

PROPERTIES OF FIRST EIGENVECTORS AND FIRST EIGENVALUES OF NONSINGULAR WEIGHTED DIRECTED GRAPHS*

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Abstract. The class of nonsingular connected weighted directed graphs with an unweighted undirected branch is considered in this article. This paper investigates the monotonicity properties of the first eigenvectors of such graphs along certain paths. The paper describes how the first eigenvalue of such graphs changes under some perturbation. It is shown that replacing a branch which is a tree by a path on the same number of vertices will not increase the first eigenvalue, while replacing the tree by a star on the same number of vertices will not decrease the first eigenvalue. As an application the paper characterizes the graphs minimizing the first eigenvalue over certain classes of such graphs.

Key words. Laplacian matrix, Weighted directed graph, First eigenvalue, First eigenvector.

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1. Introduction. In [2, 6], the authors introduced the concept of a *weighted directed graph*: it is a directed graph with a simple underlying undirected graph and edges having complex weights of unit modulus. Thus, there are no digons in a weighted directed graph; that is, in a weighted directed graph both the edges (i, j) and (j, i) cannot be present simultaneously. Furthermore, the presence of an edge (i, j) of weight w is as good as the presence of the edge (j, i) of weight \bar{w} , the complex conjugate of w . Weighted directed graphs with edge weights of unit modulus are also known as *T-gain graphs* (see [7]), where \mathbb{T} stands for the complex numbers of unit modulus. The adjacency matrix, the vertex edge incidence matrix, and the Laplacian matrix of the weighted directed graphs were introduced in [2, 6]. In this article, we use w_{ij} to denote the weight of an edge (i, j) .

Let G be a weighted directed graph on vertices $1, 2, \dots, n$. At times, we use $V(G)$ and $E(G)$ to denote the set of vertices and the set of edges of a graph G , respectively. Sometimes it is convenient to denote ' $(i, j) \in E(G)$ or $(j, i) \in E(G)$ ' by ' $i \sim j$ '. The *adjacency matrix* (see [2]), $A(G) = [a_{ij}]$ of G is an $n \times n$ matrix with entries $a_{ij} = w_{ij}$ or \bar{w}_{ji} or 0, depending on whether $(i, j) \in E(G)$ or $(j, i) \in E(G)$ or

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otherwise, respectively. The *degree* $d(i)$ of a vertex i in a weighted directed graph G is the number of edges incident with i . It may be viewed as the sum of the absolute values of the weights of the edges incident with the vertex i . The *Laplacian matrix* $L(G)$ of G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix with $d(i)$ as the i -th diagonal entry. If weights of all the edges in G are 1, then $L(G)$ coincides with the usual Laplacian matrix of a graph. In view of this fact, henceforth, a weighted directed graph with all the edges having weight 1 is said to be an *unweighted undirected* graph. If the weights of all edges in G are ± 1 , then (viewing the edges with weight 1 as directed and the edges with weight -1 as undirected) $L(G)$ coincides with the Laplacian matrix of a mixed graph (see [1]). If the weights of all edges in G are -1 , then $L(G)$ coincides with the well studied signless Laplacian (see [4]) of a graph. Thus, the study of the adjacency and the Laplacian matrices of weighted directed graphs includes the study of those matrices for unweighted undirected graphs, mixed graphs and the study of the signless Laplacian matrix as special cases. It was observed and proved in [2] that $L(G)$ is a positive semidefinite matrix. So its eigenvalues are nonnegative.

Let G be a weighted directed graph on vertices $1, \dots, n$. Note that for any vector $x \in \mathbb{C}^n$ we have

$$x^* L(G) x = \sum_{(i,j) \in E(G)} |x_i - w_{ij} x_j|^2. \tag{1.1}$$

Furthermore, unlike the Laplacian matrix of an unweighted undirected graph, the Laplacian matrix of a weighted directed graph is sometimes nonsingular (see [2]). We call a connected weighted directed graph *singular* (resp. *nonsingular*) if its corresponding Laplacian matrix is singular (resp., nonsingular).

In defining subgraph, walk, path, component, connectedness, and degree of a vertex in G we focus only on the underlying unweighted undirected graph of G .

DEFINITION 1.1. (See [2]) An i_1 - i_k -walk W in a weighted directed graph G is a finite sequence i_1, i_2, \dots, i_k of vertices such that $i_p \sim i_{p+1}$ for every $1 \leq p \leq k-1$. We call $w_W = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k}$, the *weight* of the walk W , where a_{ij} are the entries of $A(G)$.

DEFINITION 1.2. (See [2]) Let G be a weighted directed graph on vertices $1, \dots, n$ and D be a diagonal matrix with $|d_{ii}| = 1$, for each i . Then $D^* L(G) D$ is the Laplacian matrix of another weighted directed graph which we denote by ${}^D G$. If $(i, j) \in E(G)$ has a weight w_{ij} , then it has the weight $\bar{d}_{ii} w_{ij} d_{jj}$ in ${}^D G$. Let H and G be two weighted directed graphs on vertices $1, \dots, n$. We say H is *D-similar* to G , if there exists a diagonal matrix D (with $|d_{ii}| = 1$, for each i) such that $H = {}^D G$.

Below we state few results due to Bapat et al. (see [2]) which shall be used for

further development.

LEMMA 1.3. [2, Lemma 2] *Let G be a connected weighted directed graph. Then G is singular if and only if it is D -similar to an unweighted undirected graph.*

LEMMA 1.4. [2, Corollary 9] *Let G be a connected weighted directed graph. Then G is nonsingular if and only if it contains a cycle of weight different from 1. In particular, a weighted directed graph whose underlying graph is a tree is always singular.*

A cycle C in a weighted directed graph is said to be *nonsingular* if its weight $w_C \neq 1$ (see [2]). Otherwise we call it a *singular* cycle.

