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MATRIX FUNCTIONS THAT PRESERVE THE STRONG PERRON-FROBENIUS PROPERTY*

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Abstract. In this note, matrix functions that preserve the strong Perron-Frobenius property are characterized, using the *real Jordan canonical form* of a real matrix.

Key words. Matrix function, Real Jordan canonical form, Perron-Frobenius Theorem, Eventually positive matrix.

AMS subject classifications. 15A16, 15B48, 15A21.

1. Introduction.

A real matrix has the *Perron-Frobenius property* if its spectral radius is a positive eigenvalue corresponding to an entrywise nonnegative eigenvector. The *strong Perron-Frobenius property* further requires that the spectral radius is simple; that it dominates in modulus every other eigenvalue; and that it has an entrywise positive eigenvector.

In [17], Micchelli and Willoughby characterized matrix functions that preserve *doubly nonnegative matrices*. In [7], Guillot et al. used these results to solve the *critical exponent conjecture* established in [12]. In [1], Bharali and Holtz characterized entire functions that preserve nonnegative matrices of a fixed order and, in addition, they characterized matrix functions that preserve nonnegative block triangular, circulant, and symmetric matrices. In [4], Elhashash and Szyld characterized entire functions that preserve sets of generalized nonnegative matrices.

In this work, using the characterization of a matrix function via the *real Jordan canonical form* established in [16], we characterize matrix functions that preserve the strong Perron-Frobenius property. Although our results are similar to those presented in [4], the assumption of entirety of a function is dropped in favor of analyticity in some domain containing the spectrum of a matrix.

2. Notation.

Denote by $M_n(\mathbb{C})$ (respectively, $M_n(\mathbb{R})$) the algebra of complex (respectively, real) $n \times n$ matrices. Given $A \in M_n(\mathbb{C})$, the *spectrum* of A is denoted by $\sigma(A)$, and the *spectral radius* of A is denoted by $\rho(A)$.

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The *direct sum* of the matrices $A_1, \dots, A_k$, where $A_i \in M_{n_i}(\mathbb{C})$, denoted by $A_1 \oplus \cdots \oplus A_k$, $\bigoplus_{i=1}^k A_i$, or $\operatorname{diag}(A_1, \dots, A_k)$, is the $n \times n$ matrix

$$\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{bmatrix}, \quad n = \sum_{i=1}^k n_i.$$

For $\lambda \in \mathbb{C}$, $J_n(\lambda)$ denotes the $n \times n$ *Jordan block* with eigenvalue λ. For $A \in M_n(\mathbb{C})$, denote by $J = Z^{-1}AZ = \bigoplus_{i=1}^t J_{n_i}(\lambda_i) = \bigoplus_{i=1}^t J_{n_i}$, where $\sum n_i = n$, a Jordan canonical form of A. Denote by $\lambda_1, \dots, \lambda_s$ the *distinct* eigenvalues of A, and, for $i = 1, \dots, s$, let m_i denote the *index* of λ_i, i.e., the size of the largest Jordan block associated with λ_i. Denote by i the imaginary unit, i.e., $\mathrm{i} := \sqrt{-1}$.

A domain $\mathcal{D}$ is any open and connected subset of $\mathbb{C}$. We call a domain *self-conjugate* if $\bar{\lambda} \in \mathcal{D}$ whenever $\lambda \in \mathcal{D}$ (i.e., $\mathcal{D}$ is symmetric with respect to the real-axis). Given that an open and connected set is also path-connected, it follows that if $\mathcal{D}$ is self-conjugate, then $\mathbb{R} \cap \mathcal{D} \neq \emptyset$.

3. Background. Although there are multiple ways to define a matrix function (see, e.g., [9]), our preference is via the Jordan Canonical Form.

Definition 3.1. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a function and denote by $f^{(j)}$ the jth derivative of f. The function f is said to be *defined on the spectrum of* A if the values

$$f^{(j)}(\lambda_i), \quad j = 0, \dots, m_i - 1, \quad i = 1, \dots, s,$$

called *the values of the function* f *on the spectrum of* A, exist.

Definition 3.2 (Matrix function via Jordan canonical form). If f is defined on the spectrum of $A \in M_n(\mathbb{C})$, then

$$f(A) := Zf(J)Z^{-1} = Z\left(\bigoplus_{i=1}^t f(J_{n_i})\right)Z^{-1},$$

where

$$f(J_{n_i}) := \begin{bmatrix} f(\lambda_i) & f'(\lambda_i) & \cdots & \frac{f^{(n_i-1)}(\lambda_i)}{(n_i-1)!} \\ & f(\lambda_i) & \ddots & \vdots \\ & & \ddots & f'(\lambda_i) \\ & & & f(\lambda_i) \end{bmatrix}. \tag{3.1}$$

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The following theorem is well-known (for details see, e.g., [11], [14]; for a complete proof, see, e.g., [6]).

