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NOTE ON THE JORDAN FORM OF AN IRREDUCIBLE EVENTUALLY NONNEGATIVE MATRIX*

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Abstract. A square complex matrix A is eventually nonnegative if there exists a positive integer k_0 such that for all $k \geq k_0$, $A^k \geq 0$; A is strongly eventually nonnegative if it is eventually nonnegative and has an irreducible nonnegative power. It is proved that a collection of elementary Jordan blocks is a Frobenius Jordan multiset with cyclic index r if and only if it is the multiset of elementary Jordan blocks of a strongly eventually nonnegative matrix with cyclic index r. A positive answer to an open question and a counterexample to a conjecture raised by Zaslavsky and Tam are given. It is also shown that for a square complex matrix A with index at most one, A is irreducible and eventually nonnegative if and only if A is strongly eventually nonnegative.

Key words. Irreducible eventually nonnegative, Strongly eventually nonnegative, Eventually reducible, Eventually r-cyclic, Cyclic index, Frobenius collection, Frobenius Jordan multiset, Jordan multiset, Jordan form.

AMS subject classifications. 15B48, 15A21, 15A18.

1. Introduction.

A square complex matrix A is *eventually nonnegative* (respectively, *eventually positive*) if there exists a positive integer k_0 such that for all $k \geq k_0$, $A^k \geq 0$ (respectively, $A^k > 0$); for an eventually nonnegative matrix, the least such k_0 is the *power index* of A. Eventually nonnegative matrices and their subclasses have been studied extensively since their introduction in [7] by Friedland; see [3, 4, 6, 8, 11, 12, 13] and the references therein.

A matrix is *strongly eventually nonnegative* if it is eventually nonnegative and has an irreducible nonnegative power. Eventually positive matrices and strongly eventually nonnegative matrices retain much of the Perron-Frobenius structure of positive and irreducible nonnegative matrices, respectively. A matrix A is *eventually reducible* if there exists a positive integer k_0 such that for all $k \geq k_0$, A^k is reducible [9]. For an eventually nonnegative matrix A with power index k_0, if for some $k \geq k_0$, A^k is irreducible, then A is clearly strongly eventually nonnegative. Otherwise, A is

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eventually reducible. So we have:

OBSERVATION 1.1. If $n \geq 2$ and $A \in \mathbb{C}^{n\times n}$ is eventually nonnegative, then A is eventually reducible or A is strongly eventually nonnegative.

In Section 2, we show that a collection of elementary Jordan blocks is a Frobenius Jordan multiset with cyclic index r if and only if it is the multiset of elementary Jordan blocks of a strongly eventually nonnegative matrix with cyclic index r (see Theorem 2.10). We also establish a positive answer to a question of Zaslavsky and Tam [13] (see Theorem 2.6), present a counterexample to a conjecture in the same paper (see Example 2.8), and prove other results related to strongly eventually nonnegative matrices and their Jordan forms. The remainder of this introduction contains notation and definitions.

We adopt much of the terminology of [8], [9], and [13], introducing new terms as needed (we follow the convention in [8] that the 1×1 zero matrix is reducible, rather than the usage in [9] and [13] that all 1×1 matrices are irreducible). Note that while the results in [8] are stated for real matrices, they are actually true for complex matrices (and with the same proofs). For $r \geq 2$, a square matrix $A \in \mathbb{C}^{n\times n}$ is called *r-cyclic* (or *r-cyclic with partition* Π) if there is an ordered partition $\Pi = (V_1, \ldots, V_r)$ of $\{1, \ldots, n\}$ into r nonempty sets such that $A[V_i|V_j] = 0$ unless $j \equiv i + 1 \mod r$ (where $A[R|C]$ denotes the *submatrix* of A whose rows and columns are indexed by R and C, respectively). The largest r such that A is r-cyclic is the *cyclic index* of A; if A is not r-cyclic for any $r \geq 2$, then the cyclic index of A is 1.

