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SIGN PATTERNS THAT REQUIRE EVENTUAL EXPONENTIAL NONNEGATIVITY[*]

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Abstract. Sign patterns that require exponential nonnegativity are characterized. A set of conditions necessary for a sign pattern to require eventual exponential nonnegativity are established. It is shown that these conditions are also sufficient for an upper triangular sign pattern to require eventual exponential nonnegativity and it is conjectured that these conditions are both necessary and sufficient for any sign pattern to require eventual exponential nonnegativity. It is also shown that the maximum number of negative entries in a sign pattern that requires eventual exponential nonnegativity is $\frac{(n-1)(n-2)}{2} + 2$.

Key words. Matrix exponential, Exponential nonnegativity, Eventual exponential nonnegativity, Requires eventual nonnegativity, Sign pattern.

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1. Introduction. A real square matrix A is eventually exponentially nonnegative (positive) if there exists some $\tau_0 \geq 0$ such that $e^{\tau A}$ is an entrywise nonnegative (positive) matrix for all $\tau > \tau_0$. If $e^{\tau A}$ is entrywise nonnegative (positive) for all $\tau > 0$, then A is called exponentially nonnegative (positive). Ellison, Hogben, and Tsatsomeros [4] showed that a sign pattern $\mathcal{A}$ requires exponential positivity if and only if $\mathcal{A}$ requires eventual exponential positivity and characterized such sign patterns.

In Section 2, we establish some eigenstructure of eventually exponentially nonnegative matrices, which is analogous to the eigenstructure of eventually exponentially positive matrices established in [1]. In Section 3, we develop a set of necessary conditions for a sign pattern to require eventual exponential nonnegativity and conjecture that these conditions are also sufficient for a sign pattern to require eventual exponential nonnegativity. We utilize the Hermite interpolation method for evaluating $e^{\tau A}$ in confirming this conjecture in the case that the sign pattern is permutationally similar to an upper triangular sign pattern. The remainder of the current section contains definitions, notation, and results cited throughout this paper.

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1.1. Definitions and notation. A *sign pattern* $\mathcal{A}$ is a matrix with entries in $\{+,-,0\}$. The class of all real matrices for which $\mathrm{sgn}(A) = \mathcal{A}$ is called the *qualitative class* of $\mathcal{A}$, denoted $\mathcal{Q}(\mathcal{A})$, and a *realization* of $\mathcal{A}$ is a real matrix $A \in \mathcal{Q}(\mathcal{A})$. Sign pattern $\mathcal{A}$ *allows* property $\mathcal{P}$ if there exists a realization $A \in \mathcal{Q}(\mathcal{A})$ which has property $\mathcal{P}$. The sign pattern $\mathcal{A}$ *requires* property $\mathcal{P}$ if every realization $A \in \mathcal{Q}(\mathcal{A})$ has property $\mathcal{P}$.

Let $A = [a_{ij}]$ be an $n \times n$ matrix, we denote the (i,j)-entry of A^ℓ by $a_{ij}^{(\ell)}$ and use similar notation for the (i,j)-entry of the power of a sign pattern.

DEFINITION 1.1. An $n \times n$ matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ is

- *eventually nonnegative (positive)* if there exists $k_0 \in \mathbb{Z}^+$ such that for all $k \geq k_0$, $A^k \geq 0$ $(A^k > 0)$, where the inequality is entrywise;

- *exponentially nonnegative (positive)* if for all $\tau > 0$, $e^{\tau A} = \sum_{k=0}^{\infty} \dfrac{\tau^k A^k}{k!} \geq 0$ $(e^{\tau A} > 0)$;

- *eventually exponentially nonnegative (positive)* if there exists $\tau_0 \geq 0$ such that for all $\tau > \tau_0$, $e^{\tau A} = \sum_{k=0}^{\infty} \dfrac{\tau^k A^k}{k!} \geq 0$ $(e^{\tau A} > 0)$;

- *essentially nonnegative* if $a_{ij} \geq 0$ for all $i \neq j$.

Another (equivalent) definition of an eventually exponentially nonnegative matrix is: for all $i,j \in \{1,2,\ldots,n\}$, if $(e^{\tau A})_{ij} \neq 0$ then $\lim_{\tau \to \infty}(e^{\tau A})_{ij} > 0$. The *dominant term* (or *dominating term*) of $(e^{\tau A})_{ij}$ is the term which determines $\lim_{\tau \to \infty}(e^{\tau A})_{ij}$. So if A is eventually exponentially nonnegative, either $(e^{\tau A})_{ij} = 0$ or the dominating term for the (i,j)-entry is positive for all $i,j \in \{1,\ldots,n\}$.

DEFINITION 1.2. Given an $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$, we denote by $\mathcal{A}_{D(+)} = [\widehat{\alpha}_{ij}]$ the $n \times n$ sign pattern such that $\widehat{\alpha}_{ij} = \alpha_{ij}$ for $i \neq j$ and $\widehat{\alpha}_{ii} = +$ for $i,j \in \{1,\ldots,n\}$. $\mathcal{A}_{D(0)}$ and $\mathcal{A}_{D(-)}$ are defined analogously, with zero and negative diagonal, respectively.

A square matrix (or sign pattern) A is called *reducible* if there exists some permutation matrix P such that PAP^T is upper block triangular with square diagonal blocks. If no such permutation exists, then A is *irreducible*.

We say that the digraph $G = (V,E)$ is *associated with* the matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$, or that A is associated with G, when $a_{ij} \neq 0$ if and only if $(i,j) \in E$. Therefore, A has a nonzero diagonal element a_{ii} if and only if G contains the loop (i,i). We denote the weighted digraph associated with A by $\Gamma(A)$, that is, (i,j) is an arc in $\Gamma(A)$ with weight a_{ij} if and only if $a_{ij} \neq 0$. Likewise, $\Gamma(\mathcal{A})$ is the signed digraph associated with $\mathcal{A}$.

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Let $G = (V, E)$ be a digraph and $u, v \in V$. A vertex $u \in V$ *has access to* vertex $v \in V$ if there exists a u-v walk in G or $u = v$. For $v \in V$ we define $\text{In}(v)$ to be the set of vertices which have access to v and define $\text{Out}(v)$ to be the set of vertices to which v has access. The *product of a u-v walk* in $\Gamma(A)$ is the product of the weights of the arcs in the walk. The *sign of a u-v walk* in $\Gamma(A)$ (or $\Gamma(\mathcal{A})$), is the sign of the product of the u-v walk. An *arc-positive walk* (*path*) is a walk (path) that uses only positive arcs. An arc-positive walk W may pass through a vertex that has a negative loop, so long as the negative loop is not included in W.

The *spectral abscissa* of matrix A is defined as $\alpha(A) := \max\{\text{Re}(\lambda) : \lambda \in \text{spec}(A)\}$. Eigenvalue $\gamma \in \text{spec}(A)$ is a *rightmost eigenvalue of A* if $\text{Re}(\gamma) = \alpha(A)$. In [1] an eigenvalue is called a rightmost eigenvalue if it is real and equal to the spectral abscissa, we allow for a rightmost eigenvalue to be complex (in which case it would not be the unique rightmost eigenvalue). Note that if $\alpha(A)$ is an eigenvalue of A, then it is a rightmost eigenvalue of A. Furthermore, if the spectral radius $\rho(A)$ is an eigenvalue of A, then $\rho(A) = \alpha(A)$ is a rightmost eigenvalue of A.

It is well known that e^{λ} is an eigenvalue of e^A if and only if λ is an eigenvalue of A. Suppose that A is eventually exponentially nonnegative. Then $(e^A)^k = e^{kA} \geq 0$ for large enough integers k, therefore e^A is eventually nonnegative and either e^A is nilpotent (which is not possible) or both e^A and $(e^A)^T = e^{A^T}$ have the Perron-Frobenius property (i.e., $\rho(e^A) \in \text{spec}(e^A)$, and $\rho(e^A)$ has corresponding nonnegative left and right eigenvectors).

1.2. Results cited.

THEOREM 1.3. [5, p. 323] *Let $A, B \in \mathbb{R}^{n \times n}$. If λ is a simple eigenvalue of A and $A(\varepsilon) = A + \varepsilon B$, then in a neighborhood of the origin there exist differentiable (and hence continuous) functions $\lambda(\varepsilon)$ and $\mathbf{x}(\varepsilon)$, with $\lambda(0) = \lambda$ and $\mathbf{x}(0) = \mathbf{x}$, such that $A(\varepsilon)\mathbf{x}(\varepsilon) = \lambda(\varepsilon)\mathbf{x}(\varepsilon)$.*

LEMMA 1.4. [7, Lemma 2.2] *Let $A \in \mathbb{R}^{n \times n}$. The following are equivalent:*

(i) *A is eventually exponentially nonnegative.*
(ii) *There exists $a \in \mathbb{R}$ such that $A + aI$ is eventually exponentially nonnegative.*
(iii) *For all $a \in \mathbb{R}$, $A + aI$ is eventually exponentially nonnegative.*

THEOREM 1.5. [4, Theorem 2.9] *Let $\mathcal{A}$ be a square sign pattern. Then the following are equivalent.*

(i) *$\mathcal{A}$ requires eventual exponential positivity.*
(ii) *$\mathcal{A}$ is irreducible and its off-diagonal entries are nonnegative.*
(iii) *$\mathcal{A}$ requires exponential positivity.*

Note that for $\beta \geq 0$, since A and βI commute,

$$e^{\tau A} = e^{-\tau \beta I} e^{\tau(A+\beta I)} = e^{-\tau \beta} e^{\tau(A+\beta I)},$$

so A may have negative diagonal entries and be (eventually) exponentially nonnegative. In fact, the sign pattern $\begin{bmatrix} - & + \\ + & - \end{bmatrix}$ requires (eventual) exponential positivity by Theorem 1.5.

THEOREM 1.6. [4, Theorem 2.6] *The sign pattern $\mathcal{A} = [\alpha_{ij}]$ requires eventual nonnegativity if and only if for every s,t such that $\alpha_{st} = -$, $\mathcal{A}[\text{In}(s)]$ and $\mathcal{A}[\text{Out}(t)]$ require nilpotence.*

Theorem 1.6 can be rephrased in graph theory language as follows:

THEOREM 1.7. *The sign pattern $\mathcal{A} = [\alpha_{ij}]$ requires eventual nonnegativity if and only if for every s,t such that $\alpha_{st} = -$, every directed walk in $\Gamma(\mathcal{A})$ that contains the arc (s,t) is a path.*

2. Eventually exponentially nonnegative matrices. This section introduces some results on eventually exponentially nonnegative matrices that are interesting in themselves. We will use them primarily as tools in proving sign pattern results in Section 3.

