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ON THE LAPLACIAN CHARACTERISTIC POLYNOMIALS OF MIXED GRAPHS*

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Abstract. Let G be a mixed graph and $L(G)$ be the Laplacian matrix of G. In this paper, the coefficients of the Laplacian characteristic polynomial of G are studied. The first derivative of the characteristic polynomial of $L(G)$ is explicitly expressed by means of Laplacian characteristic polynomials of its edge deleted subgraphs. As a consequence, it is shown that the Laplacian characteristic polynomial of a mixed graph is reconstructible from the collection of the Laplacian characteristic polynomials of its edge deleted subgraphs. Then, it is investigated how graph modifications affect the mixed Laplacian characteristic polynomial. Also, a connection between the Laplacian characteristic polynomial of a non-singular connected mixed graph and the signless Laplacian characteristic polynomial is provided, and it is used to establish a lower bound for the spectral radius of $L(G)$. Finally, using Coates digraphs, the perturbation of the mixed Laplacian spectral radius under some graph transformations is discussed.

Key words. Mixed graphs, Laplacian matrix, Laplacian characteristic polynomial, Laplacian spectral radius, Coates digraphs.

AMS subject classifications. 05C50, 05C90.

1. Introduction.

Let G be a mixed graph having n vertices and m edges, obtained from an undirected simple graph by orienting some of its edges. The underlying graph of G is denoted by $\overline{G}$. The vertex set and the edge set of G are denoted by $V(G)$ and $E(G)$, respectively. The incidence matrix of G is a $\{-1,0,1\}$-matrix, whose rows and columns are labelled by vertices and edges, respectively, and is denoted by $M(G) = (m_{ij})_{n\times m}$. The entry $m_{ij} = 1$ if e_j is an unoriented edge incident with v_i, or if e_j is an oriented edge with head v_i, $m_{ij} = -1$ if e_j is an oriented edge with tail v_i, and $m_{ij} = 0$ otherwise. The Laplacian matrix of the mixed graph G is defined as $L(G) = M(G)M(G)^t$, so that $L(G)$ is positive semi-definite. Hence, its eigenvalues can be arranged as $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) \geq 0$. One may see that, to obtain $L(G)$, we only check that an edge is oriented or not, and do not care which vertex is the head and which one is the tail of an oriented edge. So, for each $e \in E(G)$, we could consider a sign function which is denoted by $sgn(e)$ and defined as $sgn(e) = 1$ if e is

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unoriented in G and $sgn(e) = -1$ otherwise. The adjacency matrix of G is denoted by $A(G) = (a_{ij})_{n\times n}$, where $a_{ij} = sgn(v_iv_j)$ if $v_iv_j \in E(G)$, and $a_{ij} = 0$ otherwise. So that, the Laplacian matrix of the mixed graph G is $L(G) = \mathcal{D}(G) + A(G)$, where $\mathcal{D}(G) = diag(d_G(v_1), \ldots, d_G(v_n))$ is a diagonal matrix, and $d_G(v)$ denotes the degree of the vertex v in $\overline{G}$. It should be noted that the standard Laplacian matrix defined for simple graphs is equal to the Laplacian matrix of any all-oriented mixed graph, for bibliographies on the standard Laplacian matrix of simple graphs, the reader is referred to [9]. Further, the Laplacian matrix of a mixed graph with no directed edges is called the signless Laplacian matrix (e.g. see [4]). Also, the path, the cycle, and the star on n vertices are denoted by P_n, C_n, and S_n, respectively.

In the present paper, using the principal minor version of the Matrix-Tree Theorem for a mixed graph, we study the Laplacian characteristic polynomials of mixed graphs. Next, we explicitly express the first derivative of the Laplacian characteristic polynomial of a mixed graph by means of Laplacian characteristic polynomials of its edge deleted subgraphs. We prove that the Laplacian characteristic polynomials of mixed graphs are reconstructible from the collection of mixed Laplacian polynomials of their edge deleted subgraphs. We then investigate how graph modifications affect the mixed Laplacian characteristic polynomial. Also, we provide a connection between the Laplacian characteristic polynomial of a non-singular connected mixed graph and the standard Laplacian and the signless Laplacian characteristic polynomial of a simple graph obtained from it, and so we obtain a lower bound for the spectral radius of $L(G)$. Finally, using Coates digraphs, we discuss the perturbation of the mixed Laplacian spectral radius under some graph transformations.

2. Graphical interpretation of a determinant. In this section, we give a general overview how to use graphs in the situations when we would like to prove results on coefficients of the Laplacian characteristic polynomial. Noting that the main reference for this section is [3].

Let $A = (a_{ij})_{n\times n}$ be a matrix of order n with eigenvalues $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$. The characteristic polynomial of A is denoted by $\chi(A, x) = \det(xI - A)$. Associated with a matrix A, one may correspond a weighted digraph $D(A)$ with n vertices, such that there is an edge from vertex v_i to vertex v_j of weight a_{ij}, for $1 \le i, j \le n$. Also, the edges of weight zero, corresponding to the zero entries of A, can be removed from $D(A)$. The digraph $D(A)$ is called the *Coates digraph* of the matrix A, for more details the reader is referred to [3, page 65].

Recall that a digraph with equal number of vertices and edges is called a *cycle digraph*, when its vertices can be labelled as $v_1, v_2, \ldots, v_n$; in such a way that its edge set consists of the edges directed from vertex v_i to vertex v_{i+1}, for $1 \le i \le n-1$, and an edge from vertex v_n to vertex v_1. In addition, a *linear subdigraph* of a digraph D

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is a subdigraph of D such that the indegree is equal to the outdegree of each vertex and is equal to one, in which the indegree of a vertex v in D is the number of arcs with head v, and the outdegree of v is the number of arcs with tail v. Thus, a linear subdigraph consists of disjoint union of cycle digraphs. Let L be a linear subdigraph of the digraph $D(A)$. The product of the weights of the edges of L is called the weight of L, and is denoted by $\omega(L)$. The number of cycles contained in L is denoted by $c(L)$. Also, the set of all linear subdigraphs of the Coates digraph $D(A)$ with exactly i vertices is denoted by $\mathfrak{L}_i(A)$. So, we have the following theorem which is based on graphical interpretation of the determinant of matrices.