Let G be a weighted directed graph. Henceforth, by an eigenvalue (resp., eigenvector) of G we mean an eigenvalue (resp., eigenvector) of the Laplacian matrix $L(G)$ of G . By $\lambda_i(G)$ we denote the i -th smallest eigenvalue of G . The smallest eigenvalue $\lambda_1(G)$ of G is said to be the *first eigenvalue* of G and an eigenvector corresponding to $\lambda_1(G)$ is said to be a *first eigenvector* of G . By Rayleigh's theorem (see [9]),

$$\lambda_1(G) = \min_{\substack{x \in \mathbb{C}^n \\ x \neq 0}} \frac{x^* L(G) x}{x^* x}. \quad (1.2)$$

Thus, for an arbitrary unit vector $x \in \mathbb{C}^n$ we have $\lambda_1(G) \leq x^* L(G) x$ and equality holds if and only if x is a first eigenvector of G .

If λ is an eigenvalue of G with an eigenvector x , then by an *eigenvector equation* of x at a vertex v we mean the following equation

$$[d(v) - \lambda]x(v) = \sum_{v \sim j} a_{vj}x(j), \text{ where } a_{vj} \text{ is the } vj\text{-th entry of } A(G). \quad (1.3)$$

Let G_1, G_2 be two vertex disjoint weighted directed graphs, and let v_1 and v_2 be two distinct vertices of G_1 and G_2 , respectively. By $G_1(v_1) \diamond G_2(v_2)$ we denote the weighted directed graph obtained from G_1 and G_2 by identifying v_1 with v_2 and forming the new vertex u . Sometimes the graph $G_1(v_1) \diamond G_2(v_2)$ is also denoted by $G_1(u) \diamond G_2(u)$. This graph operation is popularly known as a *coalescence* of G_1 and G_2 . If a connected weighted directed graph G can be expressed as $G = G_1(u) \diamond G_2(u)$ for some weighted directed graphs G_1 and G_2 , then G_1 is called a *branch of G with root u* . Note that G_2 is also a branch of G according to the above definition. A branch H of a weighted directed graph G is said to be *unweighted undirected* if all its edges have weight 1.

The article is organized as follows. In Section 2, we discuss monotonicity properties of the first eigenvectors along paths in the unweighted undirected branches of a nonsingular weighted directed graph. In Section 3, we investigate how the first eigenvalue of a nonsingular weighted directed graph changes under perturbation of

an unweighted undirected branch. Furthermore, we show that replacing a branch which is a tree by a path on the same number of vertices will not increase the first eigenvalue, while replacing the tree by a star on the same number of vertices will not decrease the first eigenvalue. In Section 4, as an application we characterize the nonsingular weighted directed graph minimizing the first eigenvalue over the class of connected weighted directed graphs with fixed order which contains a given nonsingular weighted directed graph as an induced subgraph. Moreover, we obtain the nonsingular weighted directed graph minimizing the first eigenvalue over the class of connected weighted directed graphs with exactly one nonsingular cycle.

2. Properties of a first eigenvector. In this section, we consider the class of nonsingular weighted directed graphs having unweighted undirected branch and describe various properties of the first eigenvectors of such graphs. Analogous results for the Fiedler vector (see [8]) of unweighted undirected graphs can be found in [8, 3].

The following lemma is crucial for further developments.

LEMMA 2.1. *Let $G = G_1(u) \diamond H(u)$ be a weighted directed graph and let L_H be the principal submatrix of $L(G)$ corresponding to H . If H is a connected unweighted and undirected graph, then the smallest eigenvalue of L_H is simple and the corresponding eigenvector is positive, unique up to a scalar multiple.*

Proof. Since H is connected unweighted and undirected, we see that L_H is a real irreducible matrix and has non-positive off diagonal entries. Consider the matrix $M = kI - L_H$, where $k > 0$, large enough such that each entry of M is nonnegative. Thus, M is a nonnegative, symmetric irreducible matrix. By the Perron-Frobenius Theorem (see [10]), the largest eigenvalue of M has multiplicity 1 and the corresponding eigenvector is positive and unique up to a scalar multiple. The rest of the proof is trivial. \square

By $N_G(u)$ we mean the set of neighbors of the vertex u in a weighted directed graph G . Note that $d(v) = |N_G(v)|$. For $x \in \mathbb{C}^n$, by $\operatorname{Re} x$ and $\operatorname{Im} x$ we mean the real part and imaginary part of x , respectively.

Let x be an eigenvector of G corresponding to an eigenvalue λ . A branch H of G is said to be *real, purely imaginary, nonnegative, positive or a zero branch* with respect to x if the valuations of every vertex in H are real, purely imaginary, nonnegative, positive, or zero with respect to x , respectively.

The next result says that an unweighted undirected branch H of a nonsingular weighted directed graph G with root u is real (resp., purely imaginary, zero, nonnegative) with respect to a first eigenvector of G if the valuation of the first eigenvector

at the root is real (resp., purely imaginary, zero, nonnegative).

LEMMA 2.2. *Let G be a nonsingular connected weighted directed graph which contains an unweighted undirected branch H with root u . Let x be a first eigenvector of G .*

- (i) *If $x(u)$ is real, then H is real with respect to x .*
- (ii) *If $x(u)$ is purely imaginary, then H is purely imaginary with respect to x .*
- (iii) *If $x(u) = 0$, then H is a zero branch with respect to x .*
- (iv) *If $x(u)$ is real and $x(u) > 0$, then H is nonnegative with respect to x .*
- (v) *If $x(u) \neq 0$, then $x(v) \neq 0$ for all $v \in V(H)$.*

Proof. (i) Assume that $x^*x = 1$. Suppose that $\text{Im } x(u) = 0$, but $\text{Im } x(w) \neq 0$ for some $w \in V(H) - \{u\}$. Let x be partitioned conformally as $x = \begin{bmatrix} x' \\ x_H \end{bmatrix}$, where x_H is the sub-vector of x corresponding to the vertices of H . Let $y = \begin{bmatrix} x' \\ \bar{x}_H \end{bmatrix}$. We see that $x^*x = 1 = y^*y$ and $x^*L(G)x = y^*L(G)y$. Thus, y is also a first eigenvector of G . Hence $\hat{y} = \frac{1}{2}(x - y) = \begin{bmatrix} 0 \\ \text{Im } x_H \end{bmatrix}$ is a first eigenvector of G . Let L_H be the principal sub-matrix of $L(G)$ corresponding to H . Since $x(u)$ is real, we see that $\hat{y}(u) = 0$. Thus, $\lambda_1(G)$ is an eigenvalue of L_H with $\text{Im } x_H$ as a corresponding eigenvector. So $\lambda_1(L_H) \leq \lambda_1(G)$. By the interlacing theorem, we have $\lambda_1(G) \leq \lambda_1(L_H)$, which implies $\lambda_1(L_H) = \lambda_1(G)$. By Lemma 2.1, $\text{Im } x_H > 0$. Thus, \hat{y} is nonnegative. By the eigenvector equation of \hat{y} at the vertex u , we have $\sum_{v \in N_G(u)} \hat{y}(v) = 0$, which implies $\hat{y}(v) = 0$ for all $v \in N_G(u)$. Again by considering the eigenvector equation of \hat{y} at an arbitrary vertex $v \in N_G(u)$, we see that $\hat{y}(v') = 0$ for all $v' \in N_H(v)$. Repeating the same argument we see that $\hat{y}(v) = 0$ for all $v \in V(H) - \{u\}$. Thus, $\hat{y} = 0$, a contradiction to $\text{Im } x(w) \neq 0$.