THEOREM 3.3 (Real Jordan canonical form). *If $A \in M_n(\mathbb{R})$ has r real eigenvalues (including multiplicities) and c complex conjugate pairs of eigenvalues (including multiplicities), then there exists an invertible matrix $R \in M_n(\mathbb{R})$ such that*

$$R^{-1}AR = \begin{bmatrix} \bigoplus_{k=1}^{r} J_{n_k}(\lambda_k) & \\ & \bigoplus_{k=r+1}^{r+c} C_{n_k}(\lambda_k) \end{bmatrix}, \tag{3.2}$$

where:

1.
$$C_j(\lambda) := \begin{bmatrix} C(\lambda) & I_2 & & \\ & C(\lambda) & \ddots & \\ & & \ddots & I_2 \\ & & & C(\lambda) \end{bmatrix} \in M_{2j}(\mathbb{R}); \tag{3.3}$$

2.
$$C(\lambda) := \begin{bmatrix} \Re(\lambda) & \Im(\lambda) \\ -\Im(\lambda) & \Re(\lambda) \end{bmatrix} \in M_2(\mathbb{R}); \tag{3.4}$$

3. *$\Im(\lambda_k) = 0$, $k = 1, \dots, r$; and*
4. *$\Im(\lambda_k) \neq 0$, $k = r+1, \dots, r+c$.*

PROPOSITION 3.4 ([16, Corollary 2.11]). *Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$, and let f be a function defined on the spectrum of $J_k(\lambda) \oplus J_k(\bar{\lambda})$. For j a nonnegative integer, let $f_\lambda^{(j)}$ denote $f^{(j)}(\lambda)$. If $C_k(\lambda)$ and $C(\lambda)$ are defined as in (3.3) and (3.4), respectively, then*

$$f(C_k(\lambda)) = \begin{bmatrix} f(C_\lambda) & f'(C_\lambda) & \cdots & \frac{f^{(k-1)}(C_\lambda)}{(k-1)!} \\ & f(C_\lambda) & \ddots & \vdots \\ & & \ddots & f'(C_\lambda) \\ & & & f(C_\lambda) \end{bmatrix} \in M_{2k}(\mathbb{C}),$$

and, moreover,

$$f(C_k(\lambda)) = \begin{bmatrix} C(f_\lambda) & C(f'_\lambda) & \cdots & C\left(\frac{f_\lambda^{(k-1)}}{(k-1)!}\right) \\ & C(f_\lambda) & \ddots & \vdots \\ & & \ddots & C(f'_\lambda) \\ & & & C(f_\lambda) \end{bmatrix} \in M_{2k}(\mathbb{R})$$

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if and only if $\overline{f_{\lambda}^{(j)}} = f_{\bar{\lambda}}^{(j)}$.

We recall the Perron-Frobenius theorem for positive matrices (see [11, Theorem 8.2.11]).

THEOREM 3.5. *If* $A \in M_n(\mathbb{R})$ *is positive, then*

(i) $\rho := \rho(A) > 0$*;*
(ii) $\rho \in \sigma(A)$*;*
(iii) there exists a positive vector x *such that* $Ax = \rho x$*;*
(iv) ρ *is a simple eigenvalue of* A*; and*
(v) $|\lambda| < \rho$ *for every* $\lambda \in \sigma(A)$ *such that* $\lambda \neq \rho$.

One can verify that the matrix

$$B = \begin{bmatrix} 2 & 1 \\ 2 & -1 \end{bmatrix} \tag{3.5}$$

satisfies properties (i) through (v) of Theorem 3.5, but obviously contains a negative entry. This motivates the following concept.

DEFINITION 3.6. A matrix $A \in M_n(\mathbb{R})$ is said to *possess the strong Perron-Frobenius property* if A satisfies properties (i) through (v) of Theorem 3.5.

It can also be shown that the matrix B given in (3.5) satisfies $B^k > 0$ for $k \geq 4$, which leads to the following generalization of positive matrices.

DEFINITION 3.7. A matrix $A \in M_n(\mathbb{R})$ is *eventually positive* if there exists a nonnegative integer p such that $A^k > 0$ for all $k \geq p$.

The following theorem relates the strong Perron-Frobenius property with eventually positive matrices (see [8, Lemma 2.1], [13, Theorem 1], or [18, Theorem 2.2]).