We use the definition of eventually r-cyclic given in [9]: For a positive integer $r \geq 2$, a matrix $M \in \mathbb{C}^{n\times n}$ is *eventually r-cyclic* if there exists a positive integer k_0 such that $k \geq k_0$ and $k \equiv 1 \mod r$ implies M^k is r-cyclic. For an ordered partition $\Pi = (V_1, \ldots, V_r)$ of $\{1, \ldots, n\}$ into r nonempty sets, the *cyclic characteristic matrix* $C_\Pi = [c_{ij}]$ of Π is the $n \times n$ matrix such that $c_{ij} = 1$ if there exists $\ell \in \{1, \ldots, r\}$ such that $i \in V_\ell$ and $j \in V_{\ell+1}$ (where V_{r+1} is interpreted as V_1), and $c_{ij} = 0$ otherwise. For matrices $A = [a_{ij}], B = [b_{ij}] \in \mathbb{C}^{n\times n}$, A *conforms* to B if for all $i, j = 1, \ldots, n$, $b_{ij} = 0$ implies $a_{ij} = 0$. A matrix A is *eventually r-cyclic with partition* Π if there is an ordered partition Π of $\{1, \ldots, n\}$ into $r \geq 2$ nonempty sets, and a positive integer k_0 such that for all $k \geq k_0$, A^k conforms to ${C_\Pi}^k$. Note that [8] defined eventually r-cyclic to mean there exists a partition Π such that A is eventually r-cyclic with partition Π. It was shown in [9] that the two definitions are equivalent, and now that it is available we choose to use the more natural definition from [9].

Let $\sigma = \{\lambda_1, \ldots, \lambda_t\}$ be a multiset of complex numbers, $\omega \in \mathbb{C}$, and $m \in \mathbb{Z}^+$. Define $\omega\sigma := \{\omega\lambda_1, \ldots, \omega\lambda_t\}$ and $\sigma^m := \{{\lambda_1}^m, \ldots, {\lambda_t}^m\}$. If $\sigma = \omega\sigma$, then σ is *ω-invariant.* The *radius* of σ is $\rho(\sigma) := \max\{|\lambda| : \lambda \in \sigma\}$ and the *periphery* or *boundary* of σ is $\partial(\sigma) := \sigma \cap \{z \in \mathbb{C} : |z| = \rho(\sigma)\}$. If $\rho(\sigma) > 0$, the *cyclic index* of σ is the largest positive integer r such that σ is $e^{2\pi i/r}$-invariant. The *conjugate* of σ is $\overline{\sigma} := \{\overline{\lambda}_1, \ldots, \overline{\lambda}_t\}$ and σ is *self-conjugate* if $\overline{\sigma} = \sigma$. We say σ is a *Frobenius multiset* if for $r = |\partial(\sigma)|$, $\omega = e^{2\pi i/r}$, and $Z_r = \{1, \omega, \omega^2, \ldots, \omega^{r-1}\}$ we have

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1. $\rho(\sigma) > 0$,
2. $\partial(\sigma) = \rho(\sigma)Z_r$, and
3. σ is ω-*invariant*.

In this case, necessarily r is equal to the cyclic index of σ.

A *Jordan multiset* (called a *Jordan collection* in [13]) is a finite multiset $\mathcal{J} = \{J_{k_1}(\lambda_1), \ldots, J_{k_t}(\lambda_t)\}$ of elementary Jordan blocks. Throughout this paper $\mathcal{J}$ will denote a Jordan multiset. The *nonsingular part* of $\mathcal{J}$ is the multiset $\{J_k(\lambda) : J_k(\lambda) \in \mathcal{J} \text{ and } \lambda \neq 0\}$. Define the *conjugate* of $\mathcal{J}$, denoted $\overline{\mathcal{J}}$, to be the multiset of elementary Jordan blocks $J_k(\overline{\lambda})$ where $J_k(\lambda)$ ranges over the elements of $\mathcal{J}$; $\mathcal{J}$ is *self-conjugate* if $\overline{\mathcal{J}} = \mathcal{J}$. For $\omega \in \mathbb{C}$, $\omega\mathcal{J}$ is defined to be the multiset of Jordan blocks $J_k(\omega\lambda)$ where $J_k(\lambda)$ ranges over the elements of $\mathcal{J}$; $\mathcal{J}$ is ω-invariant if $\omega\mathcal{J} = \mathcal{J}$. For any square complex matrix A, $\mathcal{J}(A)$ is the Jordan multiset of elementary Jordan blocks in a Jordan form of A. If $\mathcal{J}$ is a Jordan multiset and $m \in \mathbb{Z}^+$, then $\mathcal{J}^m := \mathcal{J}(A^m)$ whenever A is a square complex matrix with $\mathcal{J} = \mathcal{J}(A)$. Note that the nonsingular part of $\mathcal{J}^m$ is the multiset $\{J_k(\lambda^m) : J_k(\lambda) \in \mathcal{J} \text{ and } \lambda \neq 0\}$. The *radius* of a Jordan multiset $\mathcal{J}$ is $\rho(\mathcal{J}) := \max\{|\lambda| : J_k(\lambda) \in \mathcal{J}\}$ and the *periphery* or *boundary* of $\mathcal{J}$ is $\partial(\mathcal{J}) := \{J_k(\lambda) \in \mathcal{J} : |\lambda| = \rho(\mathcal{J})\}$. Note also that for any square complex matrix A, $\rho(A) = \rho(\mathcal{J}(A))$. We say $\mathcal{J}$ is a *Frobenius Jordan multiset* if for $r = |\partial(\mathcal{J})|$ and $\omega = e^{2\pi i/r}$ we have

