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GENERALIZATION OF GRACIA'S RESULTS*

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Abstract. Let α be a linear transformation of the $m \times n$-dimensional vector space $M_{m\times n}(\mathbb{C})$ over the complex field $\mathbb{C}$ such that $\alpha(X) = AX - XB$, where A and B are $m \times m$ and $n \times n$ complex matrices, respectively. In this paper, the dimension formulas for the kernels of the linear transformations α^2 and α^3 are given, which generalizes the work of Gracia in [J.M. Gracia. Dimension of the solution spaces of the matrix equations $[A, [A, X]] = 0$ and $[A\,[A, [A, X]]] = 0$. *Linear and Multilinear Algebra*, 9:195–200, 1980.].

Key words. Linear transformation, Dimension formulas, Jordan canonical form.

AMS subject classifications. 15A24, 15A27.

1. Introduction. The notation used in this paper is standard, see [2, 3] for example. Let $\mathbb{C}$ be the complex field. Suppose $A \in M_{m\times m}(\mathbb{C})$ and $B \in M_{n\times n}(\mathbb{C})$. Let α_{AB} be a linear transformation of $M_{m\times n}(\mathbb{C})$ defined by

$$\alpha_{AB}(X) = AX - XB, \text{ for } X \in M_{m\times n}(\mathbb{C}).$$

If $A = B$, then we will write α_A instead of α_{AA} for brevity. In the case of no confusion, we write $\alpha = \alpha_{AB}$ for short.

The well known dimension formula of the kernel $\ker\alpha_A$ is due to Frobenius [3, Theorem VII.1]. Then Gracia has obtained the dimension formulas of $\ker\alpha_A^2$ and $\ker\alpha_A^3$ in [1].

It is obvious that the kernels of the liner transformations α^2 and α^3 are the solutions of the matrix equations $A(AX - XB) - (AX - XB)B = 0$ and $A[A(AX - XB) - (AX - XB)B] - [A(AX - XB) - (AX - XB)B]B = 0$, respectively. In this paper, we obtain the dimensions of $\ker\alpha^2$ and $\ker\alpha^3$, which generalizes Gracia's results.

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For convenience, we introduce the following notations. Suppose that the elementary divisors of A and B are $(\lambda-\lambda_1)^{p_1},\ (\lambda-\lambda_2)^{p_2},\dots,(\lambda-\lambda_u)^{p_u}$ and $(\lambda-\mu_1)^{q_1},\ (\lambda-\mu_2)^{q_2},\dots,(\lambda-\mu_v)^{q_v}$, respectively. Let E_n be the unit matrix of size n and let

$$N_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

be the square matrix of size n in which all the elements of the first superdiagonal are 1 and all the other elements are 0. Let

$$J_A = \begin{pmatrix} \lambda_1 E_{p_1} + N_{p_1} & & & \\ & \lambda_2 E_{p_2} + N_{p_2} & & 0 \\ 0 & & \ddots & \\ & & & \lambda_u E_{p_u} + N_{p_u} \end{pmatrix}$$

and

$$J_B = \begin{pmatrix} \mu_1 E_{q_1} + N_{q_1} & & & \\ & \mu_2 E_{q_2} + N_{q_2} & & 0 \\ 0 & & \ddots & \\ & & & \mu_v E_{q_v} + N_{q_v} \end{pmatrix}$$

be respectively the Jordan normal forms of A and B. For $1 \le \alpha \le u$ and $1 \le \beta \le v$. Let $\mathrm{F}^{(2)}_{\alpha\beta}$ and $\mathrm{F}^{(3)}_{\alpha\beta}$ be defined by the following:

$$\mathrm{F}^{(2)}_{\alpha\beta} = \begin{cases} 0 & \text{if } \lambda_\alpha \neq \mu_\beta; \\ 2\min(p_\alpha,\ q_\beta) - 1 & \text{if } \lambda_\alpha = \mu_\beta \text{ and } p_\alpha = q_\beta; \\ 2\min(p_\alpha,\ q_\beta) & \text{if } \lambda_\alpha = \mu_\beta \text{ and } p_\alpha \neq q_\beta \end{cases}$$

and

$$\mathrm{F}^{(3)}_{\alpha\beta} = \begin{cases} 0 & \text{if } \lambda_\alpha \neq \mu_\beta; \\ 3\min(p_\alpha,\ q_\beta) - 2 & \text{if } \lambda_\alpha = \mu_\beta \text{ and } p_\alpha = q_\beta; \\ 3\min(p_\alpha,\ q_\beta) - 1 & \text{if } \lambda_\alpha = \mu_\beta \text{ and } |p_\alpha - q_\beta| = 1; \\ 3\min(p_\alpha,\ q_\beta) & \text{if } \lambda_\alpha = \mu_\beta \text{ and } |p_\alpha - q_\beta| \ge 2. \end{cases}$$

The main results are the following:

THEOREM 1.1. *Let $\alpha : M_{m\times n}(\mathbb{C}) \to M_{m\times n}(\mathbb{C})$ be a linear transformation such that $\alpha(X) = AX - XB$, where A and B are $m\times m$ and $n\times n$ complex matrices,*

