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FURTHER RESULTS ON GENERALIZED INVERSES IN RINGS WITH INVOLUTION*

N. CASTRO-GONZÁLEZ†, JIANLONG CHEN‡, AND LONG WANG‡

Abstract. Let $\mathcal{R}$ be a unital ring with an involution. Necessary and sufficient conditions for the existence of the Bott-Duffin inverse of $a \in \mathcal{R}$ relative to a pair of self-adjoint idempotents (e, f) are derived. The existence of a $\{1, 3\}$-inverse, $\{1, 4\}$-inverse, and the Moore-Penrose inverse of a matrix product is characterized, and explicit formulas for their computations are obtained. Some applications to block matrices over a ring are given.

Key words. Ring, outer inverse, Bott-Duffin-inverse, $\{1, 3\}$-inverse, $\{1, 4\}$-inverse, Moore-Penrose inverse.

AMS subject classifications. 16W10, 15A09.

1. Introduction. Let $\mathcal{R}$ be an associative ring with unity 1. The set of all idempotent elements of $\mathcal{R}$ will be denoted by $E(\mathcal{R})$. Let $a \in \mathcal{R}$ and $e \in E(\mathcal{R})$ such that $ae + 1 - e$ is invertible. Then the Bott-Duffin e-inverse of a (see [3, Chapter 2, Section 10]) is defined as the element $y = e(ae + 1 - e)^{-1}$. It is an outer inverse for a, i.e., $yay = y$.

Let $e, f \in E(\mathcal{R})$. Djordjević and Wei introduced a type of outer inverse by prescribing the idempotens ya and ay in [8]: The (e, f)-outer generalized inverse of a is the unique element $y \in \mathcal{R}$, whenever it exists, satisfying

$$yay = y, \quad ya = e, \quad ay = 1 - f.$$

A characterization of the existence of the (e, f)-outer generalized inverse was given in [8, Theorem 2.1].

For $a \in \mathcal{R}$, we associate the image and kernel ideals:

$$a\mathcal{R} = \{ax : x \in \mathcal{R}\}, \quad a^0 = \{x \in \mathcal{R} : ax = 0\}.$$

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†ETSI Informática, Universidad Politécnica de Madrid, 28660 Madrid, Spain (nieves@fi.upm.es). The research was supported by Project MTM2010-18057, "Ministerio de Economía y Competitividad" of Spain.

‡Department of Mathematics, Southeast University, Nanjing, 210096, China (jlchen@seu.edu.cn, wanglseu@hotmail.com). The research was supported by the NNSF of China (11371089), NSF of Jiangsu Province (BK 20141327) and Specialized Research Fund for the Doctoral Program of Higher Education (20120092110020).

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Kantún-Montiel [14] explored the idea of prescribing the image ideal $ya\mathcal{R}$ and the kernel ideal $(ay)^0$ related to the outer inverse: The image-kernel (e,f)-inverse of a is the unique element $y \in \mathcal{R}$, whenever it exists, satisfying

$$yay = y, \quad ya\mathcal{R} = e\mathcal{R}, \quad (ay)^0 = (1-f)^0.$$

If y is the (e,f)-outer generalized inverse of a, then it is the image-kernel (e,f)-inverse of a. The converse part is as follows. If y is the image-kernel (e,f)-inverse of a then y is the $(eya, f(1-ay))$-outer generalized inverse of a. Elements with equal idempotents related to their image-kernel (e,f)-inverses are characterized in [18]. The representation and approximation for the outer inverse having prescribed range and null space in the setting of complex matrices were given in [23].

Drazin in [9, Definition 3.2] introduced the following generalization of the Bott-Duffin inverse relative of a pair of idempotents: The Bott-Duffin (e,f)-inverse of a is the unique element $y \in \mathcal{R}$, when it exists, such that

$$y = ey = yf, \quad yae = e, \quad fay = f. \tag{1.1}$$

We abbreviate Bott-Duffin to B-D. It was showed in [14, Proposition 3.4] that y is the the image-kernel (e,f)-inverse of a if and only if it is the B-D $(e,1-f)$-inverse of a.

By [9, Theorem 2.2], we know that there exists a B-D (e,f)-inverse of a if and only if $e \in \mathcal{R}fae$ and $f \in fae\mathcal{R}$.

On account of the above result, for $e=f$ the equations in (1.1) have a common solution iff $e \in \mathcal{R}eae \cap eae\mathcal{R}$. This is equivalent to the invertibility of $ea+1-e$, see Lemma 2.1. In fact, the element for which (1.1) holds is precisely the classical Bott-Duffin e-inverse of a, $y = e(ae+1-e)^{-1}$.

We ask whether the existence of B-D (e,f)-inverse can be characterized in terms of classical invertibility. We present a result in Section 2 to answer this question in the setting of a ring with an involution under the assumption that both e and f are self-adjoint idempotents.

We recall that $*$ is an involution in $\mathcal{R}$ if it is a map $* : \mathcal{R} \to \mathcal{R}$ such that for all $a,b \in \mathcal{R}$: $(a^*)^* = a$, $(a+b)^* = a^*+b^*$, $(ab)^* = b^*a^*$. The set of all idempotent self-adjoint elements of $\mathcal{R}$ ($e = e^2 = e^*$) will be denoted by $E^*(\mathcal{R})$.

Let $\mathcal{M}_{m\times n}(\mathcal{R})$ denote the set of $m \times n$ matrices over $\mathcal{R}$ and let $\mathcal{M}_m(\mathcal{R})$ denote the ring of $m \times m$ matrices over $\mathcal{R}$. For any matrix $A = (a_{ij}) \in \mathcal{M}_{m\times n}(\mathcal{R})$, $A^* \in \mathcal{M}_{n\times m}(\mathcal{R})$ stands for $(\overline{A})^T$ where $\overline{A} = (a_{ij}^*)$.