OBSERVATION 2.1. Let A be a block triangular matrix with square diagonal blocks $A_1, A_2, \ldots, A_m$. Then $e^{\tau A}$ is block triangular and the diagonal blocks of $e^{\tau A}$ are $e^{\tau A_1}, e^{\tau A_2}, \ldots, e^{\tau A_m}$.

The following lemma appears as an aside in [7] and is a result of the application of [6, Theorem 1.36] to the matrix exponential function.

LEMMA 2.2. *Let $A \in \mathbb{R}^{n \times n}$. Then $\mathbf{x} \in \mathbb{R}^n$ is an eigenvector of e^A if and only if $\mathbf{x}$ is an eigenvector of A.*

In [1], it was shown that matrix A is eventually exponentially positive if and only if the spectral abscissa of A is a simple (real) eigenvalue of A with corresponding positive left and right eigenvectors. Eventually exponentially nonnegative matrices also have a special eigenstructure.

THEOREM 2.3. *Let $A \in \mathbb{R}^{n \times n}$ be eventually exponentially nonnegative. Then the spectral abscissa of A is an eigenvalue with corresponding nonnegative left and right eigenvectors. Equivalently, A has a real rightmost eigenvalue with corresponding nonnegative left and right eigenvectors.*

Proof. Since A is eventually exponentially nonnegative, there exists $\tau_0 \geq 0$ such that for all $\tau > \tau_0$, $e^{\tau A} \geq 0$. Let $k_0 = \lceil \tau_0 \rceil$, then for all $k \in \mathbb{Z}^+$, $k \geq k_0$, $\left(e^A\right)^k = e^{kA} \geq$

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0 and hence e^A is eventually nonnegative. Therefore, $\rho\left(e^A\right)$ is a (nonzero) eigenvalue of e^A with corresponding nonnegative left and right eigenvectors. By Lemma 2.2, there exists $\mu \in \mathrm{spec}(A)$ such that $e^\mu = \rho\left(e^A\right)$ and μ has corresponding nonnegative left and right eigenvectors. Moreover, since

$$\left|e^{a+ib}\right| = e^a\left|\cos(b) + i\sin(b)\right| = e^a\left(\cos^2(b) + \sin^2(b)\right) = e^a,$$

$\rho(e^A) = e^{\mathrm{Re}(\mu)} \geq e^{\mathrm{Re}(\lambda)}$ for all $\lambda \in \mathrm{spec}(A)$, and since $e^z : \mathbb{R} \to \mathbb{R}$ is nonnegative and strictly monotonically increasing, this implies that $\mathrm{Re}(\mu) \geq \mathrm{Re}(\lambda)$ for all $\lambda \in \mathrm{spec}(A)$. Therefore, the spectral abscissa of A is $\mathrm{Re}(\mu)$, and μ has nonnegative left and right eigenvectors.

Moreover, since A is real and μ has corresponding nonnegative (and therefore real) left and right eigenvectors, μ must be real. ☐

However, as the following example shows, a matrix can have a simple real rightmost eigenvalue with corresponding nonnegative left and right eigenvectors without being eventually exponentially nonnegative.

EXAMPLE 2.4. Let $A = \begin{bmatrix} 0 & 4 & 1 & 0 & 0 \\ 9 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$. Then $\mathrm{spec}(A) = \{-6, -1, 0, 1, 6\}$ and $\begin{bmatrix} 2 & 3 & 0 & 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 6 & 4 & 1 & 1 & 0 \end{bmatrix}^T$ are right and left eigenvectors, respectively, corresponding to the eigenvalue 6. However, the $(1,5)$-entry of $e^{\tau A}$ is

$$-\frac{1}{35}e^{-6\tau} + \frac{e^{-\tau}}{10} - \frac{e^\tau}{14},$$

which is negative for $\tau > 0$.

3. Sign patterns that require exponential nonnegativity or eventual exponential nonnegativity. In [4] Ellison, Hogben, and Tsatsomeros showed that a sign pattern requires eventual exponential positivity if and only if it requires exponential positivity. This is not the case for nonnegativity. If a matrix A is exponentially nonnegative, then A is also eventually exponentially nonnegative; therefore the class of sign patterns that require exponential nonnegativity is contained in the class of sign patterns that require eventual exponential nonnegativity. However, as Example 3.3 below shows, a sign pattern can require eventual exponential nonnegativity without requiring exponential nonnegativity, so the two classes of sign patterns are not equivalent. It is well known that matrix A is exponentially nonnegative if and only if A has no negative off-diagonal entries (i.e., is essentially nonnegative, see, e.g., [2, Chapter 6, Theorem 3.12], [7]). This immediately leads to a classification of those

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sign patterns that require exponential nonnegativity.

PROPOSITION 3.1. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern. $\mathcal{A}$ requires exponential nonnegativity if and only if $\alpha_{ij} \neq -$ for $i \neq j$.*

So if $\mathcal{A}$ is irreducible and requires exponential nonnegativity, then $\mathcal{A}$ requires (eventual) exponential positivity by Theorem 1.5.

It is clear from the power series definition of $e^{\tau A}$ that for a sign pattern to require eventual exponential nonnegativity, the following necessary condition must hold.

OBSERVATION 3.2. *Let $\mathcal{A}$ be a sign pattern that requires eventual exponential nonnegativity. If $\Gamma(\mathcal{A})$ has a negative i-j walk of length k, then there must exist a positive i-j walk of length greater than k.*

While it is true that if $\mathcal{A}$ requires eventual exponential nonnegativity, then $\mathcal{A}_{D(+)}$ allows eventual exponential nonnegativity; as the following example shows, it is not necessarily the case that $\mathcal{A}_{D(+)}$ requires eventual exponential nonnegativity.

EXAMPLE 3.3. Consider the matrix

$$A = \begin{bmatrix} 0 & a_{12} & -a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix},$$

where $a_{12}, a_{13}, a_{23} > 0$. Clearly

$$e^{\tau A} = \begin{bmatrix} 1 & \tau a_{12} & -\tau a_{13} + \tau^2 a_{12} a_{23} \\ 0 & 1 & \tau a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

and for $\tau > \dfrac{a_{13}}{a_{12} a_{23}}$, we have $e^{\tau A} \geq 0$, therefore $\mathcal{A} = \mathrm{sgn}(A)$ requires eventual exponential nonnegativity. However, the matrix

$$A = \begin{bmatrix} 1 & 1 & -10 \\ 0 & \frac{1}{10} & 1 \\ 0 & 0 & \frac{1}{10} \end{bmatrix},$$

which is in $\mathcal{Q}(\mathcal{A}_{D(+)})$, is not eventually exponentially nonnegative. The $(1,3)$-entry of $e^{\tau A}$ is

$$\frac{10 e^{\tau/10} \left(80 - 80 e^{9\tau/10} - 9\tau \right)}{81}$$

which is negative for $\tau > 0$. Therefore, $\mathcal{A}_{D(+)}$ does not require eventual exponential nonnegativity.

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In [4] it was shown that a sign pattern requires exponential positivity if and only if it requires eventual exponential positivity. The preceding example also illustrates that a sign pattern may require eventual exponential nonnegativity without requiring exponential nonnegativity.

The following proposition gives a condition which is sufficient for the sign pattern $\mathcal{A}$ to require eventual exponential nonnegativity.

PROPOSITION 3.4. *Let $\mathcal{A}$ be an $n \times n$ sign pattern such that*

1. $\mathcal{A}$ *requires eventual nonnegativity, and*
2. *if there is a negative (directed) s-t walk of length k in $\Gamma(\mathcal{A})$, then there exists an $\ell > k$ such that every s-t walk of length ℓ is positive.*

Then $\mathcal{A}$ requires eventual exponential nonnegativity.

Proof. Let $A = [a_{ij}] \in \mathcal{Q}(\mathcal{A})$. Then, denoting the entries of A^m by $a_{ij}^{(m)}$, we have

$$
\left(e^{\tau A}\right)_{ij} = \begin{cases} 1 + \tau a_{ii} + \dfrac{\tau^2}{2} a_{ii}^{(2)} + \dfrac{\tau^3}{3!} a_{ii}^{(3)} + \cdots & \text{if } j = i, \\[2mm] \tau a_{ij} + \dfrac{\tau^2}{2} a_{ij}^{(2)} + \dfrac{\tau^3}{3!} a_{ij}^{(3)} + \cdots & \text{if } j \neq i. \end{cases}
$$

Suppose that $a_{st}^{(m)} < 0$ for some $m \in \mathbb{Z}^+$ and let m_0 be the greatest such integer (which exists since $\mathcal{A}$ requires eventual nonnegativity). Then by hypothesis, there exists some $\ell > m_0$ such that $a_{st}^{(\ell)} > 0$. Denote the degree ℓ Maclaurin polynomial for $\left(e^{\tau A}\right)_{st}$ by $p_\ell(\tau)$. This polynomial has a finite number of roots and a positive leading coefficient, therefore there exists $\tau_0(s,t) \geq 0$ such that $p_\ell(\tau) > 0$ for all $\tau > \tau_0(s,t)$ (namely, $\tau_0(s,t) = 0$ if all the roots of $p_\ell(\tau)$ are nonpositive, otherwise $\tau_0(s,t)$ is the greatest real positive root of $p_\ell(\tau)$). Then $\left(e^{\tau A}\right)_{st} = p_\ell(\tau) + r(\tau)$, where $r(\tau) \geq 0$ for all $\tau > 0$, and hence $\left(e^{\tau A}\right)_{st} > 0$ for $\tau > \tau_0(s,t)$.

If $a_{st}^{(m)} \geq 0$ for all $m \in \mathbb{Z}^+$, define $\tau_0(s,t) = 0$. Then $e^{\tau A} \geq 0$ for $\tau \geq \max\limits_{1 \leq s,t \leq n} \{\tau_0(s,t)\}$ and $\mathcal{A}$ requires eventual exponential nonnegativity. $\square$

For matrix $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ (or an $n \times n$ sign pattern) with associated digraph $\Gamma(A) = (V,E)$ let $\widehat{V}(s,t) := \{v \in V : v \in \mathrm{Out}(s) \cap \mathrm{In}(t)\}$. Then the *embedding* $\widetilde{A[\widehat{V}]} = [\widetilde{a}_{ij}]$ of A is the $n \times n$ matrix defined by $\widetilde{a}_{ij} = a_{ij}$ if $i,j \in \widehat{V} = \widehat{V}(s,t)$ and $\widetilde{a}_{ij} = 0$ otherwise. Note that the (i,j)-entry of $e^{\tau A}$ is only affected by the nonzero entries in $\widetilde{A[\widehat{V}]}$ (where $\widehat{V} = \widehat{V}(i,j)$), that is,

$$
\left(e^{\tau A}\right)_{ij} = \left(e^{\tau \widetilde{A[\widehat{V}]}}\right)_{ij}
$$

for $i,j \in \{1,\ldots,n\}$.