1. $\rho(\mathcal{J}) > 0$,
2. $\partial(\mathcal{J})$ is the set $\{J_1(\rho(\mathcal{J})\omega^j) : j = 0, \ldots, r-1\}$, and
3. $\mathcal{J}$ is ω-invariant.

In [13], the *cyclic index* of a Jordan multiset $\mathcal{J}$ with $\rho(\mathcal{J}) > 0$ is defined to be the maximum r such that $\mathcal{J}$ is $e^{2\pi i/r}$-invariant, and it is observed there that if $\mathcal{J}$ is Frobenius, then $r = |\partial(\mathcal{J})|$ is equal to the cyclic index of $\mathcal{J}$. In this case, $\mathcal{J}$ is referred to as a Frobenius multiset with cyclic index r.

2. Main results. In their study of eventually nonnegative matrices, Zaslavsky and Tam ask the following question.

QUESTION 2.1. [13, Question 6.3] Let A be an irreducible matrix with cyclic index r and $\rho(A) > 0$, that is eventually nonnegative, and suppose that the singular blocks in $\mathcal{J}(A)$, if any, are all 1×1. Does it follow that $\mathcal{J}(A)$ is a self-conjugate Frobenius Jordan multiset with cyclic index r?

The hypothesis that the singular blocks in $\mathcal{J}(A)$, if any, are all 1×1 is equivalent to $\operatorname{rank} A^2 = \operatorname{rank} A$, and also to the statement that the index of A is less than or equal to one. Thus, this question is equivalent to the following: Let A be an irreducible matrix with cyclic index r, $\rho(A) > 0$, and $\operatorname{rank} A^2 = \operatorname{rank} A$, that is eventually nonnegative. Does it follow that $\mathcal{J}(A)$ is a self-conjugate Frobenius Jordan multiset with cyclic index r? This question is answered affirmatively by Theorem 2.6.

In Zaslavsky and Tam's proof that a multiset σ is a union of Frobenius multisets whenever all sufficiently large powers of σ are unions of Frobenius multisets [13,

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Theorem 3.1], it is implicit that one obtains the analogous result for a single Frobenius multiset; this is explicitly established in [9, Theorem 4.8], where it is shown that for a multiset of complex numbers σ with $r = |\partial(\sigma)|$ if there exists a positive integer ℓ_0 such that for all $\ell \geq \ell_0$, $\sigma^{\ell r+1}$ is a Frobenius multiset, then σ is a Frobenius multiset. Zaslavsky and Tam observe that their proof of Theorem 3.1 in [13] extends to the analogous result for unions of Frobenius Jordan multisets [13, Theorem 3.3]. By similar reasoning, [9, Theorem 4.8] extends to a single Frobenius Jordan multiset.

THEOREM 2.2. *Let $\mathcal{J}$ be a Jordan multiset and let $r = |\partial(\mathcal{J})|$. If there exists a positive integer ℓ_0 such that for all $\ell \geq \ell_0$, $\mathcal{J}^{\ell r+1}$ is a Frobenius Jordan multiset, then $\mathcal{J}$ is a Frobenius Jordan multiset.*

PROPOSITION 2.3. *Let A be a strongly eventually nonnegative matrix with $r \geq 2$ dominant eigenvalues and power index k_0. Then $A^{\ell r+1}$ is irreducible and nonnegative whenever $\ell r + 1 \geq k_0$.*