THEOREM 3.8. *A real matrix* A *is eventually positive if and only if* A *and* $A^\top$ *possess the strong Perron-Frobenius property.*

4. Main results.

Before we state our main results, we begin with the following definition.

DEFINITION 4.1. A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined on a self-conjugate domain $\mathcal{D}$, $\mathcal{D} \cap \mathbb{R}^+ \neq \emptyset$, is called *Frobenius*[1] if

(i) $\overline{f(\lambda)} = f(\bar{\lambda})$, $\lambda \in \mathcal{D}$;
(ii) $|f(\lambda)| < f(\rho)$, whenever $|\lambda| < \rho$, and λ, $\rho \in \mathcal{D}$.

[1] We use the term 'Frobenius' given that such a function preserves *Frobenius multi-sets*, introduced by Friedland in [5].

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REMARK 4.2. Condition (i) implies $f(r) \in \mathbb{R}$, whenever $r \in \mathcal{D} \cap \mathbb{R}$; and condition (ii) implies $f(r) \in \mathbb{R}^+$, whenever $r \in \mathcal{D} \cap \mathbb{R}^+$.

The following theorem is our first main result.

THEOREM 4.3. *Let $A \in M_n(\mathbb{R})$ and suppose that A is diagonalizable and possesses the strong Perron-Frobenius property. If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is a function defined on the spectrum of A, then $f(A)$ possesses the strong Perron-Frobenius property if and only if f is Frobenius.*

Proof. Suppose that f is Frobenius. For convenience, denote by f_λ the scalar $f(\lambda)$. Following Theorem 3.3 and Proposition 3.4, the matrix

$$f(A) = R \begin{bmatrix} f_{\rho(A)} & & \\ & \bigoplus_{k=2}^{r} f_{\lambda_k} & \\ & & \bigoplus_{k=r+1}^{r+c} C(f_{\lambda_k}) \end{bmatrix} R^{-1}, \tag{4.1}$$

where $R = \begin{bmatrix} x & R' \end{bmatrix}$, $x > 0$, is real. If $\sigma(A) = \{\rho(A), \lambda_2, \ldots, \lambda_n\}$, then $\sigma(f(A)) = \{f_{\rho(A)}, f_{\lambda_2}, \ldots, f_{\lambda_n}\}$ (see, e.g., [9][Theorem 1.13(d)]) and because f is Frobenius, it follows that $|f_{\lambda_k}| < f_{\rho(A)}$ for $k = 2, \ldots, n$. Moreover, from (4.1) it follows that $f(A)x = f_{\rho(A)}x$. Thus, $f(A)$ possesses the strong Perron-Frobenius property.

Conversely, if f is not Frobenius, then the matrix $f(A)$, given by (4.1), is not real (e.g, $\exists \lambda \in \sigma(A)$, $\lambda \in \mathbb{R}$ such that $f(\lambda) \notin \mathbb{R}$), or $f(A)$ does not retain the strong Perron-Frobenius property (e.g., $\exists \lambda \in \sigma(A)$ such that $|f(\lambda)| \geq f(\rho(A))$). ☐

EXAMPLE 4.4. Table 4.1 lists examples of Frobenius functions for diagonalizable matrices that possess the strong Perron-Frobenius property.

f	$\mathcal{D}$
$f(z) = z^p$, $p \in \mathbb{N}$	$\mathbb{C}$
$f(z) = \lvert z \rvert$	$\mathbb{C}$
$f(z) = z^{1/p}$, $p \in \mathbb{N}$, p even	$\{z \in \mathbb{C} : z \notin \mathbb{R}^-\}$
$f(z) = z^{1/p}$, $p \in \mathbb{N}$, p odd, $p > 1$	$\mathbb{C}$
$f(z) = \sum_{k=0}^{n} a_k z^k$, $a_k > 0$	$\mathbb{C}$
$f(z) = \exp(z)$	$\mathbb{C}$

TABLE 4.1
Examples of Frobenius functions.

For matrices that are not diagonalizable, i.e., possessing Jordan blocks of size two or greater, given (3.1) it is reasonable to assume that f is complex-differentiable, i.e., *analytic*. We note the following result, which is well known (see, e.g., [2], [3], [10,

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Theorem 3.2], [15], or [19]).

THEOREM 4.5 (Reflection Principle). *Let f be analytic in a self-conjugate domain $\mathcal{D}$ and suppose that $I := \mathcal{D} \cap \mathbb{R} \neq \emptyset$. Then $\overline{f(\lambda)} = f(\bar{\lambda})$ for every $\lambda \in \mathcal{D}$ if and only if $f(r) \in \mathbb{R}$ for all $r \in I$.*

The Reflection Principle leads immediately to the following result.