THEOREM 2.1. [3, Chapter 4] Let A be a matrix of order n, and let $\chi(A,x) = \sum_{i=0}^{n} c_i x^{n-i}$ be the characteristic polynomial of A. Then for $1 \leq i \leq n$, we have

$$c_i = \sum_{L \in \mathfrak{L}_i(A)} (-1)^{c(L)} \omega(L).$$

There are some theorems which extend some well-known formulas in the spectral graph theory to matrices as you see in the following. For $e \in E(G)$, the graph $G - e$ denotes the spanning subgraph of G with edge set $E(G) \setminus \{e\}$. Also, if $v \in V(G)$, then the graph $G - v$ is an induced subgraph of G, obtained from G by removing the vertex v and all edges incident with it. The next result states the relation between characteristic polynomial of a symmetric matrix A and symmetric matrices obtained from A by replacing some entries with zero.

THEOREM 2.2. [2, Theorem 3.4] Let A be any symmetric matrix, and let $G = D(A)$ be its Coates digraph. If $\overrightarrow{uv}$ $(u \neq v)$ is a fixed edge of G, then

$$\chi(G,x) = \chi(G - \overrightarrow{uv} - \overrightarrow{vu}, x) - a_{uv}^2 \chi(G - u - v, x) - 2 \sum_{Z \in C_{uv}} \omega(Z) \chi(G - V(Z), x),$$

where C_{uv} is the set of all undirected cycles of G of length greater than or equal to 3 containing u and v, while $\omega(Z) = \prod_{ij \in E(Z)} a_{ij}$.

Suppose that we have two weighted digraphs G and H with $v \in V(G)$ and $u \in V(H)$. The *coalescence* of G and H with respect to v and u, obtained from G and H by identifying v with u and is denoted by $GvuH$ (see e.g. [2]). Note that the weight of loop at $w(= v = u)$ in $GvuH$ is equal to the sum of weights of loops at v and u.

THEOREM 2.3. [2, Theorem 3.5] Let $GuvH$ be the coalescence of two weighted digraphs G and H with $v \in V(G)$ and $u \in V(H)$. Then we have

$$\begin{aligned}\chi(GuvH, x) = {} & \chi(G - u, x)\chi(H, x) + \chi(G, x)\chi(H - v, x) \\ & - x\chi(G - u, x)\chi(H - v, x).\end{aligned}$$

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3. Laplacian characteristic polynomials of mixed graphs. We begin this section by studying the coefficients of the Laplacian characteristic polynomial of a mixed graph G. The characteristic polynomial of $L(G)$ is denoted by $L(G,x) = \det(xI - L(G)) = \sum_{i=0}^{n} l_i x^{n-i}$. One may see that the coefficient l_i can be expressed in terms of substructures of G, for each $0 \leq i \leq n$. Indeed, in the article [1, Theorem 1], the principal minor version of the Matrix-Tree Theorem for a mixed graph is proved. For more details, we first need to consider the *substructure* of G which is obtained from G by deleting some edges or some vertices. We also could delete vertices without deleting the edges incident with them, although we assume that each undeleted edge is incident with at least one undeleted vertex. If we delete one vertex of a spanning tree of G without deleting the edges incident with that vertex, the resulting substructure is called a *rootless spanning tree*, which has $n-1$ vertices and $n-1$ edges. Also, a mixed graph G is called *singular* (or *non-singular*) if $L(G)$ is singular (or non-singular). In [1, Lemma 1], it is shown that a cycle is non-singular if and only if it contains an odd number of unoriented edges. A connected mixed graph containing exactly one cycle with that cycle being non-singular, is called a *non-singular unicyclic graph*. Thus, a non-singular unicyclic graph consists of a non-singular cycle together with trees attached to each of the vertices of the cycle. A *k-reduced spanning substructure* of G is a substructure containing $n-k$ vertices, each connected component of which contains an equal number of vertices and edges and has no singular cycles. It is easy to see that any k-reduced spanning substructure R of the mixed graph G has rootless trees and non-singular unicyclic graphs as its connected components and satisfies $|V(R)| = |E(R)| = n-k$. A mixed graph G is called *quasi-bipartite* if it does not contain any non-singular cycle. So, a mixed graph with all edges unoriented is quasi-bipartite if and only if it is bipartite, and a mixed graph with all edges oriented is always quasi-bipartite. Let $\omega(G)$ and $\omega_0(G)$ denote the number of connected components and the number of quasi-bipartite connected components of G, respectively.

THEOREM 3.1. *Let G be a mixed graph. The mixed Laplacian coefficient l_k of G is given by*

$$(-1)^k l_k = \sum_{R \in \mathfrak{R}_k} 4^{\omega_1(R)},$$

where $\mathfrak{R}_k$ is the set of all $(n-k)$-reduced substructures R of G with k edges and $\omega_1(R) = \omega(R) - \omega_0(R)$.

Proof. According to this fact that the coefficient l_k, for each $0 \leq k \leq n$, is the sum of the kth order principal minors of the matrix $L(G)$, [1, Theorem 1] yields the assertion. ∎

As we see in the following remark, one may apply more simple structures instead

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of $(n-k)$-reduced substructures, to compute the coefficient l_k.

REMARK 3.1. Let G be a mixed graph, and let $\mathfrak{S}_k(G)$ be the set of all spanning subgraphs of G with k edges whose connected components are trees or non-singular unicyclic graphs, where $1 \leq k \leq n$. Suppose that $H \in \mathfrak{S}_k(G)$ contains trees T_1, T_2, $\dots$, $T_{\omega_0(H)}$. The weight of H is defined by $W(H) = 4^{\omega_1(H)} \prod_{i=1}^{\omega_0(H)} n_i$, in which n_i is the order of T_i, and $\omega_1(H)$ is the number of non-singular unicyclic components of H. So, the previous theorem implies that $(-1)^k l_k = \sum_H W(H)$, where $H \in \mathfrak{S}_k$.

It is worth mentioning that the above result is a generalization of [4, Theorem 4.4] and a theorem attributed to Kel'mans who gave a combinatorial interpretation to coefficients of the Laplacian characteristic polynomial of simple graphs in terms of the numbers of certain subforests, for more details see [9, Theorem 4.3].