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respectively. Then the dimension formula for $\ker\alpha^2$ *is*

$$\dim(\ker\alpha^2) = \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v} \mathrm{F}_{\alpha\beta}^{(2)}. \tag{1.1}$$

THEOREM 1.2. *Let* $\alpha : M_{m\times n}(\mathbb{C}) \to M_{m\times n}(\mathbb{C})$ *be a linear transformation such that* $\alpha(X) = AX - XB$, *where* A *and* B *are* $m \times m$ *and* $n \times n$ *complex matrices, respectively. Then the dimension formula for* $\ker\alpha^3$ *is*

$$\dim(\ker\alpha^3) = \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v} \mathrm{F}_{\alpha\beta}^{(3)}. \tag{1.2}$$

2. The proof of Theorem 1.1. Before proving the theorem, we first give a lemma in the following:

LEMMA 2.1. *For a matrix* $M \in M_{l\times l}(\mathbb{C})$, *let* $\Lambda(M)$ *be the set of its different eigenvalues. If* $\Lambda(A) \cap \Lambda(B) = \emptyset$, *then* $\ker\alpha^k = 0$ *for* $k = 1, 2, 3, \ldots$

Proof. It is well known that $\Lambda(A) \cap \Lambda(B) = \emptyset$ if and only if the unique solution of the matrix equation $AX - XB = 0$ is $X = 0$. Thus, when $\Lambda(A) \cap \Lambda(B) = \emptyset$, we can prove by induction on k that if $\Lambda(A) \cap \Lambda(B) = \emptyset$, the equality $\alpha^k(X) = 0$ implies $X = 0$. In fact, if $\alpha^k(X) = 0$ then $\alpha(\alpha^{k-1}(X)) = 0$ and $A\alpha^{k-1}(X) - \alpha^{k-1}(X)B = 0$. Since $\Lambda(A) \cap \Lambda(B) = \emptyset$, it follows that $\alpha^{k-1}(X) = 0$. By hypothesis of the induction the equality $\alpha^{k-1}(X) = 0$ implies that $X = 0$. So that, $\Lambda(A) \cap \Lambda(B) = \emptyset$ implies that $\ker\alpha^k = 0$ for $k = 1, 2, 3, \ldots$ □

Proof. It is obvious that there are invertible matrices U and V such that $A = UJ_AU^{-1}$ and $B = VJ_BV^{-1}$. Assume $X \in \ker\alpha^2$, then $A(AX - XB) - (AX - XB)B = 0$. Hence,

$$UJ_AU^{-1}(UJ_AU^{-1}X - XVJ_BV^{-1}) = (UJ_AU^{-1}X - XVJ_BV^{-1})VJ_BV^{-1}.$$

Thus,

$$J_A(J_AU^{-1}XV - U^{-1}XVJ_B) = (J_AU^{-1}XV - U^{-1}XVJ_B)J_B.$$

Let $\overline{X} = U^{-1}XV$. Then, the equation is

$$J_A(J_A\overline{X} - \overline{X}J_B) = (J_A\overline{X} - \overline{X}J_B)J_B. \tag{2.1}$$

Now we partition $\overline{X}$ into blocks $(X_{\alpha\beta})$ where $X_{\alpha\beta} = (\varepsilon_{ik})_{p_\alpha\times q_\beta}$ is a $p_\alpha \times q_\beta$ matrix for $1 \le \alpha \le u$ and $1 \le \beta \le v$. Then we get uv matrix equations from (3):

$$\begin{aligned}&(\lambda_\alpha E_{p_\alpha} + N_{p_\alpha})[(\lambda_\alpha E_{p_\alpha} + N_{p_\alpha})X_{\alpha\beta} - X_{\alpha\beta}(\mu_\beta E_{q_\beta} + N_{q_\beta})]\\ &= [(\lambda_\alpha E_{p_\alpha} + N_{p_\alpha})X_{\alpha\beta} - X_{\alpha\beta}(\mu_\beta E_{q_\beta} + N_{q_\beta})](\mu_\beta E_{q_\beta} + N_{q_\beta}).\end{aligned}$$

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Write $P_\alpha := N_{p_\alpha}$ and $Q_\beta := N_{q_\beta}$. An easy calculation gives

$$\begin{aligned}(\mu_\beta - \lambda_\alpha)^2 X_{\alpha\beta} &= 2(\mu_\beta - \lambda_\alpha)(P_\alpha X_{\alpha\beta} - X_{\alpha\beta} Q_\beta) \\ &\quad + P_\alpha (X_{\alpha\beta} Q_\beta - P_\alpha X_{\alpha\beta}) - (X_{\alpha\beta} Q_\beta - P_\alpha X_{\alpha\beta}) Q_\beta. \qquad (2.2)\end{aligned}$$

If $\mu_\beta \neq \lambda_\alpha$, then $X_{\alpha\beta} = 0$ by Lemma 2.1. Next we assume that $\mu_\beta = \lambda_\alpha$. In this case, we have

$$P_\alpha (P_\alpha X_{\alpha\beta} - X_{\alpha\beta} Q_\beta) = (P_\alpha X_{\alpha\beta} - X_{\alpha\beta} Q_\beta) Q_\beta. \qquad (2.3)$$

Case 1. $p_\alpha = q_\beta$.