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Another method to calculate the matrix exponential is by use of an interpolating polynomial. We make use of the Hermite interpolation formula from [6, Chapter 1], applied to the matrix exponential function, $f(A) = e^{\tau A}$, which is reproduced below for the reader's convenience. Let $A \in \mathbb{R}^{n \times n}$ have m distinct eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$. The Hermite interpolation conditions are: $p^{(k)}(\lambda_i) = f^{(k)}(\lambda_i)$ for $0 \leq k \leq n_i - 1$ and $1 \leq i \leq m$ and the Hermite interpolating polynomial $p(z)$ is given by

$$(3.1) \qquad p(z) = \sum_{i=1}^{m} \left[\left(\sum_{k=0}^{n_i-1} \frac{1}{k!} \phi_i^{(k)}(\lambda_i)(z - \lambda_i)^k \right) \prod_{j \neq i} (z - \lambda_j)^{n_j} \right],$$

where

$$\phi_i(z) = \frac{e^{\tau z}}{\prod_{j \neq i} (z - \lambda_j)^{n_j}}$$

and n_i is the multiplicity of λ_i as a root of the minimal polynomial of A, i.e., the size of the largest Jordan block associated with λ_i in the Jordan canonical form of A. Then $e^{\tau A} = p(A)$. However, as noted by Higham in [6, Remark 1.5], it is often convenient to use a higher degree interpolating polynomial $q(z)$, for example, replacing n_i in (3.1) with the multiplicity of λ_i as a root of the characteristic polynomial rather than the minimal polynomial. This is allowed since as long as $p(z)$ divides $q(z)$, $q(z)$ also satisfies the Hermite interpolation conditions.

It is clear that the (s,t)-entry of a power of A is affected only by the entries associated with $\widehat{V} = \widehat{V}(s,t)$, so the dominating term of the (s,t)-entry of A is the same as the dominating term of the (s,t)-entry of $\widetilde{A} = \widetilde{A[\widehat{V}]}$. Let $\widehat{\lambda}_1, \widehat{\lambda}_2, \ldots$ be the distinct eigenvalues of the principal submatrix $A[\widehat{V}]$ with $\widehat{n}_1, \widehat{n}_2, \ldots$ the multiplicities of $\widehat{\lambda}_1, \widehat{\lambda}_2, \ldots$ as roots of the minimal polynomial of $A[\widehat{V}]$, respectively. Note that in the Hermite interpolation formula, τ appears only in $\phi_i^{(k)}(\widehat{\lambda}_i)$, for $0 \leq k \leq \widehat{n}_i - 1$ and $1 \leq i \leq m$. Furthermore, for $\tau > 0$, $e^{\tau \mu_1} \geq e^{\tau \mu_2}$ if and only if $\mathrm{Re}(\mu_1) \geq \mathrm{Re}(\mu_2)$. Thus, if the principal submatrix $A[\widehat{V}(s,t)]$ has a unique rightmost eigenvalue, which we denote by $\widehat{\gamma} = \widehat{\lambda}_\nu$, then the dominating term of the (s,t)-entry of $e^{\tau A}$ is precisely the (s,t)-entry of

$$(3.2) \qquad \left(\sum_{k=0}^{\widehat{n}_\nu - 1} \frac{1}{k!} \phi_\nu^{(k)}(\widehat{\gamma})(\widetilde{A} - \widehat{\gamma} I)^k \right) \prod_{j \neq \nu} (\widetilde{A} - \widehat{\lambda}_j I)^{\widehat{n}_j}.$$

Moreover, $\widehat{\gamma}$ is real, and hence $\phi_\nu(\widehat{\gamma}) = \dfrac{e^{\tau \widehat{\gamma}}}{\prod_{j \neq \nu} (\widehat{\gamma} - \widehat{\lambda}_j)^{\widehat{n}_j}}$ is real and positive since $\widehat{\gamma} \geq \mathrm{Re}(\widehat{\lambda}_j)$ for $\widehat{\lambda}_j \in \mathrm{spec}(A[\widehat{V}(s,t)])$ and any complex eigenvalues of $A[\widehat{V}(s,t)]$ come

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in conjugate pairs. Furthermore,

$$\phi'_\nu(\widehat{\gamma}) = \tau \cdot \phi_\nu(\widehat{\gamma}) - \left(\frac{\phi_\nu(\widehat{\gamma})}{\prod_{j\neq\nu}(\widehat{\gamma}-\widehat{\lambda}_j)^{\widehat{n}_j}} \right) \frac{d}{dz} \left[\prod_{j\neq\nu}(z-\widehat{\lambda}_j)^{\widehat{n}_j} \right] \Big|_{z=\widehat{\gamma}} = \phi_\nu(\widehat{\gamma})\,(\tau-K),$$

where K is a constant. Therefore, $\phi'_\nu(\widehat{\gamma}) \to \infty$ as $\tau \to \infty$. It is apparent that $\phi_\nu^{(k)}(\widehat{\gamma})$ is a k-th degree polynomial in τ with leading coefficient $\phi_\nu(\widehat{\gamma}) > 0$ and hence $\phi_\nu^{(k)}(\widehat{\gamma}) \to \infty$ as $\tau \to \infty$. Since $\phi_\nu^{(k)}(\widehat{\gamma})$ is a k-th degree polynomial (in τ) with leading coefficient $\phi_\nu(\widehat{\gamma})$, the dominating term of the (s,t)-entry of (3.2) is the (s,t)-entry of

$$\frac{e^{\tau\widehat{\gamma}}\tau^{\widehat{n}_\nu-1}}{(\widehat{n}_\nu-1)!\,\prod_{j\neq\nu}(\widehat{\gamma}-\widehat{\lambda}_j)^{\widehat{n}_j}}(\widetilde{A}-\widehat{\gamma}I)^{\widehat{n}_\nu-1}\prod_{j\neq\nu}(\widetilde{A}-\widehat{\lambda}_jI)^{\widehat{n}_j}.$$

Combining the fact that an eventually exponentially nonnegative matrix has a (real) rightmost eigenvalue (Theorem 2.3) with the above Hermite interpolation analysis, we have the following observation.

OBSERVATION 3.5. $A \in \mathbb{R}^{n\times n}$ is eventually exponentially nonnegative if and only if for $1 \leq s,t \leq n$ (i) the (s,t)-entry of $e^{\tau A}$ is 0 or (ii) the (s,t)-entry of

$$(\widetilde{A}-\widehat{\gamma}I)^{\widehat{n}_\nu-1}\prod_{j\neq\nu}(\widetilde{A}-\widehat{\lambda}_jI)^{\widehat{n}_j}$$

is positive.

3.1. Necessary conditions. In this section, we discuss the following questions: What properties does every sign pattern that requires eventual exponential nonnegativity have? What type of structure prohibits a sign pattern from requiring eventual exponential nonnegativity? In addressing these questions, we establish conditions that are necessary for a sign pattern to require eventual exponential nonnegativity.

A sign pattern is called *acyclic* if there are no (directed) cycles or loops in $\Gamma(\mathcal{A})$. An acyclic sign pattern requires nilpotence, and therefore requires eventual nonnegativity. Proposition 3.4 leads to the following result.

COROLLARY 3.6. *Let $\mathcal{A}$ be an acyclic sign pattern. $\mathcal{A}$ requires eventual exponential nonnegativity if and only if for any negative u-v path of length k, there exists a positive u-v path of length greater than k.*

Furthermore, an acyclic sign pattern allows eventual exponential nonnegativity if and only if it requires eventual exponential nonnegativity.

LEMMA 3.7. *Let $C = [c_{ij}]$ be an $n \times n$ matrix with entries in $\{-1,0,1\}$ such that $\Gamma(C)$ is a (directed) n-cycle. Then the characteristic polynomial of C is $x^n+(-1)^{m+1}$, where m is the number of -1 entries in C.*

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Proof. Without loss of generality, suppose that $c_{ij} \neq 0$ implies $j \equiv i+1 \pmod{n}$. Then by cofactor expansion along the first column, $\det(C) = (-1)^{n+1}(-1)^m$. Since each $k \times k$ principal minor of C is 0 for $k < n$, the characteristic polynomial is $p_C(x) = x^n + (-1)^n \det(C) = x^n + (-1)^n(-1)^{n+1}(-1)^m = x^n + (-1)^{m+1}$. □

PROPOSITION 3.8. *Let $\mathcal{C}$ be an $n \times n$ sign pattern such that $\Gamma(\mathcal{C})$ is a (directed) n-cycle, with $n \geq 2$. $\mathcal{C}$ requires eventual exponential nonnegativity if and only if $\mathcal{C}$ is nonnegative.*

Proof. If $\mathcal{C}$ is nonnegative, then $\mathcal{C}$ requires (eventual) exponential positivity by Theorem 1.5 and therefore $\mathcal{C}$ requires eventual exponential nonnegativity.

Suppose that $\mathcal{C}$ has at least one negative entry and let $C = [c_{ij}] \in \mathbb{R}^{n \times n}$ be the characteristic matrix of $\mathcal{C}$, that is, $C \in \mathcal{Q}(\mathcal{C})$ and $c_{ij} \in \{-1, 0, 1\}$. If there are an odd number of negative entries in C, then the characteristic polynomial of C is $x^n + 1$ and the eigenvalues of C are the n roots of -1; therefore the spectral abscissa of C is not an eigenvalue and C is not eventually exponentially nonnegative by Theorem 2.3.

If there are an even number of negative entries in C, then the characteristic polynomial of C is $x^n - 1$ and the eigenvalues of C are the n roots of unity. Therefore, 1 is the spectral abscissa of C. Suppose that $C\mathbf{x} = \mathbf{x}$, where $\mathbf{x} \geq 0$. Then

$$c_{12}x_2 = x_1$$
$$c_{23}x_3 = x_2$$
$$\vdots$$
$$c_{n1}x_1 = x_n$$

and since $c_{i,i+1} = -1$ for some $i \in \{1, 2, \ldots, n\}$, this implies $x_i = 0$ for $i = 1, 2, \ldots, n$ and hence C does not have a nonnegative eigenvector corresponding to a real rightmost eigenvalue. Therefore, C is not eventually exponentially nonnegative by Theorem 2.3. □

PROPOSITION 3.9. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern. If $\Gamma(\mathcal{A})$ has a cycle of length at least 2 that contains a negative arc, then $\mathcal{A}$ does not require eventual exponential nonnegativity.*

Proof. Let $\mathcal{A}$ be as prescribed and let $W = (i_1, i_2, \ldots, i_k, i_1)$ denote a cycle of length at least 2 where $\alpha_{st} = -$ for some $s = i_j$, $t = i_{j+1}$ (with $i_{k+1} = i_1$). Consider the matrix C obtained from the characteristic matrix of $\mathcal{A}$ by setting all of the entries not associated with W equal to zero. For $\varepsilon > 0$, consider $A(\varepsilon) = C + \varepsilon B \in \mathcal{Q}(\mathcal{A})$. As in the proof of Proposition 3.8, C does not have nonnegative left and right eigenvectors for a real rightmost eigenvalue. Therefore by Theorem 1.3, for small enough ε, $A(\varepsilon)$ does not have nonnegative left and right eigenvectors for a real rightmost eigenvalue and hence is not eventually exponentially nonnegative by Theorem 2.3. □

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Hence if an $n \times n$ sign pattern $\mathcal{A}$ is irreducible (with $n \geq 2$) one of two things is true: either $\mathcal{A}$ is essentially nonnegative and therefore requires exponential positivity by Theorem 1.5, or $\mathcal{A}$ has an off-diagonal negative entry and therefore does not require eventual exponential nonnegativity by Proposition 3.9.