In the next result, we obtain the first derivative of the Laplacian characteristic polynomial of a graph G by means of whose edge deleted subgraphs.

THEOREM 3.2. *Let G be a mixed graph of order n and size m. Then the first derivative of the Laplacian characteristic polynomial of G is equal to*

$$xL'(G,x) = (n-m)L(G,x) + \sum_{e \in E(G)} L(G-e,x).$$

Proof. If $H \in \mathfrak{S}_i(G)$, then $H \in \mathfrak{S}_i(G-e)$ for each $e \notin E(H)$. So that, Remark 3.1 deduces that $(m-i)l_i(G) = \sum_{e \in E(G)} l_i(G-e)$, for $1 \leq i \leq n$. Consequently, the following holds

$$\begin{aligned} xL'(G,x) + (m-n)L(G,x) &= \sum_{i=0}^{n} (m-i) l_i(G) x^{n-i} \\ &= \sum_{i=0}^{n} \sum_{e \in E(G)} l_i(G-e) x^{n-i} \\ &= \sum_{e \in E(G)} L(G-e,x). \qquad \square \end{aligned}$$

Suppose that $G = K_2$, and $e \in E(G)$ (either oriented or not). Then one may readily check that $L(G,x) = x^2 - 2x \leq L(G-e,x) = x^2$ for $x \geq \mu_1(G) = 2$. Next, we want to prove this observation in general.

Let G be a mixed graph and let $S(G) = \begin{bmatrix} 0 & M(G) \\ M(G)^t & 0 \end{bmatrix}$, where $M(G)$ is the incidence matrix of G. Then, the characteristic polynomial of $S(G)$ is

$$\chi(S(G),x) = x^{m-n} L(G,x^2). \tag{3.1}$$

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Note that Eq. (3.1) is independent of the orientation of edges of G.

THEOREM 3.3. [6, Page 43] *Let A be a matrix of order n, with characteristic polynomial $\chi(A,x) = \det(xI - A)$. Then $\chi'(A,x) = \sum_{i=1}^{n} \chi(A_i,x)$, where A_i is the square matrix obtained from A by removing the ith row and the ith column.*

LEMMA 3.2. *Let A be a symmetric matrix of order n such that all of whose diagonal entries are zero. Then we have $x\chi(A_i,x) - \chi(A,x) \geq 0$ for $x \geq \lambda_1(A)$, where A_i is the matrix obtained from A by removing the ith row and the ith column, for $1 \leq i \leq n$. Moreover, if $\lambda_1(A) > \lambda_1(A_i)$, then $x\chi(A_i,x) - \chi(A,x) > 0$ for $x \geq \lambda_1(A)$.*

Proof. If the ith row of the matrix A is equal to zero, then the lemma is obvious. So, suppose that the ith row of A is non-zero. Let A' be a matrix of order n, obtained by replacing all non-zero entries of ith row and ith column of A with 0. By induction on n, we prove that $f(x) = \chi(A',x) - \chi(A,x) \geq 0$ for $x \geq \lambda_1(A)$. One may readily check that the assertion holds when $n = 2$. So, suppose that $n \geq 3$. Applying Theorem 3.3, we find that

$$\begin{aligned} f'(x) &= \chi'(A',x) - \chi'(A,x) \\ &= \sum_{j=1}^{n} (\chi(A'_j,x) - \chi(A_j,x)). \end{aligned}$$

Since the ith row of the matrix A is non-zero, for some j, we have $A_j \neq A'_j$. So that, induction hypothesis implies

$$\chi(A'_j,x) - \chi(A_j,x) \geq 0, \quad \text{when } x \geq \lambda_1(A_j).$$

Moreover, Interlacing Theorem of eigenvalues [6, Theorem 4.3.15] specifies $\lambda_1(A) \geq \lambda_1(A_j) \geq \lambda_1(A'_j)$, for each j. Thus, $f'(x) \geq 0$ for $x \geq \lambda_1(A)$, consequently, $f(x)$ is a non-decreasing function on $x \geq \lambda_1(A)$. On the other hand, Interlacing Theorem of eigenvalues follows that $\lambda_1(A) \geq \lambda_1(A')$. Hence, $\chi(A',x) - \chi(A,x) \geq \chi(A',\lambda_1(A)) \geq 0$ for $x \geq \lambda_1(A)$.

For the second part, if $\lambda_1(A) > \lambda_1(A')$, then $\chi(A',x) - \chi(A,x) \geq \chi(A',\lambda_1(A)) > 0$. So, we are done. ∎

In the following, we recall the interlacing-type theorem for the Laplacian eigenvalues of mixed graphs.

LEMMA 3.4. [11, Lemma 2.2] Let G be a mixed graph of order n, and e be an edge of G. Then

$$\mu_1(G) \geq \mu_1(G-e) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) \geq \mu_n(G-e).$$

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THEOREM 3.5. *Let G be a mixed graph with n vertices and m edges, and let H be a spanning subgraph of G. Then $L(H,x) \geq L(G,x)$ for $x \geq \mu_1(G)$.*

Proof. In order to prove this theorem, we consider the first derivative of the polynomial $p(x) = x^{m-n}L(G,x^2)$, which is

$$p(x)' = (m-n)x^{m-n-1}L(G,x^2) + 2x^{m-n+1}L'(G,x^2). \tag{3.2}$$

On the other hand, using Eq. (3.1) and Theorem 3.3 together, we find that

$$\begin{aligned} p(x)' &= \chi'(S(G),x) \\ &= \sum_{e\in E(G)} \chi(S(G-e),x) + \sum_{v\in V(G)} \chi(S(G)_v,x) \\ &= \sum_{e\in E(G)} x^{m-1-n}L(G-e,x^2) + \sum_{v\in V(G)} \chi(S(G)_v,x), \end{aligned} \tag{3.3}$$

in which $S(G)_v = \begin{bmatrix} 0 & M(G)_v \\ M(G)_v^t & 0 \end{bmatrix}$ and also the matrix $M(G)_v$ obtained from $M(G)$ by removing the row corresponding to the vertex v. Thus, combining Theorem 3.2, and Eq. (3.2) and (3.3), we obtain that

$$x^{m-n+1}L'(G,x^2) = \sum_{v\in V(G)} \chi(S(G)_v,x). \tag{3.4}$$

Now, suppose that H is obtained from G by deleting a edges $e_1, e_2, \ldots, e_a$ $(a \geq 1)$. Therefore, for the subgraph H, we have

$$x^{m-a-n+1}L'(H,x^2) = \sum_{v\in V(H)} \chi(S(H)_v,x). \tag{3.5}$$

Applying Eq. (3.4) and (3.5), we find that

$$L'(H,x^2) - L'(G,x^2) = x^{n-1-m} \sum_{v\in V(G)} x^a\chi(S(H)_v,x) - \chi(S(G)_v,x).$$

