S_m$ there is a canonical relation between the irreducible characters of S_m and the partitions of m . For a partition π , we denote by χ_π the character of S_m associated with it. This relation enables us to find an explicit formula for the norm of the k -th derivative of $K_{\chi_\pi}(T)$, this is formula (4.1) in [7]:

$$\|D^k K_{\chi_\pi}(T)\| = k! p_{m-k}(\nu_{\omega(\pi)}),$$

where $\nu_1 \geq \dots \geq \nu_n$ are the singular values of the operator T and p_{m-k} is the elementary symmetric polynomial of degree $m - k$ in m variables.

There is also a classical result that characterizes the set $\Omega_{\chi_\pi} = \{\alpha \in \Gamma_{m,n} : e_\alpha^* \neq 0\}$ and is [10, Theorem 6.37]: for π a partition of m and $\alpha \in \Gamma_{m,n}$, we have $e_\alpha^* \neq 0$ if and only if $\mu(\alpha) \preceq \pi$.

When G is any subgroup of S_m this relation between partitions and irreducible characters of G does not exist. In [8], J.A. Dias da Silva and A. Fonseca introduced the notion of *multilinearity partition* which was used to generalize the previous result.

DEFINITION 7.1. Suppose ξ is an irreducible character of G . The multilinearity partition of the character ξ , $\text{MP}(\xi)$, is the least upper bound of the partitions π of m for which $(\xi, \chi_\pi)_G \neq 0$.

When $G = S_m$, the multilinearity partition is the partition usually associated with ξ . In the same paper the authors also prove the next result, which we will use later.

THEOREM 7.2. *Suppose ξ is an irreducible character of G and let $\alpha \in \Gamma_{m,n}$. If $e_\alpha^* = T(G, \xi)(e_\alpha^\otimes) \neq 0$, then $\mu(\alpha) \preceq \text{MP}(\xi)$.*

8. Norm of the k -th derivative of $K_\xi^G(T)$. Let U and V be finite dimensional Hilbert spaces. We recall that the norm of a multilinear operator $\Phi : (\mathcal{L}(V))^k \rightarrow \mathcal{L}(U)$ is given by

$$\|\Phi\| = \sup_{\|X^1\|=\dots=\|X^k\|=1} \|\Phi(X^1, \dots, X^k)\|.$$

The main result of this section is an upper bound for the norm of the map $T \rightarrow K_\xi^G(T)$. The proof of this result is inspired in the techniques used in [1]. We will now highlight the most important features of the proof.

By the polar decomposition, we know that for every $T \in \mathcal{L}(V)$ there are a positive semidefinite operator P and a unitary operator W such that $T = PW$. Moreover, the eigenvalues of P are the singular values of T .

PROPOSITION 8.1. *With the above notation, we have*

$$\|D^k K_\xi^G(T)\| = \|D^k K_\xi^G(P)\|.$$

The proof follows the lines of the one in [7], where it was done for $G = S_m$. It is based on the fact that this norm is unitarily invariant.

Now we need to estimate the norm of the operator $D^k K_\xi^G(P)$. For this, we use a result from [3], a multilinear version of the Russo-Dye theorem, which we quote here. A multilinear operator Φ is said to be *positive* if $\Phi(X^1, \dots, X^k)$ is a positive semidefinite operator whenever X^1, \dots, X^k are so.

THEOREM 8.2 (Russo-Dye multilinear version). *Let $\Phi : \mathcal{L}(V)^k \rightarrow \mathcal{L}(U)$ be a positive multilinear operator. Then*

$$\|\Phi\| = \|\Phi(I, I, \dots, I)\|.$$

We have that $D^k K_\xi^G(P)$ is a positive multilinear operator, since if X^1, \dots, X^k are positive semidefinite, then by the formula in Theorem 5.1, $D^k K_\xi^G(P)(X^1, \dots, X^k)$ is the restriction of a positive semidefinite operator to an invariant subspace, and thus is positive semidefinite.

Therefore,

$$\|D^k K_\xi^G(T)\| = \|D^k K_\xi^G(P)\| = \|D^k K_\xi^G(P)(I, I, \dots, I)\|.$$

Now, by the definition of the norm, we have to find the maximum eigenvalue of $D^k K_\xi^G(P)(I, I, \dots, I)$. First, we will find a basis of $V_\xi(G)$ formed by eigenvectors of $D^k K_\xi(P)(I, I, \dots, I)$. If $E = \{e_1, \dots, e_n\}$ is an orthonormal basis of eigenvectors for P , then $\{e_\alpha^* : \alpha \in \widehat{\Delta}\}$ will be a basis of eigenvectors for $D^k K_\xi^G(P)(I, I, \dots, I)$ (in general, it will not be orthonormal).