If $p_\alpha = 1$, then it is obvious that $X_{\alpha\beta} = (\varepsilon_{11})$.

If $p_\alpha = 2$, an easy computation gives $X_{\alpha\beta} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ 0 & \varepsilon_{22} \end{pmatrix}$.

If $p_\alpha \geq 3$, then

$$\begin{pmatrix} \varepsilon_{31} & \varepsilon_{32} - \varepsilon_{21} & \varepsilon_{33} - \varepsilon_{22} & \cdots & \varepsilon_{3,\ q_\beta} - \varepsilon_{2,\ q_\beta - 1} \\ \varepsilon_{41} & \varepsilon_{42} - \varepsilon_{31} & \varepsilon_{43} - \varepsilon_{32} & \cdots & \varepsilon_{4,\ q_\beta} - \varepsilon_{3,\ q_\beta - 1} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \varepsilon_{p_\alpha\ 1} & \varepsilon_{p_\alpha\ 2} - \varepsilon_{p_\alpha - 1,\ 1} & \varepsilon_{p_\alpha\ 3} - \varepsilon_{p_\alpha - 1,\ 2} & \cdots & \varepsilon_{p_\alpha\ q_\beta} - \varepsilon_{p_\alpha - 1,\ q_\beta - 1} \\ 0 & -\varepsilon_{p_\alpha\ 1} & -\varepsilon_{p_\alpha\ 2} & \cdots & -\varepsilon_{p_\alpha,\ q_\beta - 1} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \varepsilon_{21} & \varepsilon_{22} - \varepsilon_{11} & \cdots & \varepsilon_{2,\ q_\beta - 1} - \varepsilon_{1,\ q_\beta - 2} \\ 0 & \varepsilon_{31} & \varepsilon_{32} - \varepsilon_{21} & \cdots & \varepsilon_{3,\ q_\beta - 1} - \varepsilon_{2,\ q_\beta - 2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \varepsilon_{p_\alpha - 1,\ 1} & \varepsilon_{p_\alpha - 1,\ 2} - \varepsilon_{p_\alpha - 2,\ 1} & \cdots & \varepsilon_{p_\alpha - 1,\ q_\beta - 1} - \varepsilon_{p_\alpha - 2,\ q_\beta - 2} \\ 0 & \varepsilon_{p_\alpha\ 1} & \varepsilon_{p_\alpha\ 2} - \varepsilon_{p_\alpha - 1,\ 1} & \cdots & \varepsilon_{p_\alpha,\ q_\beta - 1} - \varepsilon_{p_\alpha - 1,\ q_\beta - 2} \\ 0 & 0 & -\varepsilon_{p_\alpha\ 1} & \cdots & -\varepsilon_{p_\alpha,\ q_\beta - 2} \end{pmatrix}.$$

This leads to the following equations:

$$\begin{aligned} &\varepsilon_{s1} = \varepsilon_{p_\alpha\ t} = 0, \\ &\varepsilon_{p_\alpha - 1,\ i} = 2\varepsilon_{p_\alpha,\ i+1},\ \varepsilon_{h2} = 2\varepsilon_{h-1,\ 1}, \\ &\varepsilon_{jk} = 2\varepsilon_{j-1,\ k-1} - \varepsilon_{j-2,\ k-2}, \end{aligned}$$

where $3 \leq s \leq p_\alpha$, $1 \leq t \leq q_\beta - 2$, $2 \leq i \leq q_\beta - 2$, $3 \leq j \leq p_\alpha$, $3 \leq k \leq q_\beta$, $-1 \leq k - j \leq q_\beta - 3$ and $3 \leq h \leq p_\alpha$. According to these equations, we have

$$\begin{cases} \varepsilon_{32} = 2\varepsilon_{21} \\ \varepsilon_{43} = 2\varepsilon_{32} - \varepsilon_{21} \\ \quad \vdots \qquad \vdots \\ \varepsilon_{p_\alpha,\ q_\beta - 1} = 2\varepsilon_{p_\alpha - 1,\ q_\beta - 2} - \varepsilon_{p_\alpha - 2,\ q_\beta - 3} \\ 2\varepsilon_{p_\alpha,\ q_\beta - 1} = \varepsilon_{p_\alpha - 1,\ q_\beta - 2}. \end{cases}$$

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Then, $\varepsilon_{21} = \varepsilon_{32} = \cdots = \varepsilon_{p_\alpha,\ q_\beta - 1} = 0$. We also have

$$\begin{cases} \varepsilon_{33} = 2\varepsilon_{22} - \varepsilon_{11} \\ \vdots \qquad\quad \vdots \\ \varepsilon_{p_\alpha,\ q_\beta} = 2\varepsilon_{p_\alpha-1,\ q_\beta-1} - \varepsilon_{p_\alpha-2,\ q_\beta-2}. \end{cases}$$