(ii) Assume that $x(u)$ is purely imaginary. Let $z = ix$, where $i = \sqrt{-1}$. Then z is a first eigenvector of G such that $z(u)$ is real. Thus, by part(i), H is a real branch of G with respect to z . Hence, H is a purely imaginary branch with respect to x .

(iii) Assume that $x(u) = 0$. By part(i), H is a real branch with respect to x . Suppose that $x(w) \neq 0$ for some vertex $w \in V(H) - \{u\}$. Let x be partitioned conformally as $x = \begin{bmatrix} x' \\ x_H \end{bmatrix}$, where x_H is the sub-vector of x corresponding to the vertices of H . Consider $y = \begin{bmatrix} x' \\ -x_H \end{bmatrix}$. Observe that $x^*L(G)x = y^*L(G)y$ and $y^*y = 1$. Thus, y is a first eigenvector of G . Hence $\hat{y} = \frac{1}{2}(x - y) = \begin{bmatrix} 0 \\ x_H \end{bmatrix}$ is a first eigenvector of G . Arguing similarly as in the proof of part(i), we arrive at a

contradiction to $x(w) \neq 0$. Hence, part(iii) of the result holds.

(iv) Assume that $x(u)$ is real and $x(u) > 0$. By part(i), $x(v)$ is real for each vertex $v \in V(H)$. We claim that $x(w_1)x(w_2) \geq 0$ whenever $w_1, w_2 \in V(H)$ and $w_1 \sim w_2$. Suppose that there exists $w_1, w_2 \in V(H)$, $w_1 \sim w_2$ such that $x(w_1)x(w_2) < 0$. Let y be the vector defined as $y(w) = |x(w)|$ if $w \in V(H)$ and $y(v) = x(v)$ otherwise. Observe that $y^*y = x^*x$,

$$|y(w_1) - y(w_2)|^2 < |x(w_1) - x(w_2)|^2$$

and for all other edges $(i, j) \in E(G)$ we have

$$|y(i) - w_{ij}y(j)|^2 \leq |x(i) - w_{ij}x(j)|^2.$$

Thus, $\lambda_1(G) \leq \frac{y^*L(G)y}{y^*y} < \frac{x^*L(G)x}{x^*x} = \lambda_1(G)$, a contradiction. So the claim holds.

As H is connected and $x(u) > 0$, it follows that $x(v) \geq 0$ for every vertex v of H . Hence, H is a nonnegative branch with respect to x .

(v) Assume that $x(u) \neq 0$. Without loss of generality, we assume that $x(u) > 0$. By part(iv), $x(v) \geq 0$ for each vertex $v \in V(H)$. Suppose that $x(w) = 0$ for some vertex w of H . Then by the eigenvector equation of x at w , we see that $\sum_{v \sim w} x(v) = 0$, which implies $x(v) = 0$ for all $v \in N_H(w)$. By repeated application of the eigenvector equation of x , we see that $x(v) = 0$ for every $v \in N_H(u)$. Again, by the eigenvector equation of x at the vertex v we have $x(u) = 0$, which is a contradiction. Hence, $x(v) \neq 0$ for all $v \in V(H)$. \square

DEFINITION 2.3. Let G be a weighted directed graph and let x be a first eigenvector of G . A path $P := v_1, v_2, \dots, v_k$ in G is said to be *strictly increasing*, *strictly decreasing* or *identically zero* with respect to x if the sequence $x(v_1), x(v_2), \dots, x(v_k)$ is strictly increasing, strictly decreasing, or a zero sequence, respectively. The path P is said to be *positive* with respect to x , if $x(v_i) > 0$ for $i = 1, \dots, k$.

The next lemma shows the existence of a strictly decreasing positive path in an unweighted undirected branch of a nonsingular weighted directed graph with respect its first eigenvector. This result is analogous to the result due to Bapat et al. [3] for the Fiedler vector of unweighted undirected graphs.

LEMMA 2.4. *Let G be a connected nonsingular weighted directed graph which contains an unweighted undirected branch H with root u . Let y be a first eigenvector of G such that $y(u) > 0$. Then there exists a strictly decreasing positive path P from v to u in H with respect to y , for each vertex $v \in V(H) - \{u\}$.*

Proof. Assume that $y(u) > 0$. By Lemma 2.2(iv) and Lemma 2.2(v), H is

positive with respect to y . From the eigenvector equation of y at the vertex v we have

$$\lambda_1(G)y(v) = d(v)y(v) - \sum_{i \sim v} y(i) = \sum_{i \sim v} [y(v) - y(i)].$$

Since $\lambda_1(G)y(v) > 0$, it follows that there exists a vertex i_1 in H such that $i_1 \sim v$ and $y(i_1) < y(v)$. If $i_1 = u$ then we are done. Assume that $i_1 \neq u$. As $i_1 \in V(H)$, we have $y(i_1) > 0$. Similarly by the eigenvector equation of y at the vertex i_1 , there exists a vertex i_2 such that $i_2 \sim i_1$ and $y(i_2) < y(i_1)$. Since the number of vertices is finite, with similar argument we see that $i_k = u$ for some k . Hence, the result holds. \square

The following theorem shows the monotonicity along every path in a spanning tree of an unweighted undirected branch of a weighted directed graph with respect to its first eigenvector. The result is analogous to [3, Theorem 2.4].