The proof of the following result closely follows that of [4, Theorem 2.5].

PROPOSITION 3.10. *Let the $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$ require eventual exponential nonnegativity. Then there exists no walk in $\Gamma(\mathcal{A})$ which includes both a negative arc (s, t), $s \neq t$, and either a positive loop or arc-positive cycle.*

Proof. We proceed by way of contradiction. Let $\mathcal{A}$ be as prescribed. Without loss of generality, suppose there exists a walk W

$$(s = 1, t = 2, 3, \ldots, u, \ldots, v, \ldots, \ell = u)$$

in which the cycle $(u, \ldots, v, \ldots, u)$ contains no negative arc (note that it is possible that $u = 2$ and/or $v = \ell = u$). Consider the matrix C obtained from the characteristic matrix of $\mathcal{A}$ by setting all entries to zero except those associated with the walk W. If we let $U = \{1, \ldots, u - 1\}$ and $V = \{u, \ldots, v, \ldots, \ell - 1\}$, then C has the block form

$$C = \begin{bmatrix} C[U] & C[U, V] & 0 \\ 0 & C[V] & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $C[U]$ is nilpotent $\rho(C[V]) = \rho(C)$, denote this by ρ. Since $C[V]$ is nonnegative and irreducible, by the Perron-Frobenius theorem, $\rho > 0$. Moreover, there exists $\mathbf{v} \in \mathbb{R}^n$ such that $C\mathbf{v} = \rho\mathbf{v}$ and $\mathbf{v}[V] > 0$. Working backwards from u, we have $v_i \neq 0$ for $i = 1, \ldots, u - 1$, and $v_k < 0$, where k is the greatest index in $\{1, \ldots, u - 1\}$ such that $c_{k,k+1} = -1$. Note that ρ is a simple eigenvalue of C, and therefore is a (real) rightmost eigenvalue of C. So for sufficiently small $\varepsilon > 0$, $A(\varepsilon) = C + \varepsilon B \in \mathcal{Q}(\mathcal{A})$ does not have a nonnegative (right) eigenvector corresponding to its spectral abscissa and hence is not eventually exponentially nonnegative. This contradicts the assumption that $\mathcal{A}$ requires eventual exponential nonnegativity; therefore W cannot contain a positive loop or arc-positive cycle of length 2 or more. The case of a positive loop or arc-positive cycle coming before the negative arc (s, t) in a walk is similar and involves considering C^T rather than C. ☐

Propositions 3.9 and 3.10 lead to the following result.

COROLLARY 3.11. *Let $\mathcal{A}$ be an $n \times n$ sign pattern that requires eventual exponential nonnegativity. No walk containing a negative non-loop arc also contains either a positive loop or any cycle of length 2 or more. Thus $\mathcal{A}_{D(0)}$ requires eventual nonnegativity.*

PROPOSITION 3.12. *Let the $n \times n$ sign pattern $\mathcal{A} = [\alpha_{ij}]$ require eventual expo-*

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nential nonnegativity. If $\alpha_{st} = -$, $s \neq t$, then there exists an arc-positive s-t path in $\Gamma(\mathcal{A})$.

Proof. Let $\mathcal{A}$ require eventual exponential nonnegativity and suppose that $\alpha_{st} = -$ for some $s, t \in \{1, \ldots, n\}$, $s \neq t$. Let W be the longest s-t walk in $\Gamma(\mathcal{A}_{D(0)})$. If W is an arc-positive walk, then it contains an arc-positive s-t path.

Suppose that W has a negative arc, (w_i, w_j). By Observation 3.2 there exists a positive w_i-w_j walk P_{w_i,w_j} in $\Gamma(\mathcal{A}_{D(0)})$ (which Corollary 3.11 implies is a path). Note that the only vertices that W and P_{w_i,w_j} share are w_i and w_j, otherwise $\Gamma(\mathcal{A}_{D(0)})$ would have a walk containing both a negative non-loop arc and a cycle, contradicting the assumption that $\mathcal{A}$ requires eventual exponential nonnegativity. Replacing the arc (w_i, w_j) in the path W with the path P_{w_i,w_j} creates a longer s-t path, contradicting the maximality of W. Therefore the longest s-t walk does not have any negative non-loop arcs and there exists an arc-positive s-t path. □

LEMMA 3.13. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern. If there exists $s \neq t$ such that $\alpha_{st} = -$, at least one of α_{ss}, α_{tt} is $-$, and each interior vertex in every arc-positive s-t walk in $\Gamma(\mathcal{A})$ has a negative loop, then $\mathcal{A}$ does not require eventual exponential nonnegativity.*

Proof. Let the sign pattern $\mathcal{A}$, with associated digraph $\Gamma(\mathcal{A}) = (V, E)$, be as hypothesized. If for some $s \neq t$, $\alpha_{st} = -$ and $\alpha_{vv} = -$ for all $v \in \widehat{V} = \widehat{V}(s,t)$ (recall that $s, t \in \widehat{V}(s,t)$), then $\mathcal{A}$ does not require eventual exponential nonnegativity since this is equivalent to $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}[\widehat{V}]}$ with $\alpha_{st} = -$, $s \neq t$, and $v \in \widehat{V}$ implies $\alpha_{vv} = +$ requiring eventual exponential nonnegativity, which contradicts Proposition 3.10.

Suppose that (i) for every $\alpha_{st} = -$, and $s \neq t$, there exists an arc-positive s-t path; (ii) no cycle of length 2 or more contains a negative arc; and (iii) no walk containing a negative arc also contains a positive loop or positive cycle of length 2 or more (otherwise $\mathcal{A}$ does not require eventual exponential nonnegativity by one of Propositions 3.12, 3.9, or 3.10). Choose s, t so that $|\widehat{V}|$ is minimized over all $\widehat{V}$ such that $i \neq j$, $\alpha_{ij} = -$, $\alpha_{ii} = 0$, $\alpha_{jj} = -$, and each interior vertex in every arc-positive i-j walk in $\Gamma(\mathcal{A})$ has a negative loop. Then for $i, j \in \widehat{V}$, $i \neq j$, $(i,j) \neq (s,t)$ implies $\alpha_{ij} \geq 0$ (due to the minimality of $|\widehat{V}|$, assumptions (i)-(iii), and the finiteness of $\Gamma(\mathcal{A})$).

Without loss of generality, let $s = 1$, $t = |\widehat{V}|$, and $v \in \widehat{V}$ imply $v \leq t$. Construct $\widetilde{A} = [a_{ij}] \in \mathcal{Q}(\widetilde{\mathcal{A}})$ by setting (1) $a_{ii} = -(t-1)$ for $i = 2, \ldots, t-1$, (2) $a_{tt} = -1$, (3) $a_{1t} = -t$, (4) $a_{it} = 1$ if $\alpha_{it} \neq 0$ for $i = 2, \ldots, t-1$, and (5) choose the remaining nonzero a_{ij} so that column j has column sum zero for $j = 2, \ldots, t-1$ (note that column 1 is a zero column). Then 0 is both a simple eigenvalue and the spectral abscissa of $\widetilde{A}[\widehat{V}]$. Let $\mathbf{x} = [x_i] \in \mathbb{R}^n$ with $x_i = 1$ for $i = 1, \ldots, t-1$, $x_t = -(t-m)$

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where $m = \sum_{i=2}^{t-1} a_{it}$ (so $m < t$), and $x_i = 0$ for $i = t+1, \ldots, n$. Then $\mathbf{x}[\widehat{V}]$ is a left eigenvector of $\widetilde{A}[\widehat{V}]$ corresponding to the simple eigenvalue 0. Since $\mathbf{x}[\widehat{V}]$ has both positive and negative entries, $\widetilde{A}[\widehat{V}]$ does not have a nonnegative left eigenvector corresponding to its spectral abscissa.

Let B be the characteristic matrix of $\mathcal{A}$ and $A(\varepsilon) = \widetilde{A} + \varepsilon B \in \mathcal{Q}(\mathcal{A})$. For small $\varepsilon > 0$, the principal submatrix $A(\varepsilon)[\widehat{V}] = \widetilde{A}[\widehat{V}] + \varepsilon B[\widehat{V}]$ of $A(\varepsilon)$ does not have a nonnegative left eigenvector corresponding to its spectral abscissa ε and hence is not eventually exponentially nonnegative. So the embedding $\widetilde{A(\varepsilon)} = \widetilde{A(\varepsilon)[\widehat{V}]}$ is not eventually exponentially nonnegative and since

$$\left(e^{\tau A(\varepsilon)}\right)_{ij} = \left(e^{\tau \widetilde{A(\varepsilon)}}\right)_{ij},$$

for $i, j \in \widehat{V}$, $A(\varepsilon)$ is not eventually exponentially nonnegative. Therefore $\mathcal{A}$ does not require eventual exponential nonnegativity.

The case for $\alpha_{ss} = -, \alpha_{tt} = 0$ is similar, considering the right eigenvector rather than the left. $\square$

LEMMA 3.14. *Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern. If there exists $s \neq t$ such that $\alpha_{st} = -$, $\alpha_{ss} = \alpha_{tt} = 0$, and each interior vertex in every arc-positive s-t walk in $\Gamma(\mathcal{A})$ has a negative loop, then $\mathcal{A}$ does not require eventual exponential nonnegativity.*

Proof. Let the sign pattern $\mathcal{A}$, with associated digraph $\Gamma(\mathcal{A}) = (V, E)$, be as hypothesized and let $A = [a_{ij}] \in \mathcal{Q}(\mathcal{A})$. Suppose that (i) for every $\alpha_{st} = -$ such that $s \neq t$, there exists an arc-positive s-t path; (ii) every cycle of length 2 or more is arc-positive; and (iii) if a walk contains a negative non-loop arc it does not contain any positive loop or positive cycle of length 2 or more (otherwise $\mathcal{A}$ does not require eventual exponential nonnegativity by one of Propositions 3.12, 3.9, or 3.10).