By induction on the subscript, we obtain $\varepsilon_{ii} = (i-1)\varepsilon_{22} - (i-2)\varepsilon_{11}$, where $3 \le i \le p_\alpha$. Note that $\varepsilon_{jk} = 2\varepsilon_{j-1,\ k-1} - \varepsilon_{j-2,\ k-2}$, where $3 \le j \le p_\alpha,\ \ 3 \le k \le q_\beta$ and $1 \le k - j \le q_\beta - 3$. By induction, we get $\varepsilon_{rh} = (r-1)\varepsilon_{2,\ h-r+2} - (r-2)\varepsilon_{1,\ h-r+1}$ where $3 \le r \le p_\alpha$, $0 \le h - r \le q_\beta - 3$ and $3 \le h \le q_\beta$. Thus, we conclude that

$$X_{\alpha\beta} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} & \cdots & \varepsilon_{1,\ q_\beta-1} & \varepsilon_{1\ q_\beta} \\ 0 & \varepsilon_{22} & \varepsilon_{23} & \cdots & \vdots & \varepsilon_{2\ q_\beta} \\ 0 & 0 & 2\varepsilon_{22}-\varepsilon_{11} & \ddots & \vdots & 2\varepsilon_{2,\ q_\beta-1}-\varepsilon_{1,\ q_\beta-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (p_\alpha-2)\varepsilon_{22}-(p_\alpha-3)\varepsilon_{11} & (p_\alpha-2)\varepsilon_{23}-(p_\alpha-3)\varepsilon_{12} \\ 0 & 0 & 0 & \cdots & 0 & (p_\alpha-1)\varepsilon_{22}-(p_\alpha-2)\varepsilon_{11} \end{pmatrix}.$$

Let $D_1 = \operatorname{diag}\{1, 0, -1, \ldots, -(p_\alpha - 2)\}$ and $D_2 = \operatorname{diag}\{0, 1, \ldots, p_\alpha - 1\}$ be diagonal matrices of size p_α. It is obvious that $X_{\alpha\beta} = \sum_{i=1}^{q_\beta-2}(\varepsilon_{1i}D_1 + \varepsilon_{2,i+1}D_2)N_{p_\alpha}^{i-1}$. Then the number of arbitrary parameters in $X_{\alpha\beta}$ is $2p_\alpha - 1$.

Case 2. $p_\alpha \ne q_\beta$. We can assume that $p_\alpha < q_\beta$, since the case $p_\alpha > q_\beta$ is analogous.

If $p_\alpha = 1$, then $X_{\alpha\beta} = (0, \ldots, 0, \varepsilon_{1,\ q_\beta-1}, \varepsilon_{1,\ q_\beta})$. In this case, $q_\beta - 2$ columns are 0.

If $p_\alpha = 2$, then $X_{\alpha\beta}$ has the following form:

$$X_{\alpha\beta} = \begin{pmatrix} 0 & \cdots & 0 & 2\varepsilon_{2,\ q_\beta-1} & \varepsilon_{1,\ q_\beta-1} & \varepsilon_{1,\ q_\beta} \\ 0 & \cdots & 0 & 0 & \varepsilon_{2,\ q_\beta-1} & \varepsilon_{2,\ q_\beta} \end{pmatrix},$$

and $q_\beta - 3$ columns are 0.

If $p_\alpha \ge 3$, then $X_{\alpha\beta}$ has the following form:

$$\begin{pmatrix} 0 \cdots 0 & \varepsilon_{1,\ q_\beta-p_\alpha} & \varepsilon_{1,\ q_\beta-p_\alpha+1} & \varepsilon_{1,\ q_\beta-p_\alpha+2} & \cdots & \varepsilon_{1,\ q_\beta-2} & \varepsilon_{1,\ q_\beta-1} & \varepsilon_{1,\ q_\beta} \\ 0 \cdots 0 & 0 & \varepsilon_{2,\ q_\beta-p_\alpha+1} & \varepsilon_{2,\ q_\beta-p_\alpha+2} & \cdots & \varepsilon_{2,\ q_\beta-2} & \varepsilon_{2,\ q_\beta-1} & \varepsilon_{2,\ q_\beta} \\ 0 \cdots 0 & 0 & 0 & \varepsilon_{3,\ q_\beta-p_\alpha+2} & \cdots & \varepsilon_{3,\ q_\beta-2} & \varepsilon_{3,\ q_\beta-1} & \varepsilon_{3,\ q_\beta} \\ \vdots\ \vdots\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 \cdots 0 & 0 & 0 & 0 & \cdots & \varepsilon_{p_\alpha-2,q_\beta-2} & \varepsilon_{p_\alpha-2,q_\beta-1} & \varepsilon_{p_\alpha-2,q_\beta} \\ 0 \cdots 0 & 0 & 0 & 0 & \cdots & \varepsilon_{p_\alpha-1,q_\beta-2} & \varepsilon_{p_\alpha-1,q_\beta-1} & \varepsilon_{p_\alpha-1,q_\beta} \\ 0 \cdots 0 & 0 & 0 & 0 & \cdots & 0 & \varepsilon_{p_\alpha,q_\beta-1} & \varepsilon_{p_\alpha,q_\beta} \end{pmatrix},$$

where $\varepsilon_{i,\ q_\beta-p_\alpha-1+i} = (p_\alpha + 1 - i)\varepsilon_{p_\alpha,\ q_\beta-1}, \varepsilon_{rh} = (r-1)\varepsilon_{2,\ h-r+2} - (r-2)\varepsilon_{1,\ h-r+1}$ and $\varepsilon_{jk} = 0, p_\alpha - q_\beta + 2 \le j - k \le p_\alpha - 1$, and $3 \le r \le p_\alpha$, $0 \le h - r \le q_\beta - 3$, $q_\beta - p_\alpha + 3 \le h \le q_\beta$, $1 \le i \le p_\alpha$. In this case, $q_\beta - p_\alpha - 1$ columns are 0.