THEOREM 2.5. *Let G be a connected nonsingular weighted directed graph which contains an unweighted undirected branch H with root u . Then there exists a first eigenvector y of G such that $y(u)$ is real and there exists a spanning tree T_H of H such that every path P in T_H which starts from the vertex u has one of the following property.*

- (a) *If $y(u) > 0$, then P is strictly increasing with respect to y .*
- (b) *If $y(u) < 0$, then P is strictly decreasing with respect to y .*
- (c) *If $y(u) = 0$, then P is identically zero with respect to y .*

Proof. Let X be a first eigenvector of G and let $y = \alpha X$, where $\alpha \in \mathbb{C}$ such that $y(u)$ is real.

(b) Assume that $y(u) > 0$. If every path starting from u in H is strictly increasing, then any spanning tree T_H of H satisfies the required property. In this case, we are done. Suppose that there is a path $Q := u, i_1, \dots, i_k, v$ from the vertex u to a vertex v in H such that $y(u) < y(i_1) < \dots < y(i_k)$, but $y(v) \leq y(i_k)$. Let $w \neq u$ be an arbitrary vertex in H . By Lemma 2.4, there exists a strictly decreasing positive path (with respect to y) from w to u in H , say $P_1 := w, w_1, \dots, w_r, u$. Let e be the edge in Q with end vertices i_k and v . Consider the subgraph $H_1 = H - e$ of H . If P_1 does not pass through e , then u, w_r, \dots, w_1, w is a strictly increasing positive path (with respect to y) starting from u to w . If P_1 passes through the edge e , then $i_k = w_{l-1}$ and $v = w_l$ for some $l, 1 \leq l \leq r$, as P_1 is strictly decreasing and $y(v) \leq y(i_k)$. Consider the path $P' := u, i_1, \dots, i_{k-1}, i_k, w_{l-2}, \dots, w_1, w$. Note that P' does not contain e and P' is a strictly increasing positive path (with respect to y) starting from u to w . Thus, there exists a strictly increasing positive path (with respect to y) starting from u to w in H_1 for each vertex $w \in V(H) - \{u\}$. If every path starting from u in H_1 is strictly increasing (with respect to y), then the proof follows by taking a spanning tree of H_1 as T_H . Otherwise, by repeated application of the above arguments we obtain a spanning tree T_H of H with the required property.

(b) If $y(u) < 0$, then the proof follows similarly as above, by considering $-y$ as a first eigenvector of G .

(c) If $y(u) = 0$, then the proof follows from Lemma 2.2 (iii). \square

The next theorem says that along every path (starting from the root) in a spanning tree of an unweighted undirected branch of a nonsingular weighted directed graph the absolute values of the valuations of a first eigenvector is strictly increasing.

THEOREM 2.6. *Let G be a connected nonsingular weighted directed graph which contains an unweighted undirected branch H with root u . Let x be a first eigenvector of G such that $x(u) \neq 0$. Then there exists a spanning tree T_H of H such that along every path P in T_H starting from the vertex u in T_H the absolute values of the valuations of x are strictly increasing.*

Proof. Let $y = \bar{\alpha}x$, where $\alpha = x(u)$. Then $y(u) > 0$. By Theorem 2.5, there exists a spanning tree T_H of H such that every path P in T_H starting from the vertex u is strictly increasing with respect to y . Hence, the result follows. \square

3. First eigenvalue under perturbation. In this section, we investigate how the first eigenvalue of a nonsingular weighted directed graph changes under perturbation of an unweighted undirected branch. Some of our results generalize the results for the least eigenvalue of the signless Laplacian matrix of a graph in [11].

Let G and H be two weighted directed graphs such that $i, j \in V(G)$, $u \in V(H)$. Consider the weighted directed graphs $G_1 = G(i) \diamond H(u)$ and $G_2 = G(j) \diamond H(u)$. Then G_2 is said to be obtained from G_1 by *relocating the branch H* with root u from the vertex i to j .

The next lemma explains how the first eigenvalue of a nonsingular weighted directed graph changes by relocating an unweighted undirected branch.

LEMMA 3.1. *Let $G = G_1(j) \diamond H(u)$ and $G^* = G_1(i) \diamond H(u)$, where G_1 is a nonsingular connected weighted directed graph, i, j are distinct vertices of G_1 , H is a connected unweighted undirected graph, and u is a vertex of H . If there exists a first eigenvector x of G such that $|x(i)| \geq |x(j)|$, then $\lambda_1(G^*) \leq \lambda_1(G)$. In particular, if $|x(i)| > |x(j)|$, then the inequality is strict.*

Proof. Let x be a first eigenvector of G such that $|x(i)| \geq |x(j)|$. Without loss of generality, we assume that $x(i)$ is real. Let \hat{x} be the vector defined on the vertices of G^* such that $\hat{x}(v) = x(v)$ if $v \in V(G_1)$ and $\hat{x}(v) = |x(v)| + x(i) - |x(j)|$ if $v \in V(H) - \{u\}$. We observe that for $v \in N_H(u)$;

$$|\hat{x}(v) - \hat{x}(i)|^2 = ||x(v)| - |x(j)||^2 \leq |x(v) - x(j)|^2,$$

for each other edge (v_1, v_2) of H ;

$$|\hat{x}(v_1) - \hat{x}(v_2)|^2 = ||x(v_1)| - |x(v_2)||^2 \leq |x(v_1) - x(v_2)|^2,$$

and for each edge (r, s) in G_1 ;

$$|\hat{x}(r) - w_{rs}\hat{x}(s)|^2 = |x(r) - w_{rs}x(s)|^2.$$

Note that

$$\hat{x}^*\hat{x} = \sum_{v \in V(G_1)} |x(v)|^2 + \sum_{v \in V(H) - \{u\}} ||x(v)| + x(i) - |x(j)||^2 \geq \sum_{v \in V(G)} |x(v)|^2 = x^*x.$$

Thus

$$\lambda_1(G^*) \leq \frac{\hat{x}^*L(G^*)\hat{x}}{\hat{x}^*\hat{x}} \leq \frac{x^*L(G)x}{x^*x} = \lambda_1(G). \quad (3.1)$$

In particular, if $|x(i)| > |x(j)|$, then $\hat{x}^*\hat{x} > x^*x$. In this case, we have a strict inequality in equation (3.1), which implies $\lambda_1(G^*) < \lambda_1(G)$. \square

The next corollary explains how the first eigenvalue changes while relocating a path.