Suppose that there exists s, t, $s \neq t$, such that $\alpha_{st} = -$, $\alpha_{ss} = \alpha_{tt} = 0$, and $\alpha_{vv} = -$ for $v \in \widehat{V}(s, t) \setminus \{s, t\}$. Let $\widehat{V} = \widehat{V}(s, t)$, $\widetilde{A} = \widetilde{A[\widehat{V}]}$, and $m = |\widehat{V}|$. It follows from assumptions (ii) and (iii) that $\text{In}(s) \cap \widehat{V} = s$ and $\text{Out}(t) \cap \widehat{V} = t$, and hence $\widetilde{A}$ has at most $m - 2$ nonzero eigenvalues, each of which is independent of a_{st}. By choosing $|a_{vv}|$ large and spread out from the other nonzero diagonal elements of $\widetilde{A}$, we can ensure that the nonzero eigenvalues of $\widetilde{A}$ are real, distinct, and negative and that for the principal submatrix $A[\widehat{V}]$ the geometric multiplicity of 0 as an eigenvalue is one. Therefore the rightmost eigenvalue of $\widetilde{A}$ is 0. Let $\widehat{\lambda}_1 = 0, \widehat{\lambda}_2, \ldots, \widehat{\lambda}_{m-1}$ be the distinct eigenvalues of $\widetilde{A}$. Note that there are $n - m + 1$ zero columns in $\widetilde{A}$, so the geometric multiplicity of 0 as an eigenvalue of $\widetilde{A}$ is $n - m + 1$. Hence $\widetilde{A}$ has $n - m + 1 + m - 2 = n - 1$ linearly independent eigenvectors and the size of the largest Jordan block corresponding to eigenvalue 0 in the Jordan canonical form of $\widetilde{A}$ is 2.

Since 0 is the rightmost eigenvalue of $\widetilde{A}$, and it is a double root of the minimal

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polynomial of $\widetilde{A}$, by Observation 3.5 it is clear that $\widetilde{A}$ is eventually exponentially nonnegative only if $\widetilde{A} \prod_{k=2}^{m-1} (\widetilde{A} - \widehat{\lambda}_k I) \geq 0$. Note that

$$\widetilde{A} \prod_{k=2}^{m-1} (\widetilde{A} - \widehat{\lambda}_k I) = \widetilde{A}^{m-1} - \mathrm{tr}(\widetilde{A})\widetilde{A}^{m-2} + \cdots + (-1)^{m-2}(\widehat{\lambda}_2 \cdots \widehat{\lambda}_{m-1})\widetilde{A}$$

$$= \widetilde{A}^{m-1} - \mathrm{tr}(\widetilde{A})\widetilde{A}^{m-2} + \cdots + |\widehat{\lambda}_2 \cdots \widehat{\lambda}_{m-1}|\widetilde{A},$$

since $\widehat{\lambda}_k < 0$ for $k = 2, \ldots, m-1$. By assumption no s-t walk of length 2 or more includes the arc (s, t), so $\widetilde{a}_{st} = a_{st}$ is not in the (s, t)-entry of $\widetilde{A}^k$ for $k = 2, \ldots, m-1$. Recall that $\widehat{\lambda}_2, \ldots, \widehat{\lambda}_{m-1}$ are independent of a_{st}, and that $a_{st} < 0$, therefore we may choose a_{st} so that the (s, t)-entry of $\widetilde{A} \prod_{k=2}^{m-1} (\widetilde{A} - \widehat{\lambda}_k I)$ is negative and hence $\widetilde{A}$ is not eventually exponentially nonnegative.

Therefore A is not eventually exponentially nonnegative and hence $\mathcal{A}$ does not require eventual exponential nonnegativity. $\blacksquare$

The previous results lead to the following necessary conditions for the sign pattern $\mathcal{A}$ to require eventual exponential nonnegativity.

THEOREM 3.15. *If the $n \times n$ sign pattern $\mathcal{A}$ requires eventual exponential nonnegativity, then*

(i) *every cycle in $\Gamma(\mathcal{A})$ of length 2 or more is arc-positive,*
(ii) *if a walk in $\Gamma(\mathcal{A})$ contains a negative non-loop arc then it does not contain any positive loop or (arc-positive) cycle of length 2 or more, and*
(iii) *for every negative arc (s, t) in $\Gamma(\mathcal{A})$, there exists an arc-positive s-t walk with an interior vertex that does not have a negative loop.*

It is interesting to note that for a sign pattern $\mathcal{A} = [\alpha_{ij}]$ that requires eventual exponential nonnegativity with $\alpha_{st} = -$, and $s \neq t$, the arc-positive s-t path with an interior negative-loop-free vertex need not be the longest arc-positive s-t path.

EXAMPLE 3.16. Consider the matrix

$$A = \begin{bmatrix} 0 & a_{12} & 0 & -a_{14} & a_{15} \\ 0 & -a_{22} & a_{23} & 0 & 0 \\ 0 & 0 & -a_{33} & a_{34} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{54} & 0 \end{bmatrix},$$

with $a_{12}, a_{14}, a_{15}, a_{22}, a_{23}, a_{33}, a_{34}, a_{54} > 0$. See Figure 3.1 for the digraph $\Gamma(\mathcal{A})$ for $\mathcal{A} = \mathrm{sgn}(A)$. Note that the principal submatrices $A[\{1, 2, 3\}]$ and $A[\{2, 3, 4, 5\}]$ are each essentially nonnegative and therefore are each eventually exponentially nonnegative. Clearly $(e^{\tau A})_{41} = (e^{\tau A})_{51} = 0$, so the only questionable entries are the $(1, 4)$- and $(1, 5)$-entries.
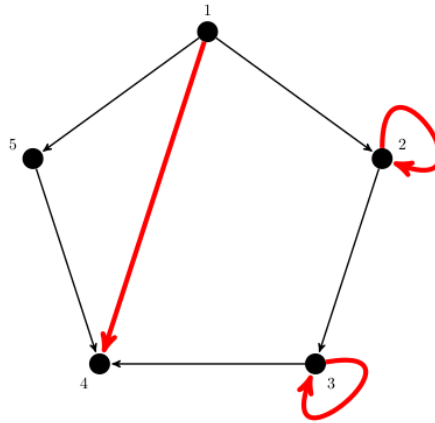
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Fig. 3.1: The digraph $\Gamma(\mathcal{A})$ for sign pattern $\mathcal{A}$ in Example 3.16. The negative arcs are represented by **thick** lines.

Note that $(A^k)_{15} = 0$ for $k \geq 2$ so by the power series for $e^{\tau A}$,

$$(e^{\tau A})_{15} = \tau a_{15},$$

which is positive for $\tau > 0$.

The distinct eigenvalues of A are $0, -a_{22}$, and $-a_{33}$. The minimal polynomial of A is the characteristic polynomial, so the size of the largest Jordan block for 0 (the rightmost eigenvalue) is three. The $(1,4)$-entry of $A^2(A + a_{22}I)(A + a_{33}I)$ is $a_{15}a_{54}a_{22}a_{33}$, which is positive. Hence by the previous analysis of the Hermite interpolating polynomial, the dominating term of $(e^{\tau A})_{14}$ is positive.

Therefore the sign pattern $\mathcal{A} = \text{sgn}(A)$ requires eventual exponential nonnegativity.

The next result is a consequence of Theorem 3.15 (conditions (i) and (iii)). It will be used in Section 3.3 in determining the maximum number of negative entries in a sign pattern that requires eventual exponential nonnegativity.

COROLLARY 3.17. *Let $\mathcal{A}$ be an $n \times n$ sign pattern $(n \geq 2)$ that requires eventual exponential nonnegativity. There exists a permutation pattern P such that for $\mathcal{A}' = P\mathcal{A}P^T = [\alpha'_{ij}]$, $\alpha'_{ij} \geq 0$ if $i > j$ or $j = i + 1$.*

Proof. We construct one such permutation by relabeling each vertex of $\Gamma(\mathcal{A}) = (V, E)$ twice, first we relabel each vertex exactly once with a label in $\mathbb{Z}$, then we shift that labeling so that each vertex is relabeled with a positive integer. (Note

that the goal of this relabeling is so that $\alpha'_{st} = -$ with $s \neq t$ implies α'_{st} is above the first super-diagonal, the location of positive entries is of no concern to us.) Let $E_N \subset E$ be the set of all non-loop negative arcs in $\Gamma(\mathcal{A})$ and for $(i,j) \in E_N$ let $L(i,j)$ denote the length of the longest arc-positive i-j path in $\Gamma(\mathcal{A})$. The rule for the intermediate relabeling of the vertices is given by: Let $V_0 = \emptyset$ and $V_k \subseteq V$ be the set of vertices that have been relabeled after iteration k. Let N_k be the set of integers used in the intermediate relabeling of V_k. Set $k = 1$ and choose $s_1, t_1 \in V$ such that $(s_1, t_1) \in E_N$ and $L(s_1, t_1) = \max\{L(s,t) : (s,t) \in E_N\}$. Choose an arc-positive path of length $L(s_1, t_1)$, denoted $P(s_1, t_1) = (s_1, p_2, \ldots, p_{L(s_1,t_1)}, t_1)$. Relabel the vertices in $P(s_1, t_1)$ as $s_1 \mapsto 0', p_2 \mapsto 1', \ldots, t_1 \mapsto (L(s_1,t_1))'$. Set $k = k + 1$. While there exists $(s,t) \in E_N$ such that $\{s,t\} \nsubseteq V_{k-1}$: Choose $s_k, t_k \in V \setminus V_{k-1}$ such that $(s_k, t_k) \in E_N$ with $L(s_k, t_k) = \max\{L(s,t) : \{s,t\} \nsubseteq V_{k-1}$ and $(s,t) \in E_N\}$ (note that $L(s_k, t_k) \geq 2$). Choose an arc-positive s_k-t_k path of length $L(s_k, t_k)$, denoted $P(s_k, t_k) = (s_k, p_2, \ldots, p_{L(s_k,t_k)}, t_k)$. The rule for relabeling the vertices in $P(s_k, t_k)$ for $k \geq 2$ depends on which (if any) vertices in $P(s_k, t_k)$ have already been relabeled: If $\{s_k, p_2, \ldots, p_{L(s_k,t_k)}, t_k\} \cap V_{k-1} = \emptyset$, then relabel the vertices in $P(s_k, t_k)$ as $s_k \mapsto (\ell_M + 1)', p_2 \mapsto (\ell_M + 2)', \ldots, t_k \mapsto (\ell_M + L(s_k, t_k) + 1)'$ where $\ell_M = \max N_{k-1}$. If $s_k \in V_{k-1}$, relabel the vertices in $\{p_2, \ldots, p_{L(s_k,t_k)}, t_k\} \cap \overline{V}_{k-1}$ sequentially along the path $P(s_k, t_k)$ as $(\ell_M + 1)', (\ell_M + 2)', \ldots, (\ell_M + c)'$, where $\ell_M = \max N_{k-1}$ and $c = \left|\{p_2, \ldots, p_{L(s_k,t_k)}, t_k\} \cap \overline{V}_{k-1}\right|$. If $s_k \notin V_{k-1}$ and $p_i \in V_{k-1}$ for some $i \in \{2, \ldots, L(s_k, t_k)\}$, then let $j = \min\{i : p_i \in V_{k-1}\}$ and relabel the vertices $\{s_k, p_2, \ldots, p_{j-1}\}$ as $s_k \mapsto (\ell_m - j + 1)', \ldots, p_{j-1} \mapsto (\ell_m - 1)'$, where $\ell_m = \min N_{k-1}$, and relabel the vertices in $\{p_{j+1}, \ldots, p_{L(s_k,t_k)}\} \cap \overline{V}_{k-1}$ sequentially along the path $P(s_k, t_k)$ as $(\ell_M + 1)', \ldots, (\ell_M + c)'$, where $\ell_M = \max N_{k-1}$, and $c = \left|\{p_{j+1}, \ldots, p_{L(s_k,t_k)}, t_k\} \cap \overline{V}_{k-1}\right|$. If $\{s_k, p_2, \ldots, p_{L(s_k,t_k)}, t_k\} \cap V_{k-1} = \{t_k\}$, relabel the vertices $\{s_k, p_2, \ldots, p_{L(s_k,t_k)}\}$ as $s_k \mapsto (\ell_m - L(s_k, t_k))', \ldots, p_{L(s_k,t_k)} \mapsto (\ell_m - 1)'$, where $\ell_m = \min N_{k-1}$. After the process has been completed (i.e., there exists no $(s,t) \in E_N$ such that $\{s,t\} \nsubseteq V_{k-1}$), the remaining vertices are mapped into $\{\ell_M + 1, \ldots, \ell_M + |V \setminus V_{k-1}|\}$, where $\ell_M = \max N_{k-1}$. Note that the vertices have now been relabeled into a set of consecutive integers, at least one of which is nonpositive. Finally, we relabel each vertex again by adding $|\ell_m| + 1$ to each intermediate label so that the final label of each vertex comes from the integers $\{1, 2, \ldots, |V|\}$. ∎