A matrix $A \in \mathcal{M}_{m\times n}(\mathcal{R})$ is said to be Moore-Penrose invertible with respect to

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the involution $*$ if the equations

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA$$

have a unique common solution. Such a solution, when exists, is denoted by $A^\dagger$.

A is called regular if there exists X satisfying (1). Let $A\{1\}$ denote the set of matrices $X \in \mathcal{M}_{n\times m}(\mathcal{R})$ which satisfy equation (1).

If X is a solution of both (1) and (3) then it is called a $\{1,3\}$-inverse of A. Similarly, if X is a solution of both (1) and (4) then it is called a $\{1,4\}$-inverse of A. We will consider the following sets:

$$A\{1,3\} = \{X \in A\{1\} : (AX)^* = AX\},$$
$$A\{1,4\} = \{X \in A\{1\} : (XA)^* = XA\}.$$

Necessary and sufficient conditions for the existence of $\{1,3\}$-inverse, $\{1,4\}$-inverse and the Moore-Penrose inverse were presented in [22, Proposition 3.10]. When A is regular, the existence of $A^\dagger$ was characterized by means of classical invertibility, see [20, Remark 3] and [19, Theorem 1.1]:

Lemma 1.1. *Let $A \in \mathcal{M}_{m\times n}(\mathcal{R})$ be regular and let $A^{(1)}$ be an arbitrary element of $A\{1\}$. Then the following conditions are equivalent:*

1. *$A^\dagger$ exists (with respect to $*$).*
2. *$U = AA^* + I_m - AA^{(1)}$ is invertible.*
3. *$V = A^*A + I_n - A^{(1)}A$ is invertible.*

In this case,

$$A^\dagger = A^*(U^*)^{-1} = (V^*)^{-1}A^*.$$

The existence of the Moore-Penrose inverse of a matrix product PAQ was studied in [10, 20]. We recall that if P and Q are both invertible then the Moore-Penrose inverse of PAQ exists if and only if PA has a $\{1,3\}$-inverse and AQ has a $\{1,4\}$-inverse, in which case

$$(PAQ)^\dagger = (AQ)^{(1,4)}A(PA)^{(1,3)}, \tag{1.2}$$

where $(PA)^{(1,3)}$ and $(AQ)^{(1,4)}$ are arbitrary elements of $(PA)\{1,3\}$ and $(AQ)\{1,4\}$, respectively.

In Section 3, Theorems 3.1, 3.4 and 3.6 provide necessary and sufficient conditions for the existence of a $\{1,3\}$-inverse of PA, a $\{1,4\}$-inverse of AQ, and the Moore-Penrose inverse of PAQ, respectively, under some conditions. We also give explicit

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formulas for the computation of these generalized inverses. In Section 4, we consider some applications of our results to block matrices.

For a treatment of generalized inverses of block matrices over a ring we refer the reader to [12].

2. Bott-Duffin inverses in involutory rings. Let $\mathcal{R}$ be a ring with unity 1 and an involution $*$. Let $e, f \in E^*(\mathcal{R})$, in this section we derive necessary and sufficient conditions for the existence of the Bott-Duffin (e,f)-inverse, as well as an explicit formula for its computation.

It will be convenient to introduce the following sets. For $e \in E(\mathcal{R})$, we consider

$$e\mathcal{R}e + 1 - e = \{exe + 1 - e : \ x \in \mathcal{R}\},$$

which is a submonoid of $\mathcal{R}$ under multiplication and the group U_e of e-units in the subring $e\mathcal{R}e$ (corner ring) given by

$$U_e = \{exe : \ exe\mathcal{R} = e\mathcal{R}, \ \mathcal{R}exe = \mathcal{R}e\}.$$

Next known result links invertible elements in $e\mathcal{R}e + 1 - e$ and elements of U_e.

Lemma 2.1. *Let $a \in \mathcal{R}$ and $e \in E(\mathcal{R})$. Then the following conditions are equivalent:*

(i) $e \in eae\mathcal{R} \cap \mathcal{R}eae$.
(ii) $eae + 1 - e$ *is invertible.*
(iii) $ae + 1 - e$ *is invertible.*
(iv) $eae \in U_e$.

In this case, the e-inverse of eae in U_e is given by

$$(eae)^{-1}_{e\mathcal{R}e} = e(eae + 1 - e)^{-1}e. \tag{2.1}$$

We can now formulate our main result of this section.

Theorem 2.2. *Let $a \in \mathcal{R}$ and $e, f \in E^*(\mathcal{R})$. Then the following conditions are equivalent:*

(a) *There exists a Bott-Duffin (e,f)-inverse of a.*
(b) $e \in \mathcal{R}(fae)^*fae$ *and* $f \in fae(fae)^*\mathcal{R}$.
(c) $u = (fae)^*fae + 1 - e$ *is invertible and* $faeu^{-1}(fae)^* = f$.
(d) $v = fae(fae)^* + 1 - f$ *is invertible and* $(fae)^*v^{-1}fae = e$.
(e) *Both* $u = (fae)^*fae + 1 - e$ *and* $v = fae(fae)^* + 1 - f$ *are invertible.*
(f) $ea^*fae \in U_e$ *and* $faea^*f \in U_f$.