COROLLARY 3.2. *Let $G = G_1(j) \diamond P(u)$ and $G^* = G_1(i) \diamond P(u)$, where G_1 is a nonsingular connected weighted directed graph, i, j are distinct vertices of G_1 , and $P := u, u_1, u_2, \dots, u_k$ is an unweighted undirected path. If there exists a first eigenvector x of G such that $0 < |x(j)| \leq |x(i)|$ then $\lambda_1(G^*) < \lambda_1(G)$.*

Proof. Let x be a first eigenvector of G such that $0 < |x(j)| \leq |x(i)|$. Assume that $x^*x = 1$. If $|x(j)| < |x(i)|$, then by Lemma 3.1, we have $\lambda_1(G^*) < \lambda_1(G)$. Next, let $0 < |x(j)| = |x(i)|$. Suppose that $\lambda_1(G^*) = \lambda_1(G)$. Define a vector $y \in \mathbb{C}^n$ as $y(v) = x(v)$ if $v \in V(G_1)$ and $y(u_l) = |x(u_l)|$ for $l = 1, \dots, k$. Observe that $y^*y = x^*x$,

$$|y(i) - y(u_1)|^2 = ||x(i)| - |x(u_1)||^2 = ||x(j)| - |x(u_1)||^2 \leq |x(j) - x(u_1)|^2,$$

$$|y(u_l) - y(u_{l+1})|^2 = ||x(u_l)| - |x(u_{l+1})||^2 \leq |x(u_l) - x(u_{l+1})|^2, \text{ for } l = 1, \dots, k - 1,$$

and for each edge (r, s) in G_1 ;

$$|y(r) - w_{rs}y(s)|^2 = |x(r) - w_{rs}x(s)|^2.$$

Thus, $\lambda_1(G^*) \leq y^*L(G^*)y \leq x^*L(G)x = \lambda_1(G)$, which implies y is a first eigenvector of G^* . By the eigenvector equation of y at the vertex j of G^* , we have

$$\lambda_1(G^*)y(j) = |N_{G_1}(j)|y(j) - \sum_{k \sim j} a_{kj}y(k) = |N_{G_1}(j)|x(j) - \sum_{k \sim j} a_{kj}x(k). \quad (3.2)$$

Again, by the eigenvector equation of x at the vertex j of G , we have

$$\begin{aligned} \lambda_1(G)x(j) &= (|N_{G_1}(j)| + 1)x(j) - x(u_1) - \sum_{k \sim j} a_{kj}x(k) \\ &= \lambda_1(G^*)y(j) + x(j) - x(u_1), \text{ (using equation (3.2))} \\ &= \lambda_1(G)x(j) + x(j) - x(u_1). \end{aligned}$$

Thus, $x(j) = x(u_1)$. By Theorem 2.6, $|x(j)| < |x(u_1)|$, which is a contradiction. Hence, the corollary holds. \square

The next theorem is one of our main results of this section. The result says that replacing a branch which is a tree by a path on the same number of vertices will not increase the first eigenvalue, while replacing the tree by a star on the same number of vertices will not decrease the first eigenvalue.

THEOREM 3.3. *Let $G_T = G(u) \diamond T(v)$, $G_S = G(u) \diamond S(v)$ and $G_P = G(u) \diamond P(v)$, where G is a nonsingular connected weighted directed graph with a vertex u , T is an unweighted undirected tree on k vertices with a vertex v , $k \geq 2$, S is an unweighted undirected star on k vertices with the center v , and P is an unweighted undirected path on k vertices with a pendent vertex v . Then $\lambda_1(G_P) \leq \lambda_1(G_T) \leq \lambda_1(G_S)$.*

Proof. Let $v = v_1, v_2, \dots, v_k$ be the vertices of T . Let x be a first eigenvector of G_T . Without loss of generality, we assume that $x(u)$ is real and $x(u) \geq 0$. By Lemma 2.2 (i) and Theorem 2.5, the valuations of x on the vertices of T are all real and nonnegative. We arrange the vertices of T in such a way that $x(u) = x(v_{i_1}) \leq x(v_{i_2}) \leq \dots \leq x(v_{i_k})$. Then

$$\begin{aligned} \lambda_1(G_T) &= \frac{x^*L(G_T)x}{x^*x} \\ &= \frac{1}{x^*x} \sum_{(i,j) \in E(G)} |x(i) - w_{ij}x(j)|^2 + \sum_{\{v_i, v_j\} \in E(T)} [x(v_i) - x(v_j)]^2 \\ &\geq \frac{1}{x^*x} \sum_{(i,j) \in E(G)} |x(i) - w_{ij}x(j)|^2 + \sum_{j=1}^{k-1} [x(v_{i_j}) - x(v_{i_{j+1}})]^2 \\ &= \frac{x^*L(G_P)x}{x^*x} \geq \lambda_1(G_P). \end{aligned}$$

If $T = S$, then there is nothing to prove. Otherwise, there exists a pendent vertex w of T adjacent to the vertex v_i of T , where $v_i \neq v$. Let $G_{T'} = G(u) \diamond T'(v)$, where T' is the tree obtained from T by removing the edge between v_i and w and adding a new edge between v and w . Observe that $G_{T'}$ can be obtained from G_T by relocating a branch from v_i to u . By Theorem 2.5, $x(v_i) \geq x(u)$. Thus, by Lemma 3.1, $\lambda_1(G_T) \leq \lambda_1(G_{T'})$. If $G_{T'} = G_S$, then we stop. Otherwise, we repeat the above discussion and after a finite number of steps we have $\lambda_1(G_T) \leq \lambda_1(G_S)$. \square

The following example shows that the inequality in Theorem 3.3 need not be strict if G_T has a first eigenvector x with $x(u) = 0$.