The following proposition gives the expression for the eigenvalues of the derivative $D^k K_\xi^G(P)(I, I, \dots, I)$, the proof can be found in [7].

PROPOSITION 8.3. *Let $\alpha \in \widehat{\Delta}$ and define*

$$\lambda(\alpha) := k! p_{m-k}(\nu_\alpha).$$

Then $\lambda(\alpha)$ is the eigenvalue of $D^k K_\xi^G(P)(I, I, \dots, I)$ associated with the eigenvector e_α^ .*

We have obtained the expression for all the eigenvalues of $D^k K_\xi(P)(I, \dots, I)$. In [7], it was possible to find the largest one, which then coincided with the norm of the operator. In the present situation, it was only possible to find an upper bound for all values $\lambda(\alpha)$, which will then be an upper bound for the norm.

LEMMA 8.4. *If $\alpha, \beta \in \widehat{\Delta}$ are in the same orbit, then $\lambda(\alpha) = \lambda(\beta)$.*

Proof. If α and β are in the same orbit, then there is $\sigma \in S_m$ such that $\alpha\sigma = \beta$. So by the definition of the elementary symmetric polynomials, we have

$$p_{m-k}(\nu_\beta) = p_{m-k}(\nu_{\alpha\sigma}) = p_{m-k}(\nu_\alpha). \quad \square$$

According to the results in Section 4, every orbit has a representative in $G_{m,n}$, and this is the first element in each orbit (for the lexicographic order). Therefore, the norm of the k -th derivative of $K_\xi^G(T)$ is attained at some $\lambda(\alpha)$ with $\alpha \in \widehat{\Delta} \cap G_{m,n}$. We now compare eigenvalues coming from different elements of $\widehat{\Delta} \cap G_{m,n}$.

LEMMA 8.5. *Let α, β be elements of $\widehat{\Delta} \cap G_{m,n}$ and π be a partition of m . We have the following results.*

1. $\lambda(\alpha) \geq \lambda(\beta)$ if and only if α precedes β in the lexicographic order.
2. If $\mu(\alpha) \preceq \pi$ then $\omega(\pi)$ precedes α in the lexicographic order.

Proof. 1. The result follows directly from the expression of the eigenvalues of $D^k K_\xi(P)(I, \dots, I)$ given in Proposition 8.3.

2. It is a matter of examining the definitions. \square

We are now ready to state and prove the main theorem.

THEOREM 8.6. *Let V be an n -dimensional Hilbert space. Let m and k be positive*

integers such that $1 \leq k \leq m \leq n$, and let ξ be an irreducible character of G . Consider the map $T \rightarrow K_\xi^G(T)$. Then

$$\|D^k K_\xi(T)\| \leq k! p_{m-k}(\nu_{\omega(\text{MP}(\xi))})$$

where p_{m-k} is the symmetric polynomial of degree $m-k$ in m variables, $\nu_1 \geq \dots \geq \nu_n$ are the singular values of T and $\text{MP}(\xi)$ the multilinearity partition of ξ .

Proof. For $\alpha \in \widehat{\Delta} \cap G_{m,n}$ we have $e_\alpha^* \neq 0$. By definition, $\text{MP}(\xi)$ majorizes $\mu(\alpha)$, where ξ is the irreducible character of G such that $T(\xi, G)(e_\alpha^\otimes) = e_\alpha^*$. On the other hand $\mu(\alpha) \preceq \text{MP}(\xi)$, so, by the previous lemma, $\omega(\text{MP}(\xi))$ precedes α in the lexicographic order. Again by the lemma, we can conclude that $\lambda(\omega(\text{MP}(\xi))) \geq \lambda(\alpha)$ for every $\alpha \in \widehat{\Delta} \cap G_{m,n}$.