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In this case, the B-D (e, f)-inverse of a is given by

$$\begin{aligned} y &= u^{-1}(fae)^* = (fae)^* v^{-1} \\ &= (ea^* fae)^{-1}_{e\mathcal{R}e} a^* f = ea^*(faea^* f)^{-1}_{f\mathcal{R}f} . \end{aligned} \tag{2.2}$$

Proof. $(a) \Rightarrow (b)$. Let $y \in \mathcal{R}$ be a common solution of equations in (1.1). Then $e = yfae$ and $f = faey$. By substituting $e = (fae)^* y^*$ into the last identity we get $f = fae(fae)^* y^* y$ and thus, $f \in fae(fae)^*\mathcal{R}$. Similarly, by substituting $f = y^*(fae)^*$ in $e = yfae$ we obtain $e = yy^*(fae)^* fae$ and $e \in \mathcal{R}(fae)^* fae$.

$(b) \Rightarrow (c)$. Suppose that there exists $s, t \in \mathcal{R}$ such that $e = s(fae)^* fae$ and $f = fae(fae)^* t$. Since $e = e^*$ we have $e = (fae)^* faes^*$ and it follows that $se = es^*$. Then

$$(se + 1 - e)((fae)^* fae + 1 - e) = ((fae)^* fae + 1 - e)(es^* + 1 - e) = 1.$$

Hence, $x = se + 1 - e$ is the inverse of $u = (fae)^* fae + 1 - e$. Further, we have $faeu^{-1}(fae)^* = faeu^{-1}(fae)^* fae(fae)^* t = fae(fae)^* t = f$.

$(c) \Leftrightarrow (d)$. We prove that (c) implies (d). Let $v = fae(fae)^* + 1 - f$. Using the relation $faeu^{-1}(fae)^* = f$, we obtain $v = 1 + fae(1 - u^{-1})(fae)^*$. Hence, v is invertible if and only if $1 + (1 - u^{-1})(fae)^* fae$ is invertible. But this last element is equal to u since $u^{-1}(fae)^* fae = e$. Now, it is easy to check that $(fae)^* v = u(fae)^*$ and, hence, $(fae)^* v^{-1} = u^{-1}(fae)^*$. Then $(fae)^* v^{-1} fae = u^{-1}(fae)^* fae = e$.

In the same manner, we can see that (d) implies (c).

$(d) \Rightarrow (e)$. On account of the above equivalence, this implication is immediate.

$(e) \Leftrightarrow (f)$. It follows by Lemma 2.1.

$(e) \Rightarrow (a)$. Suppose that both u and v are invertible. Now, we will prove that $y = (fae)^* v^{-1} = u^{-1}(fae)^*$ is a common solution of equations in (1.1). Clearly, $y = ey = yf$. Now, using this relation,

$$yae = yfae = (fae)^* v^{-1} fae = u^{-1}(fae)^* fae = e.$$

In the same manner, we see that $fay = f$, and thus y the B-D (e, f)-inverse of a. The last two identities in (2.2) are clear by (2.1). ☐

We observe that if the Moore-Penrose inverse of a exists, then it is the B-D (e, f)-inverse of a with $e = a^\dagger a$ and $f = aa^\dagger$. In this case, the element u given in item (c) and v given in item (d) of the above theorem are of the form $u = a^* a + 1 - a^\dagger a$ and $v = aa^* + 1 - aa^\dagger$, which are invertible whenever $a^\dagger$ exists, see Lemma 1.1. Koliha et al. in [15, Theorem 1] established a relation between Moore-Penrose invertible and well-supported elements in a ring with involution.

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We specialize the preceding theorem to Bott-Duffin e-inverse.

COROLLARY 2.3. *Let $a \in \mathcal{R}$ and $e \in E^*(\mathcal{R})$. Then the following conditions are equivalent:*

(a) There exists a B-D e-inverse of a.
(b) $e \in \mathcal{R}(eae)^*eae \cap eae(eae)^*\mathcal{R}$.
(c) $u = (eae)^*eae + 1 - e$ is invertible and $eaeu^{-1}(eae)^* = e$.
(d) $v = eae(eae)^* + 1 - e$ is invertible and $(eae)^*v^{-1}eae = e$.
(e) Both $u = (eae)^*eae + 1 - e$ and $v = eae(eae)^* + 1 - e$ are invertible.
(f) $ea^*eae \in U_e$ and $eaea^*e \in U_e$.

In this case, $y = u^{-1}(eae)^* = (eae)^*v^{-1}$ is the B-D e-inverse of a.

Next, we consider the product paq. We are interested in establishing a relation between the B-D (e, f)-inverse of paq and certain classes of generalized inverses of pa and aq.

THEOREM 2.4. *Let $a, p, q \in \mathcal{R}$ and let $e, f \in E(\mathcal{R})$. Then the following conditions are equivalent:*

(a) paq has a B-D (e, f)-inverse y.
(b) There exist $x, z \in \mathcal{R}$ such that:

$$x = ex, \quad x(aq)e = e, \quad fp(aq)x = fp \tag{2.3}$$

$$z = zf, \quad f(pa)z = f, \quad z(pa)qe = qe. \tag{2.4}$$

In this case, we have $y = xaz$, where x and z are any solution of (2.3) and (2.4), respectively.

Proof. First, let y be the B-D (e, f)-inverse of paq. Then

$$y = ey = yf, \quad ypaqe = e, \quad fpaqy = f. \tag{2.5}$$

We will prove that $x = yp$ satisfies (2.3). From $y = ey$ it follows $yp = eyp$, and hence, $x = ex$. Since $ypaqe = e$ we also have $xaqe = e$. Using $fpaqy = f$ it follows that $fpaqyp = fpaqx = fp$. Analogously, we can prove that $z = qy$ satisfies (2.4).