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Therefore, there are $n_2 = \sum_{\alpha=1}^{u}\sum_{\beta=1}^{v} \mathrm{F}_{\alpha\beta}^{(2)}$ linearly independent solutions $\overline{X}_j$ of (3). For each $\overline{X}$, there are $k_1, k_2, \ldots, k_{n_2}$, such that $\overline{X} = \sum_{j=1}^{n_2} k_j\overline{X}_j$. Note that $\overline{X} = U^{-1}XV$. It is straightforward to show that every solution of $A(AX - XB) - (AX - XB)B = 0$ is a linear combination of n_2 linearly independent solutions. □

Let us illustrate Theorem 1.1 with an example.

Example 2.2. Suppose that the elementary divisors of A and B are $(\lambda - \lambda_1)^4$, $(\lambda-\lambda_1)^3$, $(\lambda-\lambda_2)^2$, $(\lambda-\lambda_2)$ and $(\lambda-\lambda_1)^5$, $(\lambda-\lambda_1)^3$, $(\lambda-\lambda_2)^3$, $(\lambda-\lambda_2)^2$, $(\lambda-\lambda_3)$, respectively, where $\lambda_1 \neq \lambda_2 \neq \lambda_3$, then

$$\dim(\ker\alpha^2) = \sum_{\alpha=1}^{4}\sum_{\beta=1}^{5} \mathrm{F}_{\alpha\beta}^{(2)} = 8+6+6+5+4+3+2+2 = 36.$$

Let A be an $n \times n$ complex matrix with distinct eigenvalues $a_1, \ldots,\ a_r$ the elementary divisors of A are $\underbrace{(\lambda - a_i)^{n_{ik}}}_{j_{ik}\text{ times}}$ and Segre characteristic

$$\left[(n_{11}^{j_{11}}, n_{12}^{j_{12}}, \ldots, n_{1m_1}^{j_{1m_1}}), \ldots, (n_{r1}^{j_{r1}}, n_{r2}^{j_{r2}}, \ldots, n_{rm_r}^{j_{rm_r}})\right],$$

where $0 < n_{i1} < n_{i2} < \cdots < n_{im_i}$ for $1 \leq i \leq r$; here we write $n_{ik}^{j_{ik}}$ for $\underbrace{n_{ik}, n_{ik}, \ldots, n_{ik}}_{j_{ik}\text{ times}}$.

Then we can show the following corollaries.

Corollary 2.3 ([1]). *Let $\alpha_A : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$ be a linear transformation such that $\alpha_A(X) = AX - XA$, where A is an $n \times n$ complex matrix. Then*

$$\dim(\ker\alpha_A^2) = \sum_{i=1}^{r}\left[\sum_{k=1}^{m_i}(2n_{ik}-1)j_{ik}^2 + 4\sum_{k=1}^{m_i-1} n_{ik}\cdot j_{ik}\cdot \sum_{\beta=k+1}^{m_i} j_{i\beta}\right].$$

Proof. Let the elementary divisors of A are $\underbrace{(\lambda - a_i)^{n_{ik}}}_{j_{ik}\text{ times}}$, where $1 \leq i \leq r$ and $1 \leq k \leq m_i$. Using Theorem 1.1, we have $\dim(\ker\alpha_A^2) = \sum_{\alpha=1}^{u}\sum_{\alpha=1}^{u} \mathrm{F}_{\alpha\alpha}^{(2)}$, where $u = \sum_{i=1}^{r}\sum_{k=1}^{m_i} j_{ik}$. The result follows from that

$$\sum_{\alpha=1}^{u}\sum_{\alpha=1}^{u} \mathrm{F}_{\alpha\alpha}^{(2)} = \sum_{i=1}^{r}\left[\sum_{k=1}^{m_i}(2n_{ik}-1)j_{ik}^2 + 4\sum_{k=1}^{m_i-1} n_{ik}\cdot j_{ik}\cdot \sum_{\beta=k+1}^{m_i} j_{i\beta}\right].$$

This completes the proof. □

Corollary 2.4. *Let $\alpha_A : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$ be a linear transformation such that $\alpha_A(X) = AX - XA$, where A is an $n \times n$ complex matrix. Then $n \leq$*

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$\dim(\ker\alpha_A^2) \leq n^2$. *Moreover,* $\dim(\ker\alpha_A^2) = n$ *if and only if* A *is similar to a diagonal matrix* $\operatorname{diag}\{a_1, a_2, \ldots, a_n\}$*, where* $a_i \neq a_j$ *if* $i \neq j$.