EXAMPLE 3.4. Consider the weighted directed graphs G' , H and H' as shown below. Here the blue colored edges have weight -1 and the red colored edges have weight 1 . One can see that the Laplacian matrices of G' , H and H' are independent of the orientations of their edges. Hence, orientations of the edges are not indicated in the figures. Observe that the unweighted undirected branch T of G' with root u is neither a path nor a star, while the unweighted undirected branch of H and H' with root u is a path and a star, respectively. Using the mathematical package MATLAB, we obtain that G' has a first eigenvector x with $x(u) = 0$ and $\lambda_1(H) = \lambda_1(G') = \lambda_1(H') \approx 0.081$.

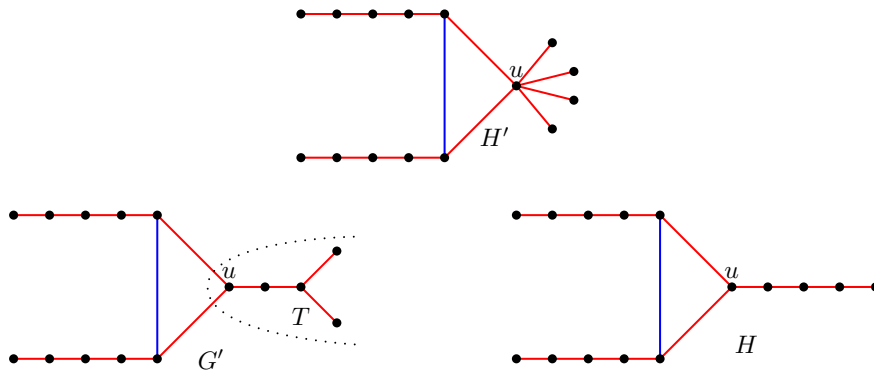


FIG. 3.1. $G' = G_T$, $H = G_P$ and $H' = G_S$, for some G .

In view of Example 3.4, it is natural to ask: Whether the inequality in Theorem 3.3 is strict for G_T , where T is neither a path nor a star, if G_T has a first eigenvector x with $x(u) \neq 0$? The following result answers this question in the affirmative.

THEOREM 3.5. Let G_T, G_S and G_P be the nonsingular weighted directed graphs as described in the statement of Theorem 3.3. If G_T has a first eigenvector x with $x(u) \neq 0$, then $\lambda_1(G_T) = \lambda_1(G_P)$ if and only if $T = P$, and $\lambda_1(G_T) = \lambda_1(G_S)$ if and only if $T = S$.

Proof. Assume that $\lambda_1(G_T) = \lambda_1(G_P)$. If $T \neq P$, then T contains two vertices i, j of degree 1, except v . Let P_{vi} and P_{vj} be the paths in T starting from v to i and from v to j , respectively. While traversing from v along P_{vi} and P_{vj} , let i_0 be the last vertex common to both these paths, and let i_1, j_1 be the vertices next to i_0 in P_{vi} and P_{vj} , respectively. Consider the weighted directed graph H obtained from G_T by relocating the branch $T_0 + e$ from i_0 to j_1 , where T_0 is the component of $T - i_0$ containing i_1 and e is the edge between i_0 and i_1 . Since $x(u) \neq 0$, we see that $|x(i_0)| < |x(j_1)|$, by Theorem 2.6. Thus, by Lemma 3.1, $\lambda_1(H) < \lambda_1(G_T)$. Observe

that $H = G(u) \diamond T'(v)$, where T' is the tree obtained from T by removing the edge e and adding a new edge between i_1 and j_1 . So by Theorem 3.3, $\lambda_1(G_P) \leq \lambda_1(H)$, which implies $\lambda_1(G_P) < \lambda_1(G_T)$, a contradiction. Hence, $T = P$.

Assume that $\lambda_1(G_T) = \lambda_1(G_S)$. If $T \neq S$, then there exists a pendent vertex w of T adjacent to a vertex v_i other than v in T . Consider the graph $G_{T'} = G(u) \diamond T'(v)$, where T' is the tree obtained from T by removing the edge between v_i and w , and adding a new edge between v and w . Since $x(u) \neq 0$, we have $|x(u)| < |x(v_i)|$, by Theorem 2.6. Observe that $G_{T'}$ can be obtained from G_T by relocating an unweighted undirected branch with root v_i from v_i to u . Thus, by Lemma 3.1, $\lambda_1(G_T) < \lambda_1(G_{T'})$. By Theorem 3.3, $\lambda_1(G_{T'}) \leq \lambda_1(G_S)$, which implies $\lambda_1(G_T) < \lambda_1(G_S)$, a contradiction. Hence, $T = S$. \square

4. Applications. In this section, we discuss some applications of our results obtained in Section 2 and Section 3. We apply these results to characterize the nonsingular weighted directed graph minimizing the first eigenvalue in certain classes of such graphs. We also prove that the first eigenvalue of the minimizing graph is simple.

The next lemma is an immediate application of interlacing theorem (see [9]).

LEMMA 4.1. *Let G be a weighted directed graph on n vertices with an edge e . Then $\lambda_1(G - e) \leq \lambda_1(G) \leq \lambda_2(G - e) \leq \lambda_2(G) \leq \dots \leq \lambda_n(G - e) \leq \lambda_n(G)$.*

Let G be a nonsingular connected weighted directed graph. By $\mathcal{G}(G, n)$ we denote the class of all connected weighted directed graph on n vertices which contain G as an induced subgraph. A weighted directed graph is said to be *minimizing graph* in a certain class of nonsingular weighted directed graphs if its first eigenvalue attains the minimum among all graphs in that class.

The following lemma is crucial for proving our main results of this section.

LEMMA 4.2. *Let G be a nonsingular connected weighted directed graph on vertices $1, \dots, k$. Suppose that H is obtained from G by attaching a singular connected weighted directed graph G_i at the vertex i of G by identifying a vertex u_i of G_i with i , for each $i = 1, \dots, k$. Then H is D -similar to ${}^D H$ such that all the edges of G_1, \dots, G_k have weight 1 in ${}^D H$.*

Proof. Let F be the set of edges in G such that $G - F$ does not contain a nonsingular cycle. Thus, the graph $H' = H - F$ is singular. By Lemma 1.3, H' is D -similar to the unweighted undirected graph ${}^D H' = {}^D H - {}^D F$, where ${}^D F$ is the set of edges in ${}^D H$ corresponding to F . Thus, the weights of all edges in G_i are 1 in ${}^D H$, for each $i = 1, \dots, k$. \square

The next theorem is one of our main results of this section, which characterizes the weighted directed graph minimizing the first eigenvalue in $\mathcal{G}(G, n)$. The proof of the following theorem is inspired by the proof of [11, Theorem 3.2].