Conversely, let x be any solution of (2.3) and let z be any solution of (2.4). Define $y = xaz$. We will prove that y is a common solution of equations in (2.5). Clearly, $y = ey = yf$. Now, $ypaqe = xaz(pa)qe = xaqe = e$ and $fpaqy = fpaqxaz = fpaz = f$. ∎

REMARK 2.5. Conditions on x in (2.3) (or conditions on z in (2.4)) are not sufficient to ensure the uniqueness as we show in the following example. Let $\mathcal{R} = \mathcal{M}_2(\mathbb{Z}_{12})$ be the ring of 2×2 matrices over $\mathbb{Z}_{12}$, $a = \begin{pmatrix} 5 & 5 \\ 0 & 0 \end{pmatrix}$, $p = q = I$, and let

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the idempotents $e = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$ be given. Then $x = \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}$ and $\widehat{x} = \begin{pmatrix} 5 & 3 \\ 0 & 0 \end{pmatrix}$ are two different solutions of (2.3).

3. Generalized inverses of a matrix product. Theorems 3.1, 3.4 and 3.6 give the existence of $(PA)^{(1,3)}$, $(AQ)^{(1,4)}$ and the Moore-Penrose invertibility of PAQ from the classical invertibility of matrices. These characterization results in the setting of matrices over a ring are news. Formulae (3.1), (3.3) and (3.4) are extensions to matrices over a ring of similar formulae obtained for matrices over the complexes in [5]. We recall that if $*$ is the conjugate transpose of a complex matrix, then a $\{1,3\}$-inverse, a $\{1,4\}$-inverse, and the Moore-Penrose inverse of a complex matrix always exist.

In what follows, E' denotes the matrix $I - E$ for any idempotent matrix E.

We begin by giving a characterization for a matrix product PA to have a $\{1,3\}$-inverse when a $A^{(1,3)}$ exists.

Theorem 3.1. *Let $A \in \mathcal{M}_{m\times n}(\mathcal{R})$ be such that a $A^{(1,3)}$ exists and let $E = AA^{(1,3)}$, and $P \in \mathcal{M}_m(\mathcal{R})$ be invertible. If $PE' = E'$ then the following conditions are equivalent:*

(a) PA has a $\{1,3\}$-inverse.
(b) $E \in \mathcal{M}_m(\mathcal{R})Z \cap Z\mathcal{M}_m(\mathcal{R})$ where $Z = EP^*PE$.
(c) $U = P^*PE + I - E$ is invertible.
(d) $I + R^*R$ is invertible with $R = E'(I - P^{-1})$.

In this case, there exists a $\{1,3\}$-inverse of PA of the form

$$(PA)^{(1,3)} = A^{(1,3)}U^{-1}P^* = A^{(1,3)}P^{-1}(I + R^*R)^{-1}(I + R^*). \tag{3.1}$$

Proof. Let us first observe that if X be an arbitrary element of $A\{1,3\}$, then $X = A^{(1,3)} + (I - A^{(1,3)}A)Z$ with $Z \in \mathcal{M}_{n\times m}(\mathcal{R})$ and, thus, $AX = AA^{(1,3)}$. Therefore, if $PE' = E'$ we also have $P(I - AX) = I - AX$.

(a)⇒(b). If PA has a $\{1,3\}$-inverse Y then $Y^*A^*P^* = PAY$ and $PA = PAYPA$. Hence, $E = AA^{(1,3)} = AYPE = P^{-1}(PAY)PE = SEP^*PE$, where $S = P^{-1}Y^*A^*$. Since $E^* = E$, then it also follows that $E = EP^*PES^* \in EP^*PE\mathcal{M}_m(\mathcal{R})$ and, (b) holds.

(b)⇔(c). This equivalence follows from Lemma 2.1.

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(c)⇔(d). Let $R = E'(I - P^{-1})$. Using $R^2 = 0$, we can write

$$\begin{aligned} I + R^*R &= I + R^* - R^*P^{-1} = (I + R^*)(I - R^*P^{-1}) \\ &= (I + R^*)(PE + (P^{-1})^*E')P^{-1} = (I + R^*)(P^{-1})^*UP^{-1}, \end{aligned}$$

where $U = P^*PE + I - E$. Since $I + R^*$ is invertible, then $I + R^*R$ is invertible if and only if U is invertible. Moreover,

$$(I + R^*R)^{-1} = PU^{-1}P^*(I - R^*).$$

From this, we also obtain that $U^{-1}P^* = P^{-1}(I+R^*R)^{-1}(I+R^*)$ whenever (c) holds and, thus, the second equality of (3.1) holds.

(c)⇒(a). Define $Y = A^{(1,3)}U^{-1}P^*$. We will prove that Y is a $\{1, 3\}$-inverse of PA. Firstly, we see that

$$YPA = A^{(1,3)}U^{-1}P^*PA = A^{(1,3)}U^{-1}(P^*PE + I - E)A = A^{(1,3)}A. \tag{3.2}$$

Then $PAYPA = PA$, and thus, Y is a $\{1\}$-inverse of PA. Since $U^*E = EU$ it follows that $(PAY)^* = P(U^{-1})^*EP^* = PEU^{-1}P^* = PAY$ and so $Y \in (PA)\{1, 3\}$. □

Remark 3.2. If we replace $A^{(1,3)}$ and $(PA)^{(1,3)}$ by $A^{(1,2,3)}$ and $(PA)^{(1,2,3)}$, respectively, in Theorem 3.1, then we obtain an analogous characterization of the existence of $\{1, 2, 3\}$-inverses of the product PA.

Some applications of previous results will be develop in Section 4. Here we include an example using the incidence matrix of a graph.

Example 3.3. Let $*$ be the conjugate transpose of a complex matrix and let A be an $m \times n$ incidence matrix of a connected graph. With an application of formula (3.1) we will derive an expression of a $\{1, 3\}$-inverse of PA when P is an $m \times m$ invertible row stochastic matrix.