Proof. By Corollary 2.3,

$$\dim(\ker\alpha_A^2) \geq \sum_{i=1}^{r}[(2n_{i1}-1)j_{i1}^2] \geq \sum_{i=1}^{r} j_{i1}^2 \geq \frac{(\sum_{i=1}^{r} j_{i1})^2}{r} = \frac{n^2}{r} \geq n$$

and this equality holds if and only if $m_1 = 1, n_{i1} = 1$ and $j_{i1} = 1, r = n$. Clearly $\ker\alpha_A^2$ is a subspace of the vector space $M_{n\times n}(\mathbb{C})$, thus $\dim(\ker\alpha_A^2) \leq n^2$. ◻

3. The proof of Theorem 1.2. This theorem can be proved by the same method as employed in the previous section.

Proof. For convenience, we still use the same notations as in the proof of Theorem 1.1. Assume $X \in \ker\alpha^3$. Then $J_A^2(J_A\overline{X} - 3\overline{X}J_B) = (\overline{X}J_B - 3J_A\overline{X})J_B^2$, where $\overline{X} = U^{-1}XV$. Consequently, we get uv matrix equations:

$$\begin{aligned}&(\lambda_\alpha E_{p_\alpha} + N_{p_\alpha})^2[(\lambda_\alpha E_{p_\alpha} + N_{p_\alpha})X_{\alpha\beta} - 3X_{\alpha\beta}(\mu_\beta E_{q_\beta} + N_{q_\beta})]\\ &= [(\lambda_\alpha E_{p_\alpha} + N_{p_\alpha})X_{\alpha\beta} - 3X_{\alpha\beta}(\mu_\beta E_{q_\beta} + N_{q_\beta})](\mu_\beta E_{q_\beta} + N_{q_\beta})^2,\end{aligned}$$

for $1 \leq \alpha \leq u$ and $1 \leq \beta \leq v$. An easy calculation gives

$$\begin{aligned}(\mu_\beta - \lambda_\alpha)^3 X_{\alpha\beta} = {}& 3(\mu_\beta - \lambda_\alpha)^2(P_\alpha X_{\alpha\beta} - X_{\alpha\beta}Q_\beta)\\ &-3(\mu_\beta - \lambda_\alpha)(P_\alpha^2 X_{\alpha\beta} + X_{\alpha\beta}Q_\beta^2 - 2P_\alpha X_{\alpha\beta}Q_\beta)\\ &+P_\alpha^2(P_\alpha X_{\alpha\beta} - 3X_{\alpha\beta}Q_\beta) + (3P_\alpha X_{\alpha\beta} - X_{\alpha\beta}Q_\beta)Q_\beta^2. \quad (3.1)\end{aligned}$$

If $\mu_\beta \neq \lambda_\alpha$ then $X_{\alpha\beta} = 0$ by Lemma 2.1. We assume that $\mu_\beta = \lambda_\alpha$. In this case, we have

$$P_\alpha^2(P_\alpha X_{\alpha\beta} - 3X_{\alpha\beta}Q_\beta) = (X_{\alpha\beta}Q_\beta - 3P_\alpha X_{\alpha\beta})Q_\beta^2. \quad (3.2)$$

Case 1. $p_\alpha = q_\beta$

If $p_\alpha < 4$, then the number of arbitrary parameters in $X_{\alpha\beta}$ is $3p_\alpha - 2$.

If $p_\alpha \geq 4$, then we obtain

$$\begin{aligned}&\varepsilon_{ij} = 0,\\ &\varepsilon_{h2} = 3\varepsilon_{h-1,\ 1},\ \varepsilon_{p_\alpha-1,\ k} = 3\varepsilon_{p_\alpha,\ k+1},\\ &\varepsilon_{h3} = 3\varepsilon_{h-1,\ 2} - 3\varepsilon_{h-2,\ 1},\\ &\varepsilon_{p_\alpha-2,\ k} = 3\varepsilon_{p_\alpha-1,\ k+1} - 3\varepsilon_{p_\alpha,\ k+2},\\ &\varepsilon_{fg} = \varepsilon_{f-3,\ g-3} - 3\varepsilon_{f-2,\ g-2} + 3\varepsilon_{f-1,\ g-1},\end{aligned}$$

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where $4 \leq i \leq p_\alpha,\ i-j \geq 3,\ 4 \leq h \leq p_\alpha,\ 1 \leq k \leq q_\beta - 3,\ 4 \leq f \leq p_\alpha$ and $4 \leq g \leq q_\beta$. Hence, we have

$$\begin{cases} \varepsilon_{42} = 3\varepsilon_{31} \\ \varepsilon_{53} = 3\varepsilon_{42} - 3\varepsilon_{31} \\ \varepsilon_{64} = 3\varepsilon_{53} - 3\varepsilon_{42} + \varepsilon_{31} \\ \quad\vdots \qquad\qquad \vdots \\ \varepsilon_{p_\alpha,\ q_\beta-2} = 3\varepsilon_{p_\alpha-1,\ q_\beta-3} - 3\varepsilon_{p_\alpha-2,\ q_\beta-4} + \varepsilon_{p_\alpha-3,\ q_\beta-5} \\ \varepsilon_{p_\alpha-2,\ q_\beta-4} = 3\varepsilon_{p_\alpha-1,\ q_\beta-3} - 3\varepsilon_{p_\alpha,\ q_\beta-2} \\ \varepsilon_{p_\alpha-1,\ q_\beta-3} = 3\varepsilon_{p_\alpha,\ q_\beta-2}. \end{cases}$$