Proof. By [5, Proposition 2.1], the dominant eigenvalues of A are $\rho(A), \rho(A)\omega, \ldots, \rho(A)\omega^{r-1}$, where $\omega := e^{2\pi i/r}$. From the definition of strongly eventually nonnegative, some power of A is both irreducible and nonnegative, and so has positive left and right eigenvectors for its simple spectral radius. Thus, for all ℓ, $\rho(A^{\ell r+1})$ is a simple eigenvalue of $A^{\ell r+1}$ with positive left and right eigenvectors. When $\ell r+1 \geq k_0$ $A^{\ell r+1} \geq 0$. So by [1, Corollary 2.3.15] $A^{\ell r+1}$ is an irreducible nonnegative matrix. □

PROPOSITION 2.4. *Let A be a strongly eventually nonnegative matrix such that $\operatorname{rank} A^2 = \operatorname{rank} A$. Then the number of dominant eigenvalues of A is equal to the cyclic index of A.*

Proof. Let r be the number of dominant eigenvalues of A. If $r = 1$, then A necessarily has cyclic index 1. Now assume $r \geq 2$. By Proposition 2.3, for every sufficiently large positive integer ℓ, $A^{\ell r+1}$ is irreducible, nonnegative, and has r dominant eigenvalues. So $A^{\ell r+1}$ is r-cyclic by [1, Theorem 2.2.20]. Thus, A is eventually r-cyclic. Then by [9, Theorem 4.1] A is eventually r-cyclic with partition Π for some Π. Since $\operatorname{rank} A^2 = \operatorname{rank} A$, A is r-cyclic by [8, Theorem 2.7]. Since r is the number of dominant eigenvalues of A, A cannot be s-cyclic for $s > r$ (see, e.g., [2, Theorem 3.4.7]). Thus, r is the cyclic index of A. □

THEOREM 2.5. *Suppose $A \in \mathbb{C}^{n\times n}$ is strongly eventually nonnegative with r dominant eigenvalues. Then $\mathcal{J}(A)$ is a Frobenius Jordan multiset with cyclic index r.*

Proof. If M is an irreducible nonnegative matrix with r dominant eigenvalues, then $\mathcal{J}(M)$ is a Frobenius Jordan multiset with cyclic index r [10, Corollary 8.4.6]. If $r = 1$, then A is eventually positive and $\mathcal{J}(A)$ is a Frobenius Jordan multiset with cyclic index 1. Now assume $r \geq 2$. By Proposition 2.3, for all ℓ sufficiently large $A^{\ell r+1}$ is irreducible and nonnegative. So for all ℓ sufficiently large, $\mathcal{J}(A)^{\ell r+1} = \mathcal{J}(A^{\ell r+1})$ is a Frobenius Jordan multiset with cyclic index r. Thus, by Theorem 2.2, $\mathcal{J}(A)$ is a Frobenius Jordan multiset with cyclic index r. □

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The next theorem answers Question 2.1.

Theorem 2.6. *If A is an irreducible eventually nonnegative matrix with cyclic index r and* $\operatorname{rank} A^2 = \operatorname{rank} A$, *then $\mathcal{J}(A)$ is a self-conjugate Frobenius Jordan multiset with cyclic index r.*

Proof. Assume the hypotheses. Since A is eventually nonnegative, $\mathcal{J}(A)$ is self-conjugate [13, Theorem 3.3]. Since A is irreducible and $\operatorname{rank} A^2 = \operatorname{rank} A$, by [9, Corollary 3.4] or [3, Theorem 3.4], A is not eventually reducible, so A is strongly eventually nonnegative. So by Proposition 2.4, the number of dominant eigenvalues of A is equal to its cyclic index r. Then by Theorem 2.5, $\mathcal{J}(A)$ is a Frobenius Jordan multiset with cyclic index r. ◻

Remark 2.7. Let $A \in \mathbb{C}^{n\times n}$ with $n \geq 2$. If A is strongly eventually nonnegative, then clearly A is irreducible and eventually nonnegative. With the proof of Theorem 2.6 (from [9, Corollary 3.4]), we established that the converse holds if, in addition, $\operatorname{rank} A^2 = \operatorname{rank} A$.

In [13, p. 328], Zaslavsky and Tam conjectured that if A is an irreducible eventually nonnegative matrix with $\rho(A) > 0$, then the elementary Jordan blocks in $\mathcal{J}(A)$ corresponding to $\rho(\mathcal{J}(A)) = \rho(A)$ must all be 1×1. The next example shows that the conjecture is not correct.