For any $A^{(1,3)}$, denote $E = AA^{(1,3)}$, we have $E' = \frac{1}{m}\mathbf{e}\mathbf{e}^T$ where $\mathbf{e}\mathbf{e}^T$ is the $m \times m$ matrix whose elements are all 1 (see [3, Ex. 109]). Since P is row stochastic, then $P\mathbf{e}\mathbf{e}^T = \mathbf{e}\mathbf{e}^T$ and, thus, $PE' = E'$ holds. Therefore, by (3.1),

$$(PA)^{(1,3)} = A^{(1,3)}U^{-1}P^* = A^{(1,3)}P^{-1}(I + R^*R)^{-1}(I + R^*),$$

where $U = P^*P - (P^*P + I)\frac{1}{m}\mathbf{e}\mathbf{e}^T$ and $R = \frac{1}{m}\mathbf{e}\mathbf{e}^T(I - P^{-1})$.

Now we state an analogue of the above theorem concerning the $\{1, 4\}$-inverse of the matrix product AQ.

Theorem 3.4. *Let $A \in \mathcal{M}_{m\times n}(\mathcal{R})$ be such that A has a $\{1, 4\}$-inverse, let $F = A^{(1,4)}A$ and let $Q \in \mathcal{M}_n(\mathcal{R})$ be invertible. If $F'Q = F'$, then the following conditions are equivalent:*

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(a) AQ has a $\{1,4\}$-inverse.
(b) $F \in \mathcal{M}_n(\mathcal{R})W \cap W\mathcal{M}_n(\mathcal{R})$ where $W = FQQ^*F$.
(c) $V = FQQ^* + I - F$ is invertible.
(d) $I + LL^*$ is invertible with $L = (I - Q^{-1})F'$.

In this case, there exists a $\{1,4\}$-inverse of AQ of the form

$$(AQ)^{(1,4)} = Q^*V^{-1}A^{(1,4)} = (I + L^*)(I + LL^*)^{-1}Q^{-1}A^{(1,4)}. \tag{3.3}$$

Proof. We first note that $(A^{(1,4)})^*$ is a $\{1,3\}$-inverse of A^* and $Q^*(I - A^*(A^{(1,4)})^*) = I - A^*(A^{(1,4)})^*$. An application of Theorem 3.1 to the product Q^*A^* shows that the following conditions are equivalent:

(a′) Q^*A^* has a $\{1,3\}$-inverse.
(b′) $F \in \mathcal{M}_n(\mathcal{R})FQQ^*F \cap FQQ^*F\mathcal{M}_n(\mathcal{R})$.
(c′) $U = QQ^*F + I - F$ is invertible.
(d′) $I + R^*R$ is invertible with $R = F'(I - (Q^*)^{-1})$.

From these relations, we conclude that (a), (b), (c), and (d) in this theorem are equivalent. Finally, by (3.1) we have $Y = (A^{(1,4)})^*U^{-1}Q = (A^{(1,4)})^*(Q^*)^{-1}(I + R^*R)^{-1}(I + R^*)$ is a $\{1,3\}$-inverse of $(AQ)^*$. Hence, Y^* is a $\{1,4\}$-inverse of AQ and, thus, (3.3) holds. ∎

Remark 3.5. If we replace $A^{(1,4)}$ and $(AQ)^{(1,4)}$ by $A^{(1,2,4)}$ and $(AQ)^{(1,4)}$, respectively, in Theorem 3.4, then we obtain an analogous characterization of the existence of $\{1,2,4\}$-inverses of the product AQ.

Based on previous Theorems, we derive a characterization of the existence of the Moore-Penrose inverse of a matrix product PAQ in the case that $A^\dagger$ exists.

Theorem 3.6. *Let $A \in \mathcal{M}_{m\times n}(\mathcal{R})$ be such that $A^\dagger$ exists, let $E = AA^\dagger$, $F = A^\dagger A$ and let $P \in \mathcal{M}_m(\mathcal{R})$ and $Q \in \mathcal{M}_n(\mathcal{R})$ be invertible matrices. If $PE' = E'$ and $F'Q = F'$, then the following are equivalent:*

(a) $(PAQ)^\dagger$ exists.
(b) $E \in \mathcal{M}_m(\mathcal{R})Z \cap Z\mathcal{M}_m(\mathcal{R})$ and $F \in \mathcal{M}_n(\mathcal{R})W \cap W\mathcal{M}_n(\mathcal{R})$, where $Z = EP^*PE$ and $W = FQQ^*F$.
(c) $U = P^*PE + I - E$ and $V = FQQ^* + I - F$ are invertible.
(d) $I+R^*R$ and $I+LL^*$ are invertible with $R = E'(I-P^{-1})$ and $L = (I-Q^{-1})F'$.

In this case,

$$\begin{aligned}(PAQ)^\dagger &= Q^*V^{-1}A^\dagger U^{-1}P^* \\ &= (I + L^*)(I + LL^*)^{-1}Q^{-1}A^\dagger P^{-1}(I + R^*R)^{-1}(I + R^*).\end{aligned} \tag{3.4}$$

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Proof. We know that the Moore-Penrose inverse of PAQ exists if and only if PA has a $\{1,3\}$-inverse and AQ has a $\{1,4\}$-inverse, in which case

$$(PAQ)^\dagger = (AQ)^{(1,4)}A(PA)^{(1,3)}.$$

Now, the proof of the theorem is a consequence of Theorems 3.1 and 3.4. $\square$

4. Applications. Several authors described generalized inverses of block matrices and their properties [1, 2, 4, 6, 7, 11, 12, 16, 17, 21].

In this section, some applications of Theorems 3.1, 3.4 and 3.6 are indicated.