COROLLARY 4.6. *An analytic function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined on a self-conjugate domain $\mathcal{D}$, $\mathcal{D} \cap \mathbb{R}^+ \neq \emptyset$, is Frobenius if and only if*

(i) $f(r) \in \mathbb{R}$, whenever $r \in \mathcal{D} \cap \mathbb{R}$; and
(ii) $|f(\lambda)| < f(\rho)$, whenever $|\lambda| < \rho$ and λ, $\rho \in \mathcal{D}$.

LEMMA 4.7. *Let f be analytic in a domain $\mathcal{D}$ and suppose that $I := \mathcal{D} \cap \mathbb{R} \neq \emptyset$. If $f(r) \in \mathbb{R}$ for all $r \in I$, then $f^{(j)}(r) \in \mathbb{R}$ for all $r \in I$ and $j \in \mathbb{N}$..*

Proof. Proceed by induction on j: when $j = 1$, note that, since f is analytic on $\mathcal{D}$, it is *holomorphic* (i.e., complex-differentiable) on $\mathcal{D}$. Thus,

$$f'(r) := \lim_{z \to r} \frac{f(z) - f(r)}{z - r}$$

exists for all $z \in \mathcal{D}$; in particular,

$$f'(r) = \lim_{x \to r} \frac{f(x) - f(r)}{x - r}, \quad x \in I,$$

and the conclusion that $f'(r) \in \mathbb{R}$ follows by the hypothesis that $f(x) \in \mathbb{R}$ for all $x \in I$.

Next, assume that the result holds when $j = k - 1 > 1$. As above, note that $f^{(k)}(r)$ exists and

$$f^{(k)}(r) = \lim_{x \to r} \frac{f^{(k-1)}(x) - f^{(k-1)}(r)}{x - r}, \quad x \in I$$

so that $f^{(k)}(r) \in \mathbb{R}$. $\square$

THEOREM 4.8. *Let $A \in M_n(\mathbb{R})$ and suppose that A possesses the strong Perron-Frobenius property. If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is an analytic function defined in a self-conjugate domain $\mathcal{D}$ containing $\sigma(A)$, then $f(A)$ possesses the strong Perron-Frobenius property if and only if f is Frobenius.*

Proof. Suppose that f is Frobenius. Following Theorem 3.3, there exists an invertible matrix R such that

$$R^{-1}AR = \begin{bmatrix} \rho(A) & & \\ & \bigoplus_{k=2}^{r} J_{n_k}(\lambda_k) & \\ & & \bigoplus_{k=r+1}^{r+c} C_{n_k}(\lambda_k) \end{bmatrix},$$

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where

$$R = \begin{bmatrix} x & R' \end{bmatrix}, \ \ x > 0.$$

Because f is Frobenius, following Theorem 4.5, $\overline{f(\lambda)} = f(\bar{\lambda})$ for all $\lambda \in \mathcal{D}$. Since f is analytic, $f^{(j)}$ is analytic for all $j \in \mathbb{N}$ and, following Lemma 4.7 $f^{(j)}(r) \in \mathbb{R}$ for all $r \in I$. Another application of Theorem 4.5 yields that $\overline{f^{(j)}(\lambda)} = f^{(j)}(\bar{\lambda})$ for all $\lambda \in \mathcal{D}$. Hence, following Proposition 3.4, the matrix

$$f(A) = R \begin{bmatrix} f(\rho(A)) & & \\ & \bigoplus_{k=2}^{r} f(J_{n_k}(\lambda_k)) & \\ & & \bigoplus_{k=r+1}^{r+c} f(C_{n_k}(\lambda_k)) \end{bmatrix} R^{-1}$$

is real and possesses the strong Perron-Frobenius property.

The proof of the converse is identical to the proof of the converse of Theorem 4.3. □

Corollary 4.9. *Let $A \in M_n(\mathbb{R})$ and suppose that A is eventually positive. If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is an analytic function defined in a self-conjugate domain $\mathcal{D}$ containing $\sigma(A)$, then $f(A)$ is eventually positive if and only if f is Frobenius.*

Proof. Follows from Theorem 4.8 and the fact that $f(A^\top) = (f(A))^\top$ ([9, Theorem 1.13(b)]). □

Remark 4.10. Aside from the function $f(z) = |z|$, which is nowhere differentiable, every function listed in Table 4.1 is analytic and Frobenius.

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