On the other hand, $S(G)_v$ is a symmetric matrix with zero diagonal, so by Lemma 3.2 $\chi(S(G)_v,x) \leq x\chi(S(G-e_1)_v,x) \leq \cdots \leq x^a\chi(S(H)_v,x)$, for each $v \in V(G)$ and $x \geq \sqrt{\mu_1(G)} \geq \lambda_1(S(G)_v)$. Hence, we have $L'(H,x) - L'(G,x) \geq 0$ for $x \geq \mu_1(G)$. Moreover, Lemma 3.4 implies that $\mu_1(G) \geq \mu_1(H)$, and so $L(H,x) - L(G,x) \geq L(H,\mu_1(G)) \geq 0$, for $x \geq \mu_1(G)$. In addition, if $\mu_1(G) > \mu_1(H)$, then $L(H,x) - L(G,x) \geq L(H,\mu_1(G)) > 0$ for $x \geq \mu_1(G)$. This completes the proof. $\square$

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Here, we consider the problem of reconstructing $L(G,x)$ from the Laplacian characteristic polynomial of edge deleted subgraphs of a mixed graph G.

THEOREM 3.6. *The Laplacian characteristic polynomial of a mixed graph G is reconstructible from the collection of the Laplacian characteristic polynomial of edge deleted subgraphs of G.*

Proof. One may see that the number of edges is reconstructible from the collection of the Laplacian characteristic polynomial of edge deleted subgraphs of G. So, Theorem 3.2 implies that

$$\sum_{i=0}^{n}(m-i)l_i(G)x^{n-i} = \sum_{e\in E(G)} L(G-e,x).$$

Consequently, the coefficients of the Laplacian characteristic polynomial are reconstructible. ◻

Let G be a mixed graph. Denote by $\overrightarrow{G}$ an all-oriented mixed graph obtained from G by assigning to each unoriented edge of G an arbitrary orientation. Also, the mixed graph with adjacency matrix $-A(G)$ is denoted by $\widehat{G}$. Let D be a *signature matrix* which is a diagonal matrix with ±1 along its diagonal. Then $D^tL(G)D$ is the Laplacian matrix of a mixed graph with the same underlying graph as that of G such that some oriented edges of G may turn to be unoriented and vice versa. We use the notation ${}^DG$ to denote the graph obtained from G by a re-signing under the signature D, and assume that the labelling of the vertices of ${}^DG$ is the same as that of G.

THEOREM 3.7. [11, Lemma 2.4] *Let G be a connected mixed graph. Then G is singular if and only if G is quasi-bipartite.*

THEOREM 3.8. [1, Theorem 4] *Let G be a connected mixed graph. Then G is quasi-bipartite if and only if there exists a signature matrix D such that $D^tL(G)D = L(\overrightarrow{G})$.*

Now, suppose that μ is an eigenvalue of the matrix $L(G)$ with the corresponding eigenvector $X = (x_1, x_2, \ldots, x_n)^t$. Then for every $v \in V(G)$, we have

$$(d(v)-\mu)x_v = \sum_{e=uv\in V(G)} (-sgn(e))x_u. \tag{3.6}$$

In the next result, we provide connections between Laplacian characteristic polynomial of a mixed graph and the Laplacian characteristic polynomial of a digraph.

THEOREM 3.9. *Let G be a non-singular connected mixed graph of order n and size m. Then there exists a digraph DG with $2n$ vertices and $2m$ edges such that*

(*i*) $L(DG,x) = L(G,x)L(\overrightarrow{G},x)$;

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(ii) $L(\overline{DG}, x) = L(\widehat{G}, x)L(\overline{G}, x)$.

Proof. Let G be a mixed graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Let $A = \{e_1, \ldots, e_k\}$ be a minimum subset of $E(G)$ whose removal from G results in a quasi-bipartite subgraph of G. By Theorem 3.8, we may find a signature matrix D such that $D^t L(G \setminus A)D = L(\overrightarrow{G \setminus A})$. Let $G' = {}^D G$, obviously two matrices $L(G)$ and $L(G')$ have the same eigenvalues. Because the set A is the minimum subset of $E(G)$ by which $G \setminus A$ is singular, the graph G' is a non-singular mixed graph such that only the edges in A are unoriented. Suppose that G'' is a copy of G' and the vertex $u_i \in V(G'')$ corresponds to the vertex $v_i \in V(G')$. Now, let DG be an all-oriented graph on $2n$ vertices which is obtained from the disjoint union $(G' \setminus A) \cup (G'' \setminus A)$ by adding edges $e'_i = u_{i_1}v_{i_2}$ and $e''_i = v_{i_1}u_{i_2}$, where $e_i = v_{i_1}v_{i_2} \in A$, and $1 \le i \le k$. Then assign an arbitrary orientation to the edges e'_i and e''_i. Also, order the vertices of DG as v_1, v_2, …, v_n, u_1, u_2, …, u_n.