First, we characterize the existence of a $\{1,3\}$-inverse of a 2×2 block matrix M over $\mathcal{R}$ of the form

$$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \tag{4.1}$$

where $a \in \mathcal{M}_m(\mathcal{R})$ is invertible, b, c and d are matrices over $\mathcal{R}$ of orders $k\times m$, $m\times l$ and $k\times l$, respectively. We denote by either I or 1_m the identity matrix in $\mathcal{M}_m(\mathcal{R})$.

Consider the factorization

$$M = \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1_m & 0 \\ ba^{-1} & 1_l \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} 1_m & a^{-1}c \\ 0 & 1_l \end{pmatrix} = PAQ. \tag{4.2}$$

THEOREM 4.1. *Let M as in (4.1) and let $s = d - ba^{-1}c$. Assume that $s\{1,3\} \neq \emptyset$ and let $s^{(1,3)} \in s\{1,3\}$ and $e = 1_k - ss^{(1,3)}$. Then $M\{1,3\} \neq \emptyset$ if and only if $u = 1_m + (ba^{-1})^*eba^{-1}$ is invertible. In this case, a $\{1,3\}$-inverse of M is given by*

$$M^{(1,3)} = \begin{pmatrix} \alpha u^{-1} & \alpha u^{-1}(ba^{-1})^*e - a^{-1}cs^{(1,3)} \\ -s^{(1,3)}ba^{-1}u^{-1} & s^{(1,3)}(1_k - ba^{-1}u^{-1}(ba^{-1})^*e) \end{pmatrix}, \tag{4.3}$$

where $\alpha = (1_m + a^{-1}cs^{(1,3)}b)a^{-1}$.

Proof. Let P, A and Q as in (4.2). It is easy to check that a $\{1,3\}$-inverse of A is of the form $A^{(1,3)} = \begin{pmatrix} a^{-1} & 0 \\ 0 & s^{(1,3)} \end{pmatrix}$. Then

$$E' = I - AA^{(1,3)} = \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix},$$

where $e = 1_k - ss^{(1,3)}$. Hence, $PE' = E'$ holds. Set

$$R = E'(I - P^{-1}) = \begin{pmatrix} 0 & 0 \\ eba^{-1} & 0 \end{pmatrix}, \quad I + R^*R = \begin{pmatrix} u & 0 \\ 0 & 1_k \end{pmatrix},$$

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where $u = 1_m + (ba^{-1})^* eba^{-1}$. By Theorem 3.1 there exists a $\{1,3\}$-inverse of PA if and only if u is invertible in the ring $\mathcal{M}_m(\mathcal{R})$. By (3.1), a $\{1,3\}$-inverse of PA is of the form $(PA)^{(1,3)} = A^{(1,3)} P^{-1} (I + R^* R)^{-1} (I + R^*)$. Therefore,

$$
\begin{aligned}
(PA)^{(1,3)} &= \begin{pmatrix} a^{-1} & 0 \\ 0 & s^{(1,3)} \end{pmatrix} \begin{pmatrix} 1_m & 0 \\ -ba^{-1} & 1_l \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1_k \end{pmatrix} \begin{pmatrix} 1_m & (ba^{-1})^* e \\ 0 & 1_k \end{pmatrix} \\
&= \begin{pmatrix} a^{-1}u^{-1} & a^{-1}u^{-1}(ba^{-1})^* e \\ -s^{(1,3)} ba^{-1} u^{-1} & s^{(1,3)} (1_k - ba^{-1} u^{-1} (ba^{-1})^* e) \end{pmatrix}. \qquad (4.4)
\end{aligned}
$$

Now, since Q is invertible then $PAQ\{1,3\} \neq \emptyset$ if and only if $PA\{1,3\} \neq \emptyset$ in which case $(PAQ)^{(1,3)} = Q^{-1} (PA)^{(1,3)}$. Pre-multiplying (4.4) by Q^{-1}, (4.3) is proved. ☐

We can now state the analogue of previous theorem for the characterization of the existence of a $\{1,4\}$-inverse of matrix M.

THEOREM 4.2. *Let M as in (4.1) and let $s = d - ba^{-1}c$. Assume that $s\{1,4\} \neq \emptyset$ and let $s^{(1,4)} \in s\{1,4\}$ and $f = 1_l - s^{(1,4)} s$. Then $M\{1,4\} \neq \emptyset$ if and only if $v = 1_m + a^{-1} cf (a^{-1}c)^*$ is invertible. In this case, a $\{1,4\}$-inverse of M is given by*

$$
M^{(1,4)} = \begin{pmatrix} v^{-1}\beta & -v^{-1}a^{-1}cs^{(1,4)} \\ f(a^{-1}c)^* v^{-1}\beta - s^{(1,4)} ba^{-1} & (1_l - f(a^{-1}c)^* v^{-1} a^{-1} c) s^{(1,4)} \end{pmatrix}, \qquad (4.5)
$$

where $\beta = a^{-1}(1_m + cs^{(1,4)} ba^{-1})$.