3.2. Sufficient conditions for upper triangular sign patterns. It is conjectured that the converse of Theorem 3.15 is also true, that is:

CONJECTURE 3.18. *The $n \times n$ sign pattern $\mathcal{A}$ requires eventual exponential non-negativity if and only if*

(i) *every cycle in $\Gamma(\mathcal{A})$ of length 2 or more is arc-positive,*
(ii) *if a walk in $\Gamma(\mathcal{A})$ contains a negative non-loop arc then it does not contain any positive loop or (arc-positive) cycle of length 2 or more, and*

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(iii) for every negative arc (s,t) in $\Gamma(\mathcal{A})$, there exists an arc-positive s-t walk with an interior vertex that does not have a negative loop.

Note that if a sign pattern has no cycles of length 2 or more, it can be simultaneously permuted into an upper triangular pattern. We will use Hermite interpolation to prove Conjecture 3.18 in the case of upper triangular sign patterns, but first we develop tools to analyze the sign of the dominating term of the (i,j)-entry of $e^{\tau A}$ for every realization $A \in \mathcal{Q}(\mathcal{A})$ when $\mathcal{A}$ is an upper triangular sign pattern that satisfies the hypotheses of Conjecture 3.18.

The technique of using König digraphs to compute the product of several matrices motivates the following terminology (see, e.g., [3] for more on König digraphs and Section 4 for a discussion of how our use of König digraphs differs slightly from that of Brualdi and Cvetković). Let $M^{(1)} = [m_{ij}^{(1)}], M^{(2)} = [m_{ij}^{(2)}], \ldots, M^{(\ell)} = [m_{ij}^{(\ell)}]$ be real $n \times n$ upper triangular matrices. The product

$$(3.3) \qquad m_{i,k_1}^{(1)} m_{k_1,k_2}^{(2)} m_{k_2,k_3}^{(3)} \cdots m_{k_{\ell-1},j}^{(\ell)}$$

is called a *loop-path product* of length ℓ. Note that the loop-path product (3.3) is the weight of the walk $(i, k_1, k_2, \ldots, k_{\ell-1}, j)$ in the composite König digraph used to compute the product $M^{(1)} M^{(2)} \cdots M^{(\ell)}$. Although König digraphs have no loops, we take the convention of calling arc (r_i, c_i) in the König digraph $K(M)$ a loop, since this corresponds to a nonzero diagonal entry m_{ii}. The *underlying path* of a loop-path product is obtained by ignoring the loops in the walk $(i, k_1, k_2, \ldots, k_{\ell-1}, j)$. For example, the loop-path product $m_{1,2}^{(1)} m_{2,4}^{(2)} m_{4,4}^{(3)} m_{4,5}^{(4)}$ has the underlying path $(1, 2, 4, 5)$.

Consider the upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}.$$

Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of A and let n_j be the multiplicity of λ_j as a root of the characteristic polynomial of A for $j = 1, \ldots, m$. Choose $\mu \in \mathrm{spec}(A)$ and let $S = \{k : a_{kk} = \mu\}$. Define

$$B^{(k)} := \begin{cases} (A - a_{kk}I) & \text{for } k \notin S, \\ I & \text{for } k \in S. \end{cases}$$

Then since any two scalar shifts of A commute with each other,

$$\prod_{\lambda_j \neq \mu} (A - \lambda_j I)^{n_j} = \prod_{k \notin S} (A - a_{kk}I) = \prod_{k \notin S} B^{(k)} = \prod_{1 \leq k \leq n} B^{(k)}.$$

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Note that this last equality is true since $B^{(k)} = I$ for $k \in S$. By definition of matrix multiplication, the $(1, n)$-entry of $\prod_{1 \leq k \leq n} B^{(k)}$ is the sum of loop-path products of the form

$$(3.4) \qquad B^{(1)}_{1,k_1} B^{(2)}_{k_1,k_2} B^{(3)}_{k_2,k_3} \cdots B^{(n)}_{k_{n-1},n}$$

where

$$B^{(k)}_{i,j} = \begin{cases} a_{ii} - a_{kk} & \text{for } j = i, \text{ and } k \notin S, \\ a_{ij} & \text{for } j \neq i, \text{ and } k \notin S, \end{cases} \quad \text{and} \quad B^{(k)}_{i,j} = \begin{cases} 1 & \text{for } j = i, \text{ and } k \in S, \\ 0 & \text{for } j \neq i, \text{ and } k \in S. \end{cases}$$

Note that in the following result, the case $S = \{1\}$ is the reason for the inclusion of the phrase "at least one of $s, t \in S$" in (iii).

LEMMA 3.19. *Suppose $\mathcal{W}$ is a nonzero loop-path product of the form (3.4). Then (i) $\mathcal{W}$ includes a factor $B^{(k)}_{kk}$ with $k \in S$, (ii) if ℓ is the least integer such that $\mathcal{W}$ includes the factor $B^{(\ell)}_{k_{\ell-1},k_\ell}$, with $k_{\ell-1} = k_\ell$, then $\ell \leq k_{\ell-1}$, and (iii) $\mathcal{W}$ includes a factor a_{st} with at least one of $s, t \in S$ and $s < t$.*

Proof. If S contains either 1 or n then (iii) is clear. Suppose $1, n \notin S$ (and therefore $S \neq \{1\}$). We show that $\mathcal{W}$ includes a factor a_{st} for some $t \in S$. Observe that $B^{(k)}$ is upper triangular for $k = 1, \ldots, n$ and $B^{(k)}_{k,k} = 0$ if $k \notin S$. Therefore $\mathcal{W}$ is nonzero only if $k_i \leq k_{i+1}$ for $1 \leq i \leq n-2$; furthermore $k_i = k_{i+1}$ implies that $k_i = k_{i+1} \in S$. By the pigeonhole principle a factor $B^{(k)}_{k,k}$ appears in $\mathcal{W}$ and hence $k \in S$. Let ℓ be the least integer $(2 \leq \ell \leq n - 1)$ such that $B^{(\ell)}_{k_{\ell-1},k_\ell}$ is in $\mathcal{W}$ with $k_{\ell-1} = k_\ell$. Due to the triangular structure, $\ell \leq k_{\ell-1}$ and if $k_{\ell-1} \notin S$, then $\ell < k_{\ell-1}$. Since $1, n \notin S$, for the loop-path product (3.4) to be nonzero, it includes a factor $B^{(\ell)}_{st} \neq 0$, with $s < t$ and $t \in S$ (and hence $\ell \notin S$, since $B^{(\ell)} = I$ for $\ell \in S$). Therefore, recalling the definition of $B^{(k)}$ for $k \notin S$, each nonzero loop-path product of the form (3.4) has a factor a_{st} appear for some $t \in S$, with $s < t$. $\square$

For an upper triangular sign pattern that satisfies the conditions of Conjecture 3.18, if the loop-path products (3.4) are positive when μ is the rightmost eigenvalue, then the sign pattern requires eventual exponential nonnegativity. However, as Example 3.20 below shows, Lemma 3.19 does not preclude a loop-path product of the form (3.4) from including a loop at a vertex not in S (i.e., a factor $B^{(\ell)}_{kk}$ with $k > \ell$), which would have a loop weight of unknown sign. Example 3.20 also shows how one can rewrite the sum of two loop-path products as a single loop-path product with the loop occurring at a vertex in S; and when μ is a rightmost eigenvalue all loops at vertices in S have nonnegative loop weights.
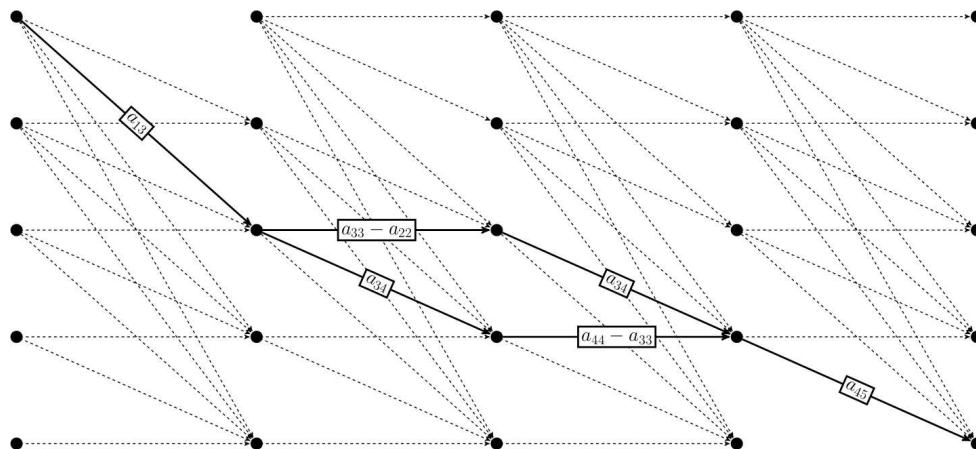
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Fig. 3.2: The composite König digraph for the matrix product $B^{(1)}B^{(2)}B^{(3)}B^{(5)}$ in Example 3.20. The loop-paths $B^{(1)}_{13}B^{(2)}_{33}B^{(3)}_{34}B^{(5)}_{45}$ and $B^{(1)}_{13}B^{(2)}_{34}B^{(3)}_{44}B^{(5)}_{45}$ are in solid lines with arc weights displayed.