To prove case (i), suppose that $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_n\}$ are two sets of orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of the matrices $L(G')$ and $L(\overrightarrow{G})$, corresponding to eigenvalues $\mu_i(G)$ and $\mu_i(\overrightarrow{G})$, respectively. Then by Eq. (3.6), $\widehat{X}_i = \begin{bmatrix} X_i \\ -X_i \end{bmatrix}$ and $\widehat{Y}_i = \begin{bmatrix} Y_i \\ Y_i \end{bmatrix}$ are eigenvectors of the matrix $L(DG)$ corresponding to the eigenvalues $\mu_i(G)$ and $\mu_i(\overrightarrow{G})$, respectively, for $1 \le i \le n$. In order to prove (i), it is enough to show that the set $\{\widehat{X}_1, \ldots, \widehat{X}_n, \widehat{Y}_1, \ldots, \widehat{Y}_n\}$ is a linearly independent set. Obviously, two sets $\{\widehat{X}_1, \ldots, \widehat{X}_n\}$ and $\{\widehat{Y}_1, \ldots, \widehat{Y}_n\}$ are linearly independent. Suppose that $\sum_{i=1}^n c_i\widehat{X}_i + \sum_{i=1}^n c'_i\widehat{Y}_i = 0$ for some c_i, $c'_i \in \mathbb{R}$. Thus, $\sum_{i=1}^n c_iX_i + \sum_{i=1}^n c'_iY_i = \sum_{i=1}^n c_iX_i - \sum_{i=1}^n c'_iY_i$. Consequently, $\sum_{i=1}^n c'_iY_i = 0$, and so $c'_i = 0$ for each $1 \le i \le n$. This completes the proof of (i).

(ii) Suppose that $\{X_1, X_2, \ldots, X_n\}$ and $\{Y_1, Y_2, \ldots, Y_n\}$ are two sets of orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of the matrices $L(\widehat{G'})$ and $L(\overline{G})$, corresponding to $\mu_i(\widehat{G})$ and $\mu_i(\overline{G})$, respectively, for $1 \le i \le n$. Then by Eq. (3.6), the vectors $\widehat{X}_i = \begin{bmatrix} X_i \\ -X_i \end{bmatrix}$ and $\widehat{Y}_i = \begin{bmatrix} Y_i \\ Y_i \end{bmatrix}$ are eigenvectors of the matrix $L(\overline{DG})$ corresponding to $\mu_i(\widehat{G})$ and $\mu_i(\overline{G})$, respectively, for $1 \le i \le n$. In a similar way, one may check that $\{\widehat{X}_1, \ldots, \widehat{X}_n, \widehat{Y}_1, \ldots, \widehat{Y}_n\}$ is a linearly independent set. ◻

It should be noted that the part (i) of the above theorem follows directly from [8, Theorem 30]. However, we are unaware of this result.

Corollary 3.3. *Let G be a non-singular mixed graph of order n, and DG be the graph constructed in the proof of Theorem* 3.9. *Then for each $e \in E(DG)$, we have $\mu_1(G) \ge \mu_1(DG - e)$.*

Proof. This is an immediate result of Lemma 3.4. ◻

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Note that the lower bound on the spectral radius of $L(G)$ in the previous corollary is better than $1 + \Delta(G)$ (see e.g [11, Lemma 3.1]). More precisely, with the notation used in proof of Theorem 3.9, we have $d_{DG}(u_i) = d_{DG}(v_i) = d_G(v_i)$. So, $\Delta(DG) = \Delta(G)$, and we then find that $\mu_1(DG - e) \geq \Delta(G) + 1$, by [11, Lemma 3.1].

A mixed graph G is called bipartite, if its underlying graph is bipartite.

Proposition 3.4. *Let G be a singular connected mixed graph. If the matrices $L(G)$ and $L(\widehat{G})$ have the same eigenvalues, then G is bipartite.*

Proof. Since G is singular, using Theorem 3.8, there exist signature matrices D_1 and D_2 such that $D_1^t L(G) D_1 = D_2^t L(\widehat{G}) D_2 = L(\overrightarrow{G})$. Then $\widehat{G} = {}^D G$, when $D = D_1 D_2$. Therefore, for each $v_i v_j \in E(G)$ we have $D_{ii} D_{jj} L_{ij}(G) = -L_{ij}(G)$. Suppose that $V_1 = \{v_i \in V(G) | D_{ii} = 1\}$ and $V_2 = \{v_i \in V(G) | D_{ii} = -1\}$. Obviously, G is a bipartite graph. □

Note that the previous proposition is not true in general. Regarding Fig. 1, one may check that G is a non-bipartite mixed graph such that $L(G, x) = L(\widehat{G}, x) = x^6 - 14x^5 + 74x^4 - 184x^3 + 217x^2 - 106x + 12$.

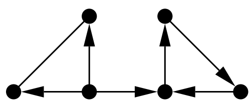

Fig. 1. The graph G.

Proposition 3.5. *Let G be a bipartite mixed graph. Then the matrices $L(G)$ and $L(\widehat{G})$ have the same eigenvalues.*

Proof. Since G is bipartite, the digraph DG is also bipartite. Thus, $L(DG, x) = L(\overline{DG}, x)$ and consequently by Theorem 3.9, $L(G, x) = L(\widehat{G}, x)$. This completes the proof. □

Lemma 3.6. *Let G be a mixed graph. Then $\mu_1(G) \leq \mu_1(\overline{G})$, and equality holds if and only if $\widehat{G}$ is a quasi bipartite graph.*

Proof. Suppose that $X = (x_1, \ldots, x_n)^t$ is an eigenvector of $L(G)$ corresponding to $\mu_1(G)$. Then by Rayleigh-Ritz Theorem [6, Theorem 4.2.2], we have

$$\begin{aligned}
\mu_1(G) &= X^t L(G) X \\
&= \sum_{e = v_i v_j \in E(\overline{G})} (x_i + sgn(e) x_j)^2 \\
&\leq \sum_{e = v_i v_j \in E(\overline{G})} (|x_i| + |x_j|)^2
\end{aligned}$$

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$$= |X|^t L(\overline{G})|X|$$
$$\leq \mu_1(\overline{G}),$$

where $|X| = (|x_1|, \ldots, |x_n|)^t$. Moreover, the equality obtained from [10, Lemma 2.3]. □

So, if G is a quasi-bipartite mixed graph, then $\mu_1(G) \leq \mu_1(\widehat{G})$. Also, by Proposition 3.5, we obtain $\mu_1(G) = \mu_1(\widehat{G})$, when G is bipartite.

4. Some transformations. As we see in Section 2, associated with a square matrix, one may correspond a weighted digraph called Coates digraph. Using the connection between the Laplacian matrix and a partitioned matrix involving the incidence matrix of a mixed graph G (see Eq. (3.1)), and also properties of Coates digraphs, in this section we discuss changes in the mixed Laplacian characteristic polynomial. More precisely, in the first part, we state some results on the Laplacian characteristic polynomial by which we could order mixed unicyclic graphs based on their Laplacian spectral radius. Finally, in the last part we focus our attention to the Laplacian characteristic polynomial of unoriented graphs, which is called signless Laplacian characteristic polynomial.