Thus, $\varepsilon_{31} = \varepsilon_{42} = \cdots = \varepsilon_{p_\alpha,\ q_\beta-2} = 0$. We also have

$$\begin{cases} \varepsilon_{43} = 3\varepsilon_{32} - 3\varepsilon_{21} \\ \varepsilon_{54} = 3\varepsilon_{43} - 3\varepsilon_{32} + \varepsilon_{21} \\ \quad\vdots \qquad\qquad \vdots \\ \varepsilon_{p_\alpha,\ q_\beta-1} = 3\varepsilon_{p_\alpha-1,\ q_\beta-2} - 3\varepsilon_{p_\alpha-2,\ q_\beta-3} + \varepsilon_{p_\alpha-3,\ q_\beta-4} \\ \varepsilon_{p_\alpha-2,\ q_\beta-3} = 3\varepsilon_{p_\alpha-1,\ q_\beta-2} - 3\varepsilon_{p_\alpha,\ q_\beta-1}. \end{cases}$$

By induction on the subscript, we obtain

$$\varepsilon_{s,\ s-1} = \left[\frac{(s-1)(s-2)(p_\alpha-2)}{(p_\alpha-1)} - (s-2)^2 + 1\right]\varepsilon_{21},$$

where $2 \leq s \leq p_\alpha$. Note that $\varepsilon_{fg} = \varepsilon_{f-3,\ g-3} - 3\varepsilon_{f-2,\ g-2} + 3\varepsilon_{f-1,\ g-1}$, where $4 \leq f \leq p_\alpha$ and $4 \leq g \leq q_\beta$. Continuing by induction, we finally have

$$\begin{aligned} \varepsilon_{fg} = {} & \frac{(f-2)(f-3)}{2}\varepsilon_{1,\ g-f+1} \\ & -[f(f-4)+3]\varepsilon_{2,\ g-f+2} + \frac{(f-1)(f-2)}{2}\varepsilon_{3,\ g-f+3}, \end{aligned}$$

where $4 \leq f \leq p_\alpha$, $4 \leq g \leq q_\beta$ and $0 \leq g-f \leq q_\beta - 4$. Then it is evident to see that the number of arbitrary parameters in $X_{\alpha\beta}$ is $3p_\alpha - 2$.

Case 2. $p_\alpha \neq q_\beta$. We can assume that $p_\alpha > q_\beta$, since the case $p_\alpha < q_\beta$ is analogous.

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If $p_\alpha - q_\beta = 1$ and $q_\beta \geq 4$, then we have

$$\begin{aligned}
\varepsilon_{ij} &= 0,\\
\varepsilon_{s,\ s-1} &= \frac{(s-1)(s-2)}{2}\varepsilon_{32} - [(s-2)^2 - 1]\varepsilon_{21},\\
\varepsilon_{fg} &= \frac{(f-2)(f-3)}{2}\varepsilon_{1,\ g-f+1}\\
&\quad -[f(f-4)+3]\varepsilon_{2,\ g-f+2} + \frac{(f-1)(f-2)}{2}\varepsilon_{3,\ g-f+3},
\end{aligned}$$

where $i - j \geq 2$, $3 \leq i \leq p_\alpha$, $2 \leq s \leq p_\alpha$, $4 \leq f \leq p_\alpha - 1$, $4 \leq g \leq q_\beta$ and $0 \leq g - f \leq q_\beta - 4$. Thus, the number of arbitrary parameters in $X_{\alpha\beta}$ is $3q_\beta - 1$.

If $p_\alpha - q_\beta = 1$ and $q_\beta < 4$, then a routine computation gives rise to the result.

If $p_\alpha - q_\beta \geq 2$ and $q_\beta \geq 4$, then we have

$$\begin{aligned}
\varepsilon_{ij} &= 0, \varepsilon_{g,\ g-2} = \frac{(g-1)(g-2)}{2}\varepsilon_{31},\\
\varepsilon_{s,s-1} &= \frac{(s-1)(s-2)}{2}\varepsilon_{32} - [(s-2)^2 - 1]\varepsilon_{21},\\
\varepsilon_{fg} &= \frac{(f-2)(f-3)}{2}\varepsilon_{1,g-f+1} - [f(f-4)+3]\varepsilon_{2,\ g-f+2}\\
&\quad + \frac{(f-1)(f-2)}{2}\varepsilon_{3,\ g-f+3},
\end{aligned}$$

where $i - j \geq 3$, $3 \leq g \leq p_\alpha$ and $2 \leq s \leq p_\alpha$ and $4 \leq f \leq p_\alpha - 1$ and $4 \leq g \leq q_\beta$, $0 \leq g - f \leq q_\beta - 4$. Thus, the number of arbitrary parameters in $X_{\alpha\beta}$ is $3q_\beta$.

If $p_\alpha - q_\beta \geq 2$ and $q_\beta < 4$, then an obvious computation gives rise to the same result.

Therefore, there are $n_3 = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \mathrm{F}_{\alpha\beta}^{(3)}$ linearly independent $X_j \in \ker\alpha^3$. For each $X \in \ker\alpha^3$, there are $k_1, k_2, \ldots, k_{n_3}$, such that $X = \sum_{j=1}^{n_3} k_j X_j$. $\square$

Let us illustrate Theorem 1.2 with an example.