EXAMPLE 3.20. Consider the upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}$$

with $\mu = a_{44}$. Then $B^{(4)} = I$ so $B^{(1)}B^{(2)}B^{(3)}B^{(4)}B^{(5)} = B^{(1)}B^{(2)}B^{(3)}B^{(5)}$ and the product $B^{(1)}_{13}B^{(2)}_{33}B^{(3)}_{34}B^{(5)}_{45} = a_{13}(a_{33}-a_{22})a_{34}a_{45}$ is not combinatorially zero (see Figure 3.2). However, the product $B^{(1)}_{13}B^{(2)}_{34}B^{(3)}_{44}B^{(5)}_{45} = a_{13}a_{34}(a_{44}-a_{33})a_{45}$ is also not combinatorially zero and summing these together we get $a_{13}a_{34}(a_{44}-a_{22})a_{45}$, which is a loop-path product utilizing a loop at vertex $\nu = 4$ with loop weight $(a_{44}-a_{22})$. Note that the underlying path of the loop-path product $a_{13}a_{34}(a_{44}-a_{22})a_{45}$ does not pass through vertex 2.

LEMMA 3.21. *Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be an upper triangular matrix. If the loop weight $(a_{ii} - a_{\ell\ell})$, with $i \notin S$, appears in a nonzero loop-path product of the form (3.4), with underlying path P, then additional loop-path products exist—also with the underlying path P—such that simplification of the sum of these loop-path products results in a single loop-path product, all of whose loops are at a vertex in S.*

Proof. Suppose $\ell \in \{1, \ldots, n\}$ is the least number such that a nonzero loop-path product in the $(1, n)$-entry of $\prod_{k \notin S}(A - a_{kk}I)$ can be written as $(a_{ii} - a_{\ell\ell})\mathcal{W}$ for some loop-path product $\mathcal{W}$. By Lemma 3.19 the loop-path product $\mathcal{W}$ passes through a

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vertex $\nu \in S$. We first consider the case $\ell < \nu$. The product $(a_{ii} - a_{\ell\ell})\mathcal{W}$ can be viewed as a loop-path product that uses a loop with weight $a_{ii} - a_{\ell\ell}$ at vertex i in conjunction with the loop-path product $\mathcal{W}$. If $i = \nu$, we are done.

Note that due to the upper triangularity of A, $i \leq \nu$ and $\ell < i$ (otherwise the loop-path product would be zero). Suppose that $i < \nu$. Let j be the next vertex (different from i) through which $\mathcal{W}$ passes. Then $(a_{jj} - a_{ii})\mathcal{W}$ is a loop-path product that differs from $(a_{ii} - a_{\ell\ell})\mathcal{W}$ by shifting the use of a loop from vertex i to vertex j. Moreover, $(a_{ii} - a_{\ell\ell})\mathcal{W} + (a_{jj} - a_{ii})\mathcal{W} = [(a_{ii} - a_{\ell\ell}) + (a_{jj} - a_{ii})]\mathcal{W} = (a_{jj} - a_{\ell\ell})\mathcal{W}$. If $j = \nu$, we are done. If not, then taking k to be the next vertex, after j, visited by $\mathcal{W}$ and simplifying the sum $(a_{jj} - a_{\ell\ell})\mathcal{W} + (a_{kk} - a_{jj})\mathcal{W}$ we get $(a_{kk} - a_{\ell\ell})\mathcal{W}$. Repeating this process, we can shift use of the loop with weight $a_{ii} - a_{\ell\ell}$ at vertex i in conjunction with $\mathcal{W}$ to the use of the loop with weight $a_{\nu\nu} - a_{\ell\ell}$ at vertex ν in conjunction with $\mathcal{W}$.

The case for $\ell > \nu$ is similar. ☐

Repeated application of this process allows for the sum of all nonzero loop-path products for the $(1,n)$-entry of $\prod_{k \neq \nu}(A - a_{kk}I)$, where A is upper triangular, to be written as loop-path products in which all loops occur at vertex ν. Note that this could require using multiple loops of different weights at vertex ν.

Theorem 3.22 follows from Lemmas 3.19 and 3.21.

THEOREM 3.22. *Let $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ be upper triangular with m distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. Then for $\mu \in \{\lambda_1, \ldots, \lambda_m\}$, $S = \{k : a_{kk} = \mu\}$, and $M = n - |S| - 1$, the $(1,n)$-entry of the product $\prod_{k \notin S}(A - a_{kk}I)$ can be expressed as*

$$\sum_{\nu \in S} \left[\left(\prod_{k \notin (S \cup \{1,n\})} (a_{\nu\nu} - a_{kk}) \right) a_{1\nu}a_{\nu n} \right.$$

$$+ \sum_{\substack{1 < \ell_1 < \ell_2 < n \\ \nu \in \{\ell_1, \ell_2\}}} \left(\prod_{k \notin (S \cup \{1, \ell_1, \ell_2, n\})} (a_{\nu\nu} - a_{kk}) \right) a_{1,\ell_1}a_{\ell_1,\ell_2}a_{\ell_2,n}$$

$$\left. + \cdots + \sum_{\substack{1 < \ell_1 < \cdots < \ell_M < n \\ \nu \in \{\ell_1, \ldots, \ell_M\}}} a_{1,\ell_1}a_{\ell_1,\ell_2} \cdots a_{\ell_M,n} \right].$$

Now we are ready to prove Conjecture 3.18 in the case that the sign pattern $\mathcal{A}$ is permutationally similar to an upper triangular sign pattern. Recall that if $\mathcal{A}$ is upper

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triangular, then $\Gamma(\mathcal{A})$ has no cycles of length 2 or more.

THEOREM 3.23. *Let $\mathcal{A}$ be an $n \times n$ upper triangular sign pattern such that (i) if a walk in $\Gamma(\mathcal{A})$ contains a negative non-loop arc then it does not contain any positive loop and that (ii) for every negative arc (s,t) in $\Gamma(\mathcal{A})$, there exists an arc-positive s-t walk with a negative-loop-free interior vertex. Then $\mathcal{A}$ requires eventual exponential nonnegativity.*

Proof. Let $A = [a_{ij}] \in \mathcal{Q}(\mathcal{A})$. Since A is upper triangular, $\mathrm{spec}(A) = \{a_{ii} : 1 \leq i \leq n\}$ and each eigenvalue of A is real. Suppose that $(e^{\tau A})_{ij} \neq 0$. We show that the dominating term of $(e^{\tau A})_{ij}$ is positive. For any given $i, j \in \{1, \ldots, n\}$,

$$\left(e^{\tau A}\right)_{ij} = \left(e^{\tau \widetilde{A}}\right)_{ij},$$

where $\widetilde{A} = \widetilde{A[\widehat{V}]}$ and $\widehat{V} = \widehat{V}(i,j) = \mathrm{Out}(i) \cap \mathrm{In}(j)$, so we need only show that the dominating term of $\left(e^{\tau \widetilde{A}}\right)_{ij}$ is positive. From the power series for $e^{\tau A}$, we have that the strictly lower triangular part of $e^{\tau A}$ is all zeros and $(e^{\tau A})_{ii} = e^{\tau a_{ii}} > 0$ for $i = 1, \ldots, n$. So we consider only $i < j$. Since $\mathcal{A}$ is upper triangular, $\widehat{V}(i,j) \subseteq \{i, i+1, \ldots, j-1, j\}$.

If the principal submatrix $A[\widehat{V}(i,j)]$ has no off-diagonal negative entry, then $A[\widehat{V}(i,j)]$ is eventually exponentially nonnegative by Proposition 3.1. Note that hypothesis (ii) implies that the first super-diagonal is nonnegative, therefore $A[\{i, i+1\}]$ is essentially nonnegative and hence eventually exponentially nonnegative for $i = 1, \ldots, n-1$. Let $j > i+1$ and suppose $A[\widehat{V}(i,j)]$ has an off-diagonal negative entry. From the interpolation method for calculating the matrix exponential, the dominating term of $\left(e^{\tau \widetilde{A}}\right)_{ij}$ is determined by the rightmost eigenvalue of the principal submatrix $A[\widehat{V}(i,j)]$.

Since by assumption there exist s, t, with $i \leq s < t \leq j$, such that $\alpha_{st} = -$, hypothesis (ii) guarantees the existence of $\kappa \in \{s+1, \ldots, t-1\}$ such that $\alpha_{\kappa\kappa} \geq 0$. Hence the spectral abscissa $\alpha(A[\widehat{V}(i,j)])$ is nonnegative; moreover, since A is a real triangular matrix, $\alpha(A[\widehat{V}(i,j)])$ is an eigenvalue of $A[\widehat{V}(i,j)]$. Let $\widehat{\gamma} = \alpha(A[\widehat{V}(i,j)]), \widehat{\lambda}_1, \ldots, \widehat{\lambda}_m$ be the distinct eigenvalues of $A[\widehat{V}(i,j)]$. Let $\widehat{n}_\nu$ be the size of the largest Jordan block for $\widehat{\gamma}$ in the Jordan canonical form of $A[\widehat{V}(i,j)]$ and $\widehat{n}_k$ be the algebraic multiplicity of $\widehat{\lambda}_k$ (for the submatrix). Let $S = \{k : i \leq k \leq j, a_{kk} = \widehat{\gamma}\}$.