4.1. Laplacian matrix of a mixed graph. For convenience, we use the notation $D(G) = D(S(G))$ for the Coates digraph of $S(G) = \begin{bmatrix} 0 & M(G) \\ M(G)^t & 0 \end{bmatrix}$, where $M(G)$ is incidence matrix of G. Noting that this matrix is defined in Section 3.

The following theorem is a generalization of Theorem 3.5 in [7] and Theorem 2.10 in [5]. Note that a pendent mixed path is a mixed path whose inner vertices all have degree 2 and one of its end vertices is a pendent vertex.

Theorem 4.1. *Let v be a vertex of a non-trivial connected mixed graph G, and for non-negative integers l, k, let $G(l,k)$ denote the graph obtained from G by adding pendent mixed paths of lengths l and k at vertex v. If $1 \leq k \leq l$, then*

$$L(G(l+1,k-1),x) - L(G(l,k),x) \geq 0,$$

for $x \geq \mu_1(G(l+1,k-1))$. In particular $\mu_1(G(l,k)) \geq \mu_1(G(l+1,k-1))$.

In order to prove Theorem 4.1, we state the following lemmas.

Lemma 4.1. *Let l,k be non-negative integers such that $1 \leq k \leq l$. Then we have $L(G(l+k,0),x) \geq L(G(l,k),x)$, for $x \geq \mu_1(G(l+k,0))$.*

Proof. Let $H = D(G(l,k))$ and $H' = D(G(l+k,0))$. By Eq. (3.1), equivalently, we want to prove that $\chi(H',x) \geq \chi(H,x)$ for $x \geq \lambda_1(H')$. Using the notation of Section 2, we have $H = D(G)vuD(P_{l+k})$, and $H' = D(G)vwD(P_{l+k})$ in which $v \in V(G)$, and

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$w, u \in V(P_{l+k})$ such that $d_{P_{l+k}}(u) = 2$ and $d_{P_{l+k}}(w) = 1$. So, applying Theorem 2.3 to the graphs H and H', we find that

$$\chi(H', x) - \chi(H, x) = \Big(x\chi(D(G) - v, x) - \chi(D(G), x)\Big)$$
$$\Big(\chi(D(P_{l+k}) - u, x) - \chi(D(P_{l+k}) - w, x)\Big).$$

According to Lemma 3.2, we have $x\chi(D(G)-v, x)-\chi(D(G), x) \geq 0$ for $x \geq \lambda_1(D(G))$. Moreover, $D(P_{l+k})-u$ is a proper spanning subdigraph of $D(P_{l+k})-w$ (i.e., $D(P_{l+k})-u$ obtained from $D(P_{l+k}) - w$ by deleting the some edges for instance st). Thus, Theorem 2.2 implies that

$$\chi(D(P_{l+k}) - u, x) - \chi(D(P_{l+k}) - w, x) = -\chi(D(P_{l+k}) - u - s - t, x)$$
$$- 2 \sum_{Z \in C_{st}} \chi(D(P_{l+k}) - u - V(Z), x).$$

Now, by Interlacing Theorem, we obtain

$$\chi(D(P_{l+k}) - u, x) - \chi(D(P_{l+k}) - w, x) \geq 0, \text{ for } x \geq \lambda_1(D(P_{l+k}) - u).$$

This completes the proof. ◻

Lemma 4.2. *Let G be a mixed graph which has a pendent vertex u adjacent to a vertex v of degree 2. Then we have*

$$L(G, x) = (x - 2)L(G - u, x) - L(G - u - v, x).$$

Proof. Recall that $L(G, x) = \sum_{i=0}^{n} l_i(G)x^{n-i}$. Obviously, $l_0(G) = l_0(G - u) = 1$, $l_1(G) = l_1(G - u) - 2l_0(G - u) = l_1(G - u) - 2 = -2m$. Moreover, in Remark 3.1, each subgraph $H \in \mathfrak{S}_i(G)$ with i edges contains $n - i$ mixed trees. Therefore, all connected components of $H \in \mathfrak{S}_n(G)$ are non-singular unicyclic graphs. So that $l_n(G) = -l_{n-1}(G - u) = l_{n-2}(G - u - v)$. Thus, to prove the lemma, it is enough to show that for each $2 \leq i \leq n - 1$,

$$l_i(G) = l_i(G - u) - 2\, l_{i-1}(G - u) - l_{i-2}(G - u - v).$$

Consider $H \in \mathfrak{S}_i(G)$, where $\mathfrak{S}_i(G)$ is the set of all spanning subgraphs of G whose connected components are trees or non-singular unicyclic graphs with i edges. Then the following cases occur:

Case (i). If $uv \notin E(H)$, let $H' = H - u \in \mathfrak{S}_i(G - u)$.

Case (ii). Suppose $uv \in E(H)$, and $vw \notin E(H)$, where w is the other neighbor of v. Let $H' = H - u \in \mathfrak{S}_{i-1}(G - u)$.

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Case (*iii*). Suppose $uv, vw \in E(H)$, and also these two edges are contained in a tree of order a in H. Let $H' = H - u \in \mathfrak{S}_{i-1}(G-u)$ and $H'' = H - u - v \in \mathfrak{S}_{i-2}(G-u-v)$.

Case (*iv*). Suppose $uv, vw \in E(H)$, and these two edges are contained in a non-singular unicyclic graph in H. Let $H' = H - u \in \mathfrak{S}_{i-1}(G-u)$ and $H'' = H - u - v \in \mathfrak{S}_{i-2}(G-u-v)$.

It should be noted that this correspondence is obviously a one-to-one correspondence. Moreover, if H contributes $(-1)^i W(H)$ toward the coefficient l_i of x^{n-i} on the left, then on the right H' and H'' also contribute the same amount in each of the above cases, as it is shown in the following:

• In case (i), H' contributes $(-1)^i W(H)$ to the coefficient $l_i(G-u)$ of x^{n-1-i} in $L(G-u, x)$. So, it supplies $(-1)^i W(H)$ toward the coefficient of x^{n-i} in $xL(G-u, x)$.