Proof. We use the factorization (4.2). Choose $A^{(1,4)} = \begin{pmatrix} a^{-1} & 0 \\ 0 & s^{(1,4)} \end{pmatrix}$, which gives $F' = I - A^{(1,4)} A = \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix}$, where $f = 1_l - s^{(1,4)} s$. Now, set

$$
L = (I - Q^{-1}) F' = \begin{pmatrix} 0 & a^{-1}cf \\ 0 & 0 \end{pmatrix}, \quad I + LL^* = \begin{pmatrix} v & 0 \\ 0 & 1_l \end{pmatrix},
$$

where $v = 1_m + a^{-1}cf(a^{-1}c)^*$. With an application of Theorem 3.4 we obtain $AQ\{1,4\} \neq \emptyset$ iff v is invertible and

$$
(AQ)^{(1,4)} = \begin{pmatrix} v^{-1}a^{-1} & -v^{-1}a^{-1}cs^{(1,4)} \\ f(a^{-1}c)^* v^{-1} a^{-1} & (1_l - f(a^{-1}c)^* v^{-1} a^{-1} c) s^{(1,4)} \end{pmatrix}. \qquad (4.6)
$$

Since $(PAQ)^{(1,4)} = (AQ)^{(1,4)} P^{-1}$. Pre-multiplying (4.6) by P^{-1}, (4.5) is proved. ☐

If $s = d - ba^{-1}c$ is regular, by Lemma 1.1 we have that $s^\dagger$ exists iff $s^*s + 1_l - s^{(1)}s$ is invertible. In this case, using previous results we can characterize the existence of the Moore-Penrose inverse of matrix M.

THEOREM 4.3. *Let M as in (4.1) and let $s = d - ba^{-1}c$. Assume that $s^\dagger$ exists and set $e = 1_k - ss^\dagger$ and $f = 1_l - s^\dagger s$. Then $M^\dagger$ exists if and only if both*

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$u = 1_m + (ba^{-1})^*eba^{-1}$ and $v = 1_m + a^{-1}cf(a^{-1}c)^*$ are invertible. In this case,

$$M^\dagger = \begin{pmatrix} \gamma & \gamma(ba^{-1})^*e - v^{-1}a^{-1}cs^\dagger \\ f(a^{-1}c)^*\gamma - s^\dagger ba^{-1}u^{-1} & \delta \end{pmatrix}, \tag{4.7}$$

where

$$\gamma = v^{-1}(a^{-1} + a^{-1}cs^\dagger ba^{-1})u^{-1},$$
$$\delta = s^\dagger + f(a^{-1}c)^*\gamma(ba^{-1})^*e - f(a^{-1}c)^*v^{-1}a^{-1}cs^\dagger - s^\dagger ba^{-1}u^{-1}(ba^{-1})^*e.$$

Proof. In view of the factorization (4.2), we have that M is Moore-Penrose invertible iff PA has a $\{1,3\}$-inverse and AQ has a $\{1,4\}$-inverse. Now, since $s^\dagger$ exists we can consider $s^{(1,3)} = s^\dagger$ in Theorem 4.1 and $s^{(1,4)} = s^\dagger$ in Theorem 4.2 to conclude that M is Moore-Penrose invertible iff both u and v defined as in the statement of this theorem are invertible. Using expressions (4.4) and (4.6), we compute $M^\dagger = (AQ)^{(1,4)}A(PA)^{(1,3)}$ and we obtain the expression (4.7). $\square$

In what sequel, let T be a matrix over $\mathcal{R}$ of the form

$$T = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}, \tag{4.8}$$

where a, b and d are matrices over $\mathcal{R}$ of orders $m \times n$, $k \times n$, and $k \times l$, respectively.

We will characterize the existence of a $\{1,3\}$-inverse of T when $a\{1,2,3\} \neq \emptyset$ and $d\{1,3\} \neq \emptyset$. Set

$$e = 1_k - dd^{(1,3)}, \quad f = 1_n - a^{(1,2,3)}a, \quad c = ebf. \tag{4.9}$$

Consider the factorization

$$T = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} = \begin{pmatrix} 1_m & 0 \\ eba^{(1,2,3)} & 1_k \end{pmatrix} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ d^{(1,3)}b & 1_l \end{pmatrix} = PAQ. \tag{4.10}$$

THEOREM 4.4. *Let e, f, and c as in (4.9) and T as in (4.10). Assume that $c\{1,3\} \neq \emptyset$ and let $c^{(1,3)} \in c\{1,3\}$. Then $T\{1,3\} \neq \emptyset$ if and only if $u = 1_m + (ba^{(1,2,3)})^*egba^{(1,2,3)}$ is invertible, where $g = 1_k - cc^{(1,3)}$. In this case,*

$$T^{(1,3)} = \begin{pmatrix} 1_n & 0 \\ -d^{(1,3)}b & 1_l \end{pmatrix} \begin{pmatrix} \sigma & \sigma(ba^{(1,2,3)})^*eg + fc^{(1,3)} \\ -\eta & d^{(1,3)} - \eta(ba^{(1,2,3)})^*eg \end{pmatrix}, \tag{4.11}$$

where $\sigma = (1 - fc^{(1,3)}eb)a^{(1,2,3)}u^{-1}$ and $\eta = d^{(1,3)}eba^{(1,2,3)}u^{-1}$.

Proof. Let P, A, and Q as in (4.10). We observe that $cc^{(1,3)} = cc^{(1,3)}e$. Using this, it is easy to check that a $\{1,3\}$-inverse of A is given by

$$A^{(1,3)} = \begin{pmatrix} a^{(1,2,3)} & fc^{(1,3)} \\ 0 & d^{(1,3)} \end{pmatrix}.$$