By the definition of $\|D^k K_\xi^G(T)\|$, we have

$$\|D^k K_\xi^G(T)\| \leq k! p_{m-k}(\nu_{\omega(\text{MP}(\xi))}). \quad \square$$

It can be shown that if $A = M(T, E)$, and, for each $1 \leq i \leq k$, $S^i = M(X^i, E)$, then $D^k K_\xi^G(A)(S^1, \dots, S^k)$ is the matrix of $D^k K_\xi^G(T)(X^1, \dots, X^k)$ with respect to the orthonormal basis \mathcal{E} . The proof is computational and a bit intricate, but straightforward. It follows along the lines of what is done at the end of Chapter 2 of [5]. This means that the upper bound we got for the norm of $D^k K_\xi^G(T)$ applies to $D^k K_\xi^G(A)$:

$$\|D^k K_\xi(A)\| \leq k! p_{m-k}(\nu_{\omega(\text{MP}(\xi))}).$$

9. Norm of the k -th derivative of d_ξ^G . We now wish to establish an upper bound for the k -th derivative of the generalized matrix function d_ξ^G . Let G be a subgroup of S_n , here we take $m = n$.

THEOREM 9.1. *Keeping with the notation established, we have that, for $k \leq n$,*

$$\|D^k d_\xi^G(A)\| \leq k! p_{n-k}(\nu_{\omega(\text{MP}(\xi))}).$$

Proof. We will adjust the arguments in [7] to our situation (even though the formula is similar, some constants are changed).

Denote $\gamma := (1, 2, \dots, n) \in Q_{n,n} \subseteq \widehat{\Delta}$ (the only element in $Q_{n,n}$). By definition, $d_\xi^G(A)$ is the (γ, γ) entry of $\text{gmm}_\xi(A)$, and, according to formula (6.1), we have

$$\text{gmm}_\xi(A) = \frac{|G|}{\xi(\text{id})} (B^*)^{-1} K_\xi^G(A) B^{-1}.$$

Since multiplication by a constant matrix is a linear map, we have

$$D^k((B^*)^{-1} K_\xi^G(A) B^{-1})(X^1, \dots, X^k) = (B^*)^{-1} D^k K_\xi^G(A)(X^1, \dots, X^k) B^{-1}.$$

We denote by C the column γ of the matrix B^{-1} :

$$C = (B^{-1})_{[\gamma]} = (b'_{\alpha\gamma}), \quad \alpha \in \widehat{\Delta}.$$

Then

$$D^k d_{\xi}^G(A)(X^1, \dots, X^k) = \frac{|G|}{\xi(\text{id})} C^* D^k K_{\xi}^G(A)(X^1, \dots, X^k) C.$$

By formula (4.1), we have that

$$\|e_{\gamma}^*\|^2 = \frac{\xi(\text{id})}{|G|}.$$

By definition of the matrix B , we have

$$e_{\gamma}^* = \sum_{\beta \in \widehat{\Delta}} b'_{\beta\gamma} v_{\beta}$$

with $C = [b'_{\beta\gamma} : \beta \in \widehat{\Delta}]$. Since the basis $(v_{\alpha} : \alpha \in \widehat{\Delta})$ is orthonormal, we have

$$\|C\|^2 = \|C\|_2^2 = \|e_{\gamma}^*\|^2 = \frac{\xi(\text{id})}{|G|},$$

where $\|C\|_2$ is the Euclidean norm of C . Therefore,

$$\begin{aligned} \|D^k d_{\xi}^G(A)\| &= \frac{|G|}{\xi(\text{id})} \|C D^k K_{\xi}^G(A) C^*\| \\ &\leq \frac{|G|}{\xi(\text{id})} \|C\|^2 \|D^k K_{\xi}^G(A)\| \\ &\leq k! p_{n-k}(\nu_{\omega(\text{MP}(\xi))}). \quad \square \end{aligned}$$

We finish by applying these results to perturbations, using Taylor's formula:

$$\|f(a+x) - f(a)\| \leq \sum_{k=1}^p \frac{1}{k!} \|D^k f(a)\| \|x\|^k.$$

COROLLARY 9.2. *According to our notation, we have, for $T, X \in \mathcal{L}(V)$ and $A, Y \in M_n(\mathbb{C})$:*

$$\|K_{\xi}^G(T) - K_{\xi}^G(T+X)\| \leq \sum_{k=1}^m p_{m-k}(\nu_{\omega(\text{MP}(\xi))}) \|X\|^k,$$

$$|d_{\xi}^G(A) - d_{\xi}^G(A+Y)| \leq \sum_{k=1}^n p_{n-k}(\nu_{\omega(\text{MP}(\xi))}) \|Y\|^k.$$

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