• In case (ii), $H' \in \mathfrak{S}_{i-1}(G-u)$ contributes $(-1)^{i-1}\frac{W(H)}{2}$ to the coefficient $l_{i-1}(G-u)$ of $x^{n-1-(i-1)}$ in $L(G-u, x)$.

• In case (iii), $H' \in \mathfrak{S}_{i-1}(G-u)$ contributes $(-1)^{i-1}(a-1)\frac{W(H)}{a}$ to the coefficient $l_{i-1}(G-u)$ of $x^{n-1-(i-1)}$ in $L(G-u, x)$ and also $H'' = H - u - v \in \mathfrak{S}_{i-2}(G-u-v)$ contributes $(-1)^{i-2}(a-2)\frac{W(H)}{a}$ to the coefficient $l_{i-2}(G-u-v)$ of $x^{n-2-(i-2)}$ in $L(G-u-v, x)$.

• In case (iv), $H' \in \mathfrak{S}_{i-1}(G-u)$ contributes $(-1)^{i-1}W(H)$ to the coefficient $l_{i-1}(G-u)$ of $x^{n-1-(i-1)}$ in $L(G-u, x)$ and also $H'' = H - u - v \in \mathfrak{S}_{i-2}(G-u-v)$ contributes $(-1)^{i-2}W(H)$ to the coefficient $l_{i-2}(G-u-v)$ of $x^{n-2-(i-2)}$ in $L(G-u-v, x)$.

Thus, the contribution of each subgraph in $\mathfrak{S}_i(G)$ to the left side is matched by the corresponding contribution on the right side by H' and H''. So we are done. □

Proof of Theorem 4.1. By induction on k, we prove that $L(G(l+1, k-1), x) \geq L(G(l, k), x)$ for $x \geq \mu_1(G(l+1, k-1))$. If $k = 1$, then by Lemma 4.1, the assertion holds. Suppose that $k \geq 2$, so that applying Lemma 4.2 to the graphs $G(l, k)$ and $G(l+1, k-1)$, we obtain that

$$L(G(l+1, k-1), x) - L(G(l, k), x) = L(G(l, k-2), x) - L(G(l-1, k-1), x).$$

Now, the induction hypothesis yields

$$L(G(l+1, k-1), x) \geq L(G(l, k), x), \quad \text{for } x \geq \mu_1(G(l, k-2)).$$

Moreover, Lemma 3.4 implies that $\mu_1(G(l+1, k-1)) \geq \mu_1(G(l, k-2))$. This completes the proof of the first part of the theorem. For the second part, we have $L(G(l, k), \mu_1(G(l+1, k-1))) \leq 0$, so that $\mu_1(G(l+1, k-1)) \leq \mu_1(G(l, k))$. □

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EXAMPLE 4.3. Suppose that G and G' are two graphs depicted in Fig. 2. By Maple, one may check that G' has the same Laplacian spectral radius as G, $\mu_1(G) = \mu_1(G') = 3+\sqrt{6}$, and the same corresponding eigenvector $X = (-2-\sqrt{6}, 0, -2-\sqrt{6}, 1, 1, 0, 0, 1, 1)^t$. So that, in the previous theorem the equality attains.

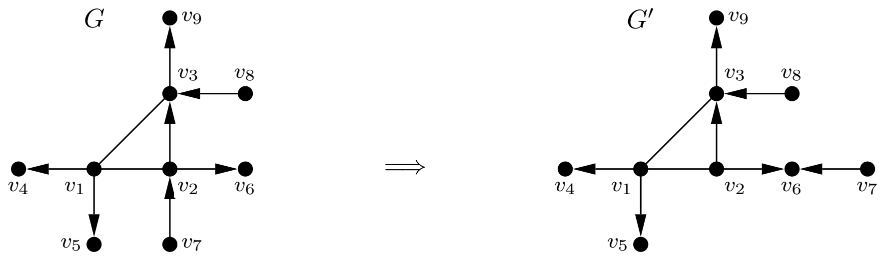

Fig. 2. The graphs G and G'.

A *starlike* is a subdivision of a star. Then a starlike has exactly one vertex of degree greater than two.

THEOREM 4.2. *Let G be a mixed graph $G \neq K_1, K_2$ and T be a mixed star like graph. Let $H = GvuT$ and $H' = GvwT$ where $v \in V(G)$, $u, w \in V(T)$ such that $d_T(u) = n-1$ and $d_T(w) = 1$. Then we have $L(H', x) \geq L(H, x)$ for $x \geq \mu_1(H')$. In particular $\mu_1(H) \geq \mu_1(H')$.*

Proof. By Eq. (3.1), equivalently, we should prove that $\chi(D(H'), x) \geq \chi(D(H), x)$ for $x \geq \lambda_1(D(H'))$. By Theorem 2.3, we conclude that

$$\begin{aligned}\chi(D(H'), x) - \chi(D(H), x) = &\left(x\chi(D(G)-v, x) - \chi(D(G), x)\right)\\ &\left(\chi(D(T)-u, x) - \chi(D(T)-w, x)\right).\end{aligned}$$

One may see that $D(T)-u$ is the spanning subdigraph of $D(T)-w$. Thus, Theorem 2.2 and Interlacing Theorem imply that

$$\chi(D(T)-u, x) - \chi(D(T)-w, x) \geq 0 \text{ for } x \geq \lambda_1(D(T)-u).$$

On the other hand, Lemma 3.2 specifies that $x\chi(D(G)-v, x) - \chi(D(G), x) \geq 0$ for $x \geq \lambda_1(D(G))$. Therefore, $L(H', x) \geq L(H, x)$ for $x \geq \mu_1(H')$. Moreover, we have $L(H, \mu_1(H')) \leq 0$, so that $\mu_1(H') \leq \mu_1(H)$. This completes the proof. ◻

Noting that, by Example 4.3, one may see that in the previous theorem, the equality case may occur.

Next, let $\mathfrak{U}_{n,g}$ be the collection of all mixed unicyclic graphs on n vertices with girth g. Suppose that the vertices of the cycle C_g are labelled by $v_1, \ldots, v_g$, ordered

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in a natural way around C_g, say in the clockwise direction. Also, assume that T_i is a rooted tree of order $n_i \geq 1$ attached to v_i, where $\sum_{i=1}^{g} n_i = n$. Then we denote U by $C(T_1, \ldots, T_g)$. In the following results, we are interested in ordering mixed unicyclic graphs with fixed girth, based on their Laplacian spectral radius.