Example 2.8. Let $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$. As observed in [3, Example 3.1], $A = B + N$ where $B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $BN = NB = N^2 = 0$, so $A^k = B^k \geq 0$ for all $k \geq 2$. Thus, A is eventually nonnegative, as well as irreducible; it is also noted in [3, Example 3.1] that the Jordan block for eigenvalue zero is 2×2. Here we observe that the Jordan block for $\rho(\mathcal{J}(A)) = 2$ is also 2×2, providing a counterexample to the conjecture of Zaslavsky and Tam. Since $\rho(A)$ is not a simple eigenvalue of A, A is not strongly eventually nonnegative. This example also shows that even if A is irreducible, eventually nonnegative, and $\rho(A) > 0$, A need not be strongly eventually nonnegative (cf. Remark 2.7).

Since [13, Question 6.5] was answered in the affirmative in [9], all of the open questions of [13] have now been resolved.

The decomposition used in Example 2.8 plays a fundamental role in understanding strongly eventually nonnegative matrices (and eventually reducible and eventually r-cyclic matrices, e.g, [9, Theorems 3.5 and 4.6]). Here we summarize the role of this

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decomposition in regard to strong eventual nonnegativity.

THEOREM 2.9. *Let A be an $n \times n$ matrix with $n \geq 2$. Let $A = B + N$ be the unique decomposition of A such that* $\operatorname{rank} B^2 = \operatorname{rank} B, BN = NB = 0$, *and N is nilpotent. The following conditions are equivalent:*

(a) A is strongly eventually nonnegative.
(b) B is strongly eventually nonnegative.
(c) B is irreducible and eventually nonnegative.

Proof. The equivalence of (b) and (c) follows from Remark 2.7. Since $B^k = A^k$ for $k \geq n$, B is eventually nonnegative if and only if A is eventually nonnegative. Clearly A and B have the same number of dominant eigenvalues, say r. Let C be one of A or B, let D be the other one of A or B, and suppose C is eventually nonnegative. By Proposition 2.3, $D^{\ell r+1} = C^{\ell r+1}$ is irreducible and nonnegative whenever $\ell r + 1 \geq \max\{k_0, n\}$, where k_0 is the power index of C. Thus, D is strongly eventually nonnegative, establishing the equivalence of (a) and (b). □

The next theorem extends [13, Theorem 5.1 and Remark 5.2].

THEOREM 2.10. *Let $\mathcal{J}$ be a multiset of elementary Jordan blocks. The following conditions are equivalent:*

(a) $\mathcal{J}$ is a self-conjugate Frobenius multiset with cyclic index r.
(b) $\mathcal{J}$ is a multiset with cyclic index r and $\mathcal{J}^r = \mathcal{J}_1 \cup \cdots \cup \mathcal{J}_r$, where $\mathcal{J}_1, \ldots, \mathcal{J}_r$ are self-conjugate Frobenius multisets with cyclic index one that have the same submultiset of non-singular elementary Jordan blocks.
(c) There exists an irreducible eventually nonnegative matrix A with cyclic index r such that $\mathcal{J}(A) = \mathcal{J}$ and A^r is permutationally similar to a direct sum of r eventually positive matrices.
(d) There exists a strongly eventually nonnegative matrix A with r dominant eigenvalues such that $\mathcal{J}(A) = \mathcal{J}$.

Proof. The equivalence of (a)–(c) was established in [13, Theorem 5.1 and Remark 5.2]. Suppose (d) is satisfied. Then by Theorem 2.5, $\mathcal{J}(A)$ is a Frobenius Jordan multiset with cyclic index r, and $\mathcal{J}(A)$ self-conjugate by [13, Theorem 3.3], so condition (a) is satisfied.

Conversely, assume (a)–(c). Suppose that $\mathcal{J} = \mathcal{J}(A)$ for some irreducible eventually nonnegative matrix A with cyclic index r as described in (c). By (a), $\rho(A) = \rho(\mathcal{J}(A))$ is simple, as is $\rho(A^{\ell r+1})$ for any ℓ. By [13, Corollary 5.4], A has positive left and right eigenvectors corresponding to $\rho(A)$. By [1, Corollary 2.3.15], $A^{\ell r+1} \geq 0$ is irreducible when $\ell r + 1$ is at least the power index. Thus, A is strongly eventually nonnegative, and r is the number of dominant eigenvalues of A by (a), so condition (d) is satisfied. □

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