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Then

$$E' = I - AA^{(1,3)} = \begin{pmatrix} 1_m - aa^{(1,2,3)} & 0 \\ 0 & eg \end{pmatrix},$$

where $g = 1_k - cc^{(1,3)}$. We can see that $PE' = E'$. Then we can apply Theorem 3.1 to the product PA. With the notation $R = \hat{E}'(I - P^{-1})$, using that $ege = ge$, we have

$$R = \begin{pmatrix} 0 & 0 \\ geba^{(1,2,3)} & 0 \end{pmatrix}, \quad R^* = \begin{pmatrix} 0 & (ba^{(1,2,3)})^* eg \\ 0 & 0 \end{pmatrix}, \quad I + R^*R = \begin{pmatrix} u & 0 \\ 0 & 1_k \end{pmatrix},$$

where $u = 1_m + (ba^{(1,2,3)})^* egba^{(1,2,3)}$. By Theorem 3.1, PA has a $\{1,3\}$-inverse if and only if $I + R^*R$ is invertible, or equivalently, u is invertible. In this case, a $\{1,3\}$-inverse of PA has the form

$$(PA)^{(1,3)} = A^{(1,3)}P^{-1}(I + R^*R)^{-1}(I + R^*).$$

Substituting into this expression the matrix products

$$A^{(1,3)}P^{-1} = \begin{pmatrix} a^{(1,2,3)} & fc^{(1,3)} \\ 0 & d^{(1,3)} \end{pmatrix} \begin{pmatrix} 1_m & 0 \\ -eba^{(1,2,3)} & 1_k \end{pmatrix} = \begin{pmatrix} (1 - fc^{(1,3)}eb)a^{(1,2,3)} & fc^{(1,3)} \\ -d^{(1,3)}eba^{(1,2,3)} & d^{(1,3)} \end{pmatrix}$$

and

$$(I + R^*R)^{-1}(I + R^*) = \begin{pmatrix} u^{-1} & u^{-1}(ba^{(1,2,3)})^* eg \\ 0 & 1 \end{pmatrix},$$

we obtain, after an easy computation,

$$(PA)^{(1,3)} = \begin{pmatrix} \sigma & \sigma(ba^{(1,2,3)})^* eg + fc^{(1,3)} \\ -\eta & d^{(1,3)} - \eta(ba^{(1,2,3)})^* eg \end{pmatrix},$$

where $\sigma = (1 - fc^{(1,3)}eb)a^{(1,2,3)}u^{-1}$ and $\eta = d^{(1,3)}eba^{(1,2,3)}u^{-1}$. Now, since Q is invertible then $PAQ\{1,3\} \neq \emptyset$ if and only if $PA\{1,3\} \neq \emptyset$ in which case $(PAQ)^{(1,3)} = Q^{-1}(PA)^{(1,3)}$. The latter establishes the formula (4.11). $\square$

In order to characterize the existence of a $\{1,4\}$-inverse of T when $a\{1,4\} \neq \emptyset$ and $d\{1,2,4\} \neq \emptyset$, set

$$e = 1_k - dd^{(1,2,4)}, \quad f = 1_n - a^{(1,4)}a, \quad c = ebf \tag{4.12}$$

and consider the factorization

$$T = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} = \begin{pmatrix} 1_m & 0 \\ ba^{(1,4)} & 1_k \end{pmatrix} \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ d^{(1,2,4)}bf & 1_l \end{pmatrix} = \hat{P}\hat{A}\hat{Q}. \tag{4.13}$$

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Similarly, with an application Theorem 3.4, we derive the analogue of previous theorem.

THEOREM 4.5. *Let* e*,* f*, and* c *as in (4.12) and* T *as in (4.13). Assume that* $c\{1,4\} \neq \emptyset$ *and let* $c^{(1,4)} \in c\{1,4\}$*. Then* $T\{1,4\} \neq \emptyset$ *if and only if* $v = 1_l + d^{(1,2,4)}bhf(d^{(1,2,4)}b)^*$ *is invertible, where* $h = 1_n - c^{(1,4)}c$*. In this case,*

$$T^{(1,4)} = \begin{pmatrix} a^{(1,4)} - hf(d^{(1,2,4)}b)^*\mu & hf(d^{(1,2,4)}b)^*\rho + c^{(1,4)}e \\ -\mu & \rho \end{pmatrix} \begin{pmatrix} 1_m & 0 \\ -ba^{(1,4)} & 1_k \end{pmatrix} \tag{4.14}$$

where $\rho = v^{-1}d^{(1,2,4)}(1 - bfc^{(1,4)}e)$ *and* $\mu = v^{-1}d^{(1,2,4)}bfa^{(1,4)}$*.*

We will characterize the Moore-Penrose invertibility of T when there exist $a^\dagger$ and $d^\dagger$. Set

$$e = 1_k - dd^\dagger, \quad f = 1_n - a^\dagger a, \quad c = ebf. \tag{4.15}$$

We begin by finding conditions for the existence of $c^\dagger$.

PROPOSITION 4.6. *Let* e, f *and* c *as in (4.15). If any of the following conditions hold, then* $c^\dagger$ *exists.*

(i) $w = cc^* + dd^*$ *is invertible.*
(ii) $z = c^*c + a^*a$ *is invertible.*

Proof. (i) First, we prove that if w is invertible then c is regular. Let $x = c^*w^{-1}$. Since $ew = cc^*$ we also have $e = cc^*w^{-1} = cx$. Using this, we get $cxc = ec = c$. Hence, x is a 1-inverse of c.

Now, choose $c^{(1)} = c^*w^{-1}$. Then $cc^{(1)} = e$ and by Lemma 1.1, $c^\dagger$ exists if and only if $v = cc^* + 1 - e$ is invertible. We only need to show that v is invertible. Since $(cc^* + 1 - e)(dd^* + e) = cc^* + dd^*$ we have that $cc^* + 1 - e$ is invertible because both w and $dd^* + e$ are invertible, the last one due to the fact that $d^\dagger$ exists.

(ii). The proof is similar to the case (i). ∎

THEOREM 4.7. *Let* T *as in (4.8) and let* e*,* f*, and* c *as in (4.15). If* $c^\dagger$ *exists, then* $T^\dagger$ *exists if and only if both* $u = 1_m + (ba^\dagger)^*egba^\dagger$ *and* $