Corollary 4.4. *Let $G = C(T_1, \ldots, T_g)$. Then for every $1 \leq i \leq n$,*

$$\mu_1(C(T_1, \ldots, P_{n_i}, \ldots, T_g)) \leq \mu_1(G) \leq \mu_1(C(T_1, \ldots, S_{n_i}, \ldots, T_g)),$$

where the roots of P_{n_i} and S_{n_i} have degree 1 and $n_i - 1$, respectively.

Proof. If $T_i \neq P_{m_i}$, then let $u \in V(T_i)$ such that $d_G(u) > 2$ and $d(u, v_i)$ is the largest one, where $d(u, v)$ denotes the length of a shortest path from u to v. Applying Theorem 4.1 to the vertex u, we obtain a unicyclic mixed graph H, for which $\mu_1(H) \leq \mu_1(G)$. Repeating this procedure, the left inequality is proved.

For the right inequality, suppose that $T_i \neq S_{m_i}$. Let w be a vertex in $V(T_i)$, such that all of its neighbors except one are pendent and $d(w, v_i)$ is also the largest one. Applying Theorem 4.2 to the vertex w, we obtain the graph H' with $\mu_1(H') \geq \mu_1(G)$. So by repeating this procedure, the right inequality is proved. □

Corollary 4.5. *Let $G = C(T_1, \ldots, T_g)$. Then*

$$\mu_1(C(P_{n_1}, \ldots, P_{n_g})) \leq \mu_1(G) \leq \mu_1(C(S_{n_1}, \ldots, S_{n_g})).$$

Proof. By applying the previous corollary to every i $(1 \leq i \leq g)$, we are done. □

4.2. Signless Laplacian matrix. In this part, we study the Laplacian matrix of a simple undirected graph G which is called signless Laplacian matrix of G, and is denoted by $Q(G)$. Also, the signless Laplacian characteristic polynomial of G is denoted by $Q(G, x) = det(xI - Q(G))$. It is worth mentioning that part (ii) of Theorem 3.9, establishes the connection between the Laplacian characteristic polynomial of a mixed graph and the signless Laplacian characteristic polynomial of the graph DG.

Also, the following relations are well-known (see for instance [4])

$$Q(G) = M(G)M(G)^t, \quad M(G)^t M(G) = A(\mathcal{L}(G)) + 2I_m, \tag{4.1}$$

where $\mathcal{L}(G)$ is the line graph of G. Since non-zero eigenvalues of $M(G)M(G)^t$ and $M(G)^t M(G)$ are identical, Eq. (4.1) implies that the adjacency characteristic polynomial of the graph $\mathcal{L}(G)$, denoted by $A(\mathcal{L}(G), x)$, equals

$$A(\mathcal{L}(G), x) = (x + 2)^{m-n} Q(G, x + 2). \tag{4.2}$$

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In the following result, we state the relation between the Laplacian characteristic polynomials of G and $G-e$, whenever $e \in E(G)$.

THEOREM 4.3. *Let G be a simple graph of order n and size m. Then for each $e \in E(G)$, we have*

$$\begin{aligned} Q(G,x) = & \frac{x-2}{x} Q(G-e,x) - \frac{1}{x^2} \sum_{e' \in N(e)} Q(G-e-e',x) \\ & -2 \sum_{Z \in C_e} \frac{1}{x^{|E(Z)|}} Q(G-E(Z),x), \end{aligned}$$

where $N(e) \subseteq E(G)$ is the set of edges incident with the edge e in G.

Proof. Using Eq. (4.2) and Theorem 2.2, we obtain that

$$\begin{aligned} (x+2)^{m-n} Q(G,x+2) &= A(\mathcal{L}(G),x) \\ &= xA(\mathcal{L}(G-e),x) - \sum_{e' \in N(e)} A(\mathcal{L}(G-e-e'),x) \\ &\quad -2 \sum_{Z \in C_e} A(\mathcal{L}(G-E(Z)),x) \\ &= x(x+2)^{m-1-n} Q(G-e,x+2) \\ &\quad -(x+2)^{m-2-n} \sum_{e' \in N(e)} Q(G-e-e',x+2) \\ &\quad - \sum_{Z \in C_e} (x+2)^{m-|E(Z)|-n} Q(G-E(Z),x+2). \end{aligned}$$

This completes the proof. ☐

Accordingly, the following results hold.

COROLLARY 4.6. *Let H be a spanning subgraph of G, then for $x \geq \mu_1(G)$*

$$Q(H,x) - \frac{2}{x} Q(H,x) \geq Q(G,x).$$

If $\mu_1(G) > \mu_1(H)$ (in particular, if G is connected and H is a spanning subgraph of G), then $Q(H,x) - \frac{2}{x}Q(H,x) > Q(G,x)$ for $x \geq \mu_1(G)$.

Proof. Let $E(H) = E(G) \setminus \{e_1, e_2, \ldots, e_a\}$. According to Theorem 4.3, we obtain for $x \geq \mu_1(G)$

$$Q(G-e_1-e_2-\cdots-e_{a-1},x) - Q(H,x) + \frac{2}{x} Q(H,x) \leq 0.$$

On the other hand, for every $1 \leq i \leq a-2$, Theorem 3.5 yields that

$$Q(G-e_1-e_2-\cdots-e_i,x) - Q(G-e_1-e_2-\cdots-e_i-e_{i+1},x) \leq 0$$

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for $x \geq \mu_1(G)$. This completes the proof. □

REMARK 4.7. If H is a spanning subgraph of G, the previous corollary together with Theorem 3.5 imply that for $x \geq \mu_1(G)$

$$Q(G,x) + \frac{2}{x}Q(G,x) \;\leq\; Q(G,x) + \frac{2}{x}Q(H,x) \;\leq\; Q(H,x).$$

If $\mu_1(H) < \mu_1(G)$ (in particular, if G is connected and H is a spanning subgraph of G), then $Q(G,x) + \frac{2}{x}Q(G,x) < Q(H,x)$ for $x \geq \mu_1(G)$.

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