By Observation 3.5, the dominating term of $\left(e^{\tau \widetilde{A}}\right)_{ij}$ is positive if and only if the (i,j)-entry of

$$(\widetilde{A} - \widehat{\gamma}I)^{\widehat{n}_\nu - 1} \prod_{i=1}^{m} (\widetilde{A} - \widehat{\lambda}_i I)^{\widehat{n}_i} = (\widetilde{A} - \widehat{\gamma}I)^{\widehat{n}_\nu - 1} \prod_{\substack{i \leq k \leq j \\ k \notin S}} (\widetilde{A} - a_{kk}I)$$

is positive. Let $M := j - (i-1) - |S| - 1 = j - i - |S|$. By Theorem 3.22, and the
fact that $a_{kk} - \widehat{\gamma} = 0$ for $k \in S$, the (i,j)-entry of this product can be expressed as

$$\sum_{\nu \in S} \left[\left(\prod_{\substack{i<k<j \\ k \notin \{i,\nu,j\}}} (a_{\nu\nu} - a_{kk}) \right) a_{i\nu} a_{\nu j} \right.$$

$$+ \sum_{\substack{i<\ell_1<\ell_2<j \\ \nu \in \{\ell_1,\ell_2\}}} \left(\prod_{\substack{i<k<j \\ k \notin \{i,\ell_1,\ell_2,j\}}} (a_{\nu\nu} - a_{kk}) \right) a_{i,\ell_1} a_{\ell_1,\ell_2} a_{\ell_2,j}$$

$$\left. + \cdots + \sum_{\substack{i<\ell_1<\cdots<\ell_M<j \\ \nu \in \{\ell_1,\ldots,\ell_M\}}} a_{i,\ell_1} a_{\ell_1,\ell_2} \cdots a_{\ell_M,j} \right].$$

Since $a_{\nu\nu} = \widehat{\gamma} = \alpha(A[\widehat{V}(i,j)])$, $a_{\nu\nu} - \widehat{\lambda}_k > 0$ for $k = 1,\ldots,m$. By hypothesis (i), if
$a_{\nu\nu} = \alpha(A[\widehat{V}(i,j)]) > 0$, every i-ν-j path is arc-positive and hence each product in the
sum is positive. If $a_{\nu\nu} = \alpha(A[\widehat{V}(i,j)]) = 0$, then there could be an i-ν-j path that uses
a negative arc (x,y); however, in using arc (x,y), the path avoids an x-y arc-positive
path with a negative-loop-free vertex z. So any loop-path product which includes a_{xy}
would then be multiplied by $(a_{\nu\nu} - a_{zz}) = 0$ (since $0 \leq a_{zz} \leq \alpha(A[\widehat{V}(i,j)]) = a_{\nu\nu} = 0$).
Therefore the nonzero products in the sum are positive.

Hence either $\left(e^{\tau A}\right)_{ij} = 0$ or the dominating term of $\left(e^{\tau A}\right)_{ij}$ is positive for any
$i,j \in \{1,\ldots,n\}$ and A is eventually exponentially nonnegative. Therefore $\mathcal{A}$ requires
eventual exponential nonnegativity. $\square$

3.3. Maximum number of negative entries. When studying generalizations
of positive or nonnegative matrices, one often considers the minimum number of posi-
tive entries or the maximum number of negative entries (the former is not relevant for
generalizations of nonnegativity). In this section we determine the maximum number
of negative entries in a sign pattern that requires eventual exponential nonnegativity.

It is clear that for $n = 1$, the answer is one.

EXAMPLE 3.24. For $n \geq 2$, define

$$\mathcal{T}_n := \begin{bmatrix} - & + & - & \cdots & \cdots & - \\ 0 & 0 & + & - & \cdots & - \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & + & - \\ 0 & \cdots & \cdots & \cdots & 0 & + \\ 0 & \cdots & \cdots & \cdots & 0 & - \end{bmatrix}.$$

By Theorem 3.23, $\mathcal{T}_n$ requires eventual exponential nonnegativity. Therefore the maximum number of negative entries in a sign pattern that requires eventual exponential nonnegativity is at least $\frac{(n-1)(n-2)}{2}+2$ for $n \geq 2$. Note that if any of the 2nd through $(n-1)$st entries of the main diagonal of $\mathcal{T}_n$ are changed to $-$, the resulting sign pattern does not require eventual exponential nonnegativity since it would violate condition (iii) of Theorem 3.15.

THEOREM 3.25. *If the $n \times n$ sign pattern $\mathcal{A}$ ($n \geq 2$) requires eventual exponential nonnegativity, then $\mathcal{A}$ has at most $\frac{(n-1)(n-2)}{2}+2$ negative entries.*

Proof. Let $\mathcal{A} = [\alpha_{ij}]$ be an $n \times n$ sign pattern ($n \geq 2$) that requires eventual exponential nonnegativity. By Corollary 3.17, there exists a permutation pattern P such that for $\mathcal{A}' = P\mathcal{A}P^T = [\alpha'_{ij}]$, $\alpha'_{ij} \geq 0$ if $i > j$ or $j = i + 1$.

Since $j \geq i + 2$ if $\alpha'_{ij} = -$, $\mathcal{A}'$ has at most $\frac{(n-1)(n-2)}{2}+n$ negative entries. By part (iii) of Theorem 3.15, if $\alpha'_{st} = -$ (where $s \neq t$), then there exists $k \in \widehat{V}(s, t)$ such that $\alpha'_{kk} \geq 0$. Specifically, if $\alpha'_{k-1,k+1} = -$, then $\alpha'_{kk} \geq 0$ and conversely, if $\alpha'_{kk} = -$, then $\alpha'_{k-1,k+1} \geq 0$. So for $k = 2, \ldots, n - 1$, at most one of α'_{kk} and $\alpha'_{k-1,k+1}$ is negative. Therefore $\mathcal{A}$ has at most $\frac{(n-1)(n-2)}{2}+n-(n-2)=\frac{(n-1)(n-2)}{2}+2$ negative entries. $\square$

It is interesting to note that $\mathcal{T}_n$ is not the unique (even when taking into account graph isomorphism) way for a sign pattern to have $\frac{(n-1)(n-2)}{2}+2$ negative entries and to require eventual exponential nonnegativity. For example,

$$\begin{bmatrix} - & + & - & - & - \\ 0 & 0 & + & 0 & - \\ 0 & 0 & - & + & - \\ 0 & 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 & - \end{bmatrix}$$

has $\frac{(5-1)(5-2)}{2}+2=8$ negative entries and requires eventual exponential nonnegativity by Theorem 3.23.

COROLLARY 3.26. *For $n \geq 2$ the maximum number of negative entries in a sign pattern that requires eventual exponential nonnegativity is $\frac{(n-1)(n-2)}{2}+2$.*

4. Appendix: König digraphs. The following is known (see, e.g., [3]) but is included here to aid the reader. Brualdi and Cvetković [3] use a slightly different (but mathematically equivalent) method in using König digraphs to determine a matrix product. The specific difference is noted after we define our method. Let $M = [m_{ij}]$ be a (real) $m \times n$ matrix. The *König digraph* of M, denoted $K(M)$, is a weighted bipartite digraph on $m + n$ vertices, with vertices $V_r = \{r_1, \ldots, r_m\}$ corresponding to the rows of M and vertices $V_c = \{c_1, \ldots, c_n\}$ corresponding to the columns of M. The ordered pair (r_i, c_j) is an arc in $K(M)$ if and only if $m_{ij} \neq 0$ and the

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weight of arc (r_i, c_j) is given by m_{ij}. Consider the matrices $X = [x_{ij}] \in \mathbb{R}^{m \times n}$ and $Y = [y_{ij}] \in \mathbb{R}^{n \times p}$ and their König digraphs $K(X)$ and $K(Y)$, with vertices $V_{X,r} \cup V_{X,c}$ and $V_{Y,r} \cup V_{Y,c}$, respectively (where $V_{X,r} = \{r_{X,1}, r_{X,2}, \dots, r_{X,m}\}$, $V_{X,c} = \{c_{X,1}, c_{X,2}, \dots, c_{X,n}\}$, $V_{Y,r} = \{r_{Y,1}, r_{Y,2}, \dots, r_{Y,n}\}$, and $V_{Y,c} = \{c_{Y,1}, c_{Y,2}, \dots, c_{Y,p}\}$). The (i, j)-entry of the product XY can be computed as follows. First, construct the *composite König digraph*: for $1 \le k \le n$, identify vertex $c_{X,k}$ with vertex $r_{Y,k}$ and rename as v_k. Second, for $1 \le k \le n$, compute the weight, w_k, of the $(r_{X,i}, v_k, c_{Y,j})$-path as $w_k = x_{ik} y_{kj}$. Finally, compute the sum $w_1 + w_2 + \dots + w_n$. By the definition of matrix multiplication, this sum is the (i, j)-entry of the product XY. This process generalizes to products of more than two matrices, as shown in Example 4.1. Note that Brualdi and Cvetković collapse what we call the composite König digraph into the König digraph for the matrix that is the result of computing the product. For example, the arc weight of the arc $(r_{X,i}, c_{Y,j})$ in $K(XY)$ would be the sum $w_1 + w_2 + \dots + w_n$.
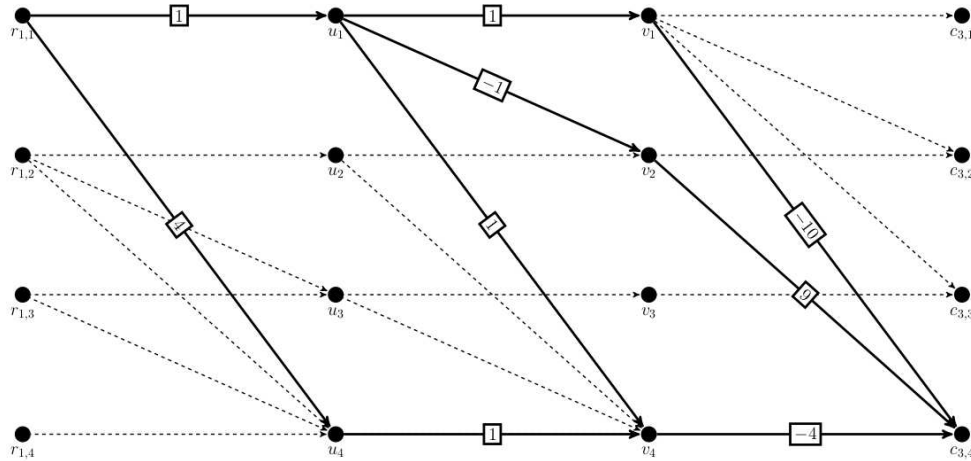

Fig. 4.1: The composite König digraph for the matrix product in Example 4.1. The arcs not belonging to any $r_{X,1}$-$c_{Z,4}$-path are shown as dashed lines, and their arc weights are omitted, as they are not used in the calculation of the $(1, 4)$-entry of the product XYZ.

EXAMPLE 4.1. Consider the matrices

$$X = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 5 & -6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 5 & 8 & -10 \\ 0 & 2 & 0 & 9 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}.$$

We can use the König digraphs, $K(X)$, $K(Y)$, and $K(Z)$ (see Figure 4.1) to determine the $(1, 4)$-entry of the product XYZ. There are four $r_{X,1}$-$c_{Z,4}$-paths in the composite

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König digraph in Figure 4.1, adding the weights of these paths, we get $(1)(1)(-10) + (1)(-1)(9) + (1)(1)(-4) + (4)(1)(-4) = -39$, which is the $(1, 4)$-entry of the product XYZ.

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