THEOREM 4.3. *Let G be a nonsingular connected weighted directed graph on k vertices. If G_0 is a minimizing graph in $\mathcal{G}(G, n)$, then $G_0 = G(u) \diamond P_{n-k+1}(v)$, where v is a pendent vertex of the path P_{n-k+1} on $n - k + 1$ vertices.*

Proof. Assume that G has the vertices $1, 2, \dots, k$. Let H be a spanning subgraph of G_0 such that H is obtained from G by attaching a weighted directed tree T_i at the vertex i of G by identifying the vertex u_i of T_i with i , for each $i = 1, \dots, k$. By Lemma 4.1, $\lambda_1(H) \leq \lambda_1(G_0)$. So we have $\lambda_1(G_0) = \lambda_1(H)$, as G_0 is a minimizing graph in $\mathcal{G}(G, n)$. By Lemma 1.4, each T_i is singular for $i = 1, \dots, k$. Thus, H is D -similar to ${}^D H$ such that all the edges of T_i have weight 1 in ${}^D H$, by Lemma 4.2. Note that $\lambda_1({}^D H) = \lambda_1(H) = \lambda_1(G_0)$.

We use \hat{T}_i to denote the unweighted undirected tree in ${}^D H$ corresponding to T_i for each $i = 1, \dots, k$. Let x be a first eigenvector of ${}^D H$. If $x(i) = 0$ for all $i, 1 \leq i \leq k$, then by Lemma 2.2 (iii), each \hat{T}_i is a zero branch of ${}^D H$ for $i = 1, \dots, k$, which implies $x = 0$, a contradiction. Thus, $x(i_0) \neq 0$ for some $i_0, 1 \leq i_0 \leq k$. Now we claim that \hat{T}_j is a trivial tree for each j with $x(j) = 0$, where $1 \leq j \leq k$. Suppose that $x(j) = 0$ and \hat{T}_j is a nontrivial tree. Let ${}^D H'$ be the weighted directed graph obtained from ${}^D H$ by relocating \hat{T}_j from j to i_0 . Since $|x(i_0)| > 0 = |x(j)|$, we see that $\lambda_1({}^D H') < \lambda_1({}^D H)$, by Lemma 3.1. Then $\lambda_1(H') < \lambda_1(H) = \lambda_1(G_0)$, which contradicts that G_0 is a minimizing graph in $\mathcal{G}(G, n)$, as $H' \in \mathcal{G}(G, n)$. Hence, the claim holds.

We assert that \hat{T}_i must be a path with u_i as one of its end vertices, for each i with $x(i) \neq 0$. Otherwise, replacing such a tree \hat{T}_i on n_i vertices by a path \hat{P}_{n_i} on n_i vertices, we obtain a graph ${}^D H''$ with $\lambda_1({}^D H'') < \lambda_1({}^D H)$, using Theorem 3.5. Thus, $\lambda_1(H'') < \lambda_1(G_0)$, a contradiction, as $H'' \in \mathcal{G}(G, n)$.

Next, assume that there are two distinct paths \hat{P}_{n_i} and \hat{P}_{n_j} attached at i and j with $x(i) \neq 0, x(j) \neq 0$, respectively. Without loss of generality, suppose that $|x(i)| \geq |x(j)| > 0$. Let ${}^D H_0$ be the graph obtained from ${}^D H$ by relocating the path \hat{P}_{n_j} from j to i . By Corollary 3.2, $\lambda_1({}^D H_0) < \lambda_1({}^D H) = \lambda_1(G_0)$. So, $\lambda_1(H_0) < \lambda_1(G_0)$, a contradiction, as $H_0 \in \mathcal{G}(G, n)$. Thus, ${}^D H = \hat{G}(u) \diamond \hat{P}_{n-k+1}$, where \hat{G} is the induced subgraph of ${}^D H$ corresponding to G and $\hat{P}_{n-k+1} = v, v_1, \dots, v_{n-k}$ is an unweighted undirected path. Hence, $H = G(u) \diamond P_{n-k+1}(v)$, where P_{n-k+1} is the weighted directed path in H corresponding to the unweighted undirected path \hat{P}_{n-k+1} in ${}^D H$.

Finally, we prove that $G_0 = H$. Consider the graphs ${}^D G_0$ and ${}^D H$. Note that ${}^D H$ is a spanning subgraph of ${}^D G_0$. Suppose that ${}^D H$ is a proper subgraph of ${}^D G_0$. Let

y be a first eigenvector of ${}^D G_0$ with $y^* y = 1$. Then we have

$$\begin{aligned} \lambda_1({}^D G_0) &= y^* L({}^D G_0) y = y^* L({}^D H) y + \sum_{(i,j) \in E({}^D G_0) \setminus E({}^D H)} |y(i) - w_{ij} y(j)|^2 \\ &\geq y^* L({}^D H) y \geq \lambda_1({}^D H). \end{aligned}$$

Since $\lambda_1({}^D H) = \lambda_1({}^D G_0)$, it follows that y is also a first eigenvector of ${}^D H$ and $y(i) = w_{ij} y(j)$, for each edge $(i, j) \in E({}^D G_0) \setminus E({}^D H)$. Thus, $|y(i)| = |y(j)|$, for each edge $(i, j) \in E({}^D G_0) \setminus E({}^D H)$. Note that $y(w) \neq 0$ for some vertex w of \hat{G} , as otherwise, by Theorem 2.5, it is easy to see that $y = 0$, which is not possible. If $y(u) = 0$, then the first eigenvalue of the graph ${}^D H_1$ obtained from ${}^D H$ by relocating \hat{P}_{n-k+1} from u to some other vertex w of \hat{G} with $y(w) \neq 0$ is smaller than $\lambda_1({}^D H)$, by Corollary 3.2. So, $\lambda_1(H_1) = \lambda_1({}^D H_1) < \lambda_1({}^D H) = \lambda_1(H) = \lambda_1(G_0)$, a contradiction, as $H_1 \in \mathcal{G}(G, n)$. Thus, $y(u) \neq 0$. So, by Theorem 2.6, $|y(u)| < |y(v_1)| < \dots < |y(v_{n-k})|$. Since any edge $e \in E({}^D G_0) \setminus E({}^D H)$ has its end vertices in $\{u, v_1, \dots, v_{n-k}\}$, it follows that $|y(i)| \neq |y(j)|$ for any edge $(i, j) \in E({}^D G_0) \setminus E({}^D H)$, a contradiction. Hence, ${}^D G_0 = {}^D H$, which implies that $H = G_0$. \square

The next result says that the multiplicity of the first eigenvalue of the minimizing graph in $\mathcal{G}(G, n)$ is one.