EXAMPLE 3.1. Suppose that the elementary divisors of A and B are $(\lambda-\lambda_1)^4$, $(\lambda-\lambda_1)^3$, $(\lambda-\lambda_2)^2$, $(\lambda-\lambda_2)$, $(\lambda-\lambda_3)$ and $(\lambda-\lambda_1)^5$, $(\lambda-\lambda_1)^3$, $(\lambda-\lambda_2)^3$, $(\lambda-\lambda_2)^2$, respectively, where $\lambda_1 \neq \lambda_2 \neq \lambda_3$, then

$$\dim(\ker\alpha^3) = \sum_{\alpha=1}^{5}\sum_{\beta=1}^{4} \mathrm{F}_{\alpha\beta}^{(3)} = 11 + 8 + 9 + 7 + 5 + 4 + 3 + 2 = 49.$$

Let A be an $n \times n$ complex matrix with distinct eigenvalues $a_1, \ldots, a_r$ and the elementary divisors of A are $\underbrace{(\lambda - a_i)^{n_{ik}}}_{j_{ik} \text{ times}}$ for $1 \leq k \leq m_i$ and $1 \leq i \leq r$. The

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difference $n_{i,h+1} - n_{ih}$ is called an h-th jump of i for $1 \leq h \leq m_i - 1$. We denote by $\mu_{i1}, \mu_{i2}, \ldots, \mu_{i\rho_i}$ the places where the jumps equal to 1 and $\mu_{i1} < \mu_{i2} < \cdots < \mu_{i\rho_i}$. Then we can show the following corollaries.

COROLLARY 3.2 ([1]). *Let $\alpha_A : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$ be a linear transformation such that $\alpha_A(X) = AX - XA$, where A is an $n \times n$ complex matrix. Then*

$$\dim(\ker\alpha_A^3) = \sum_{i=1}^{r} \left[\sum_{k=1}^{m_i} \left[(3n_{ik} - 2)j_{ik}^2 + 6n_{ik} \cdot j_{ik} \sum_{\beta=k+1}^{m_i} j_{i\beta} \right] - 2 \sum_{l=1}^{\rho_i} j_{i,\mu_{il}} \cdot j_{i,\mu_{il}+1} \right].$$

Proof. Let the elementary divisors of A are $\underbrace{(\lambda - a_i)^{n_{ik}}}_{j_{ik} \text{ times}}$, where $1 \leq i \leq r$, $1 \leq k \leq m_i$ and $0 < n_{i1} < n_{i2} < \cdots < n_{im_i}$. Using Theorem 1.2, we have $\dim(\ker\alpha_A^3) = \sum_{\alpha=1}^{u} \sum_{\alpha=1}^{u} \mathrm{F}_{\alpha\alpha}^{(3)}$, where $u = \sum_{i=1}^{r} \sum_{k=1}^{m_i} j_{ik}$. Since $\sum_{\alpha=1}^{u} \sum_{\alpha=1}^{u} \mathrm{F}_{\alpha\alpha}^{(3)}$ is equal to

$$\sum_{i=1}^{r} \left[\sum_{k=1}^{m_i} \left[(3n_{ik} - 2)j_{ik}^2 + 6n_{ik} \cdot j_{ik} \sum_{\beta=k+1}^{m_i} j_{i\beta} \right] - 2 \sum_{l=1}^{\rho_i} j_{i,\mu_{il}} \cdot j_{i,\mu_{il}+1} \right],$$

we have the dimension formula. □

COROLLARY 3.3. *Let $\alpha_A : M_{n\times n}(\mathbb{C}) \to M_{n\times n}(\mathbb{C})$ be a linear transformation such that $\alpha_A(X) = AX - XA$, where A is an $n \times n$ complex matrix. Then $n \leq \dim(\ker\alpha_A^3) \leq n^2$. Moreover, $\dim(\ker\alpha_A^3) = n$ if and only if A is similar to a diagonal matrix $\mathrm{diag}\{a_1, a_2, \ldots, a_n\}$, where $a_i \neq a_j$ if $i \neq j$.*

Proof. Since

$$\dim(\ker\alpha_A^3) = \sum_{i=1}^{r} \left[\sum_{k=1}^{m_i} \left[(3n_{ik} - 2)j_{ik}^2 + 6n_{ik} \cdot j_{ik} \sum_{\beta=k+1}^{m_i} j_{i\beta} \right] - 2 \sum_{l=1}^{\rho_i} j_{i,\mu_{il}} \cdot j_{i,\mu_{il}+1} \right],$$

it follows that $\dim(\ker\alpha_A^3) \geq \sum_{i=1}^{r} [(3n_{i1} - 2)j_{i1}^2] \geq \sum_{i=1}^{r} j_{i1}^2 \geq \frac{(\sum_{i=1}^{r} j_{i1})^2}{r} = \frac{n^2}{r} \geq n$ and the equality holds if and only if $m_1 = 1, n_{i1} = 1$ and $j_{i1} = 1, r = n$. Clearly, $\ker\alpha_A^3$ is a subspace of the $n \times n$-dimensional vector space $M_{n\times n}(\mathbb{C})$, so $\dim(\ker\alpha_A^3) \leq n^2$. □

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