THEOREM 4.4. *Let G be a nonsingular connected weighted directed graph and let G_0 be the minimizing graph in $\mathcal{G}(G, n)$. Then the first eigenvalue of G_0 is simple.*

Proof. By Theorem 4.3, G_0 is the weighted directed graph obtained from G by attaching a weighted directed path P at a vertex u of G by identifying u with a pendent vertex v of P . Assume that there are two linearly independent first eigenvectors x and y of G_0 . In view of the proof of Theorem 4.3, we see that the valuation of a first eigenvector of G_0 at the vertex u is nonzero. Thus, $x(u) \neq 0, y(u) \neq 0$. Consider the vector $z = \alpha x + \beta y$, where $\alpha = y(u), \beta = -x(u)$. Then z is a first eigenvector of G_0 with $z(u) = 0$, which is a contradiction. Hence, the result holds. \square

Next we consider the class of nonsingular weighted directed graph with **exactly** one nonsingular cycle and obtain the minimizing graph in that class.

The following lemmas are crucial in describing the structure of the weighted directed graphs containing exactly one nonsingular cycle. The proofs of these lemmas are essentially similar to those contained in [5], which describes the structure of the 3-colored digraphs (weighted directed graphs with edges having weights $\pm 1, i$) containing exactly one nonsingular cycle, however, we supply them here for completeness.

LEMMA 4.5. *Let G be a connected weighted directed graph with exactly one nonsingular cycle C . Then G is D -similar to ${}^D G$ with all edges of weight 1 except one edge on the cycle C .*

Proof. Let e be an edge on the cycle C in G . Take $G' = G - e$. Since C is the only nonsingular cycle in G , we see that G' does not contain a cycle of weight other than 1. Thus, G' is singular, by Lemma 1.4. By Lemma 1.3, each edge in ${}^D G'$ has weight 1, for some D . Consider ${}^D G$ for this D . Note that all the edges in ${}^D G$ except the edge corresponding to e have weight 1. Hence, the result holds. \square

LEMMA 4.6. *Let G be a weighted directed graph containing exactly one nonsingular cycle $C = [1, \dots, m, 1]$. Then the subgraph induced by C is C itself.*

Proof. Suppose that C has a chord joining the vertices i and j with $1 \leq i < j \leq m$. Take the cycles $C_1 = [1, \dots, i, j, \dots, m, 1]$ and $C_2 = [i, i + 1, \dots, j, i]$. Note that $w_{C_1} w_{C_2} = w_C \neq 1$, which implies one of C_1 and C_2 has weight other than 1. Hence, G contains at least two nonsingular cycles, which is a contradiction. \square

LEMMA 4.7. *Let G be a connected weighted directed graph with exactly one nonsingular cycle C . Let u be a vertex of G not on C . Then there is a vertex v on the cycle C such that $G - v$ is disconnected with at least two components, one containing u and another containing the remaining vertices of C .*

Proof. In view of Lemma 4.5, we assume that all the edges of G have weight 1 except an edge e on the cycle C . Since G is connected, let v be a vertex in the cycle for which the distance $d(v, u)$ is minimum. Let P_{uv} be a shortest u - v -path in G . Then the vertex v is on every u - w -path, for each vertex w in C . If not, suppose G contains a u - w -path, say P_{uw} which does not contain v , for some vertex w in C . Let $P_{vw}(e)$ be the v - w -path on the cycle C containing the edge e . Take the cycle $C' = P_{uw} + P_{vw}(e) + P_{uv}$. Note that $w_{C'} \neq 1$. So the cycle C' is nonsingular, which is a contradiction. Hence, $G - v$ is disconnected with at least two components, one containing u and another containing the remaining vertices of C . \square

The next lemma characterizes the structure of connected weighted directed graphs containing exactly one nonsingular cycle.

LEMMA 4.8. *Let G be a connected weighted directed graph. Then the following statements are equivalent.*

- (a) G has exactly one nonsingular cycle $C = [1, \dots, m, 1]$.
- (b) G is obtained from a nonsingular cycle $C = [1, \dots, m, 1]$ by appending a connected unweighted undirected graph G_i to the vertex i of C while identifying a vertex of G_i with i , for each $i = 1, \dots, m$.

Proof. First we assume that G has exactly one nonsingular cycle $C = [1, \dots, m, 1]$. By Lemma 4.6, the subgraph induced by C is C itself. In view of Lemma 4.7, for each $i = 1, \dots, m$, let $G_i = G - V(H_i)$, where H_i is the component of $G - i$, which contains the remaining vertices of C . As G_i does not contain a nonsingular cycle, G_i

is singular, for each $i = 1, \dots, m$. Hence, (a) \Rightarrow (b) holds. (b) \Rightarrow (a) is trivial. \square

As an application, we obtain the following result, which characterizes the weighted directed graph minimizing the first eigenvalue among all nonsingular weighted directed graphs with exactly one nonsingular cycle C of fixed length.

THEOREM 4.9. *Among all nonsingular weighted directed graph on n vertices with exactly one nonsingular cycle C of length k , the first eigenvalue is minimized by the nonsingular unicyclic weighted directed graph obtained from the path P_{n-k+1} by attaching the cycle C to one of the pendent vertex of P_{n-k+1} .*

Proof. By Lemma 4.8, we see that the class of all nonsingular weighted directed graphs on n vertices with exactly one nonsingular cycle C of length k is a subclass of $\mathcal{G}(C, n)$. Hence, proof of the theorem follows immediately from Theorem 4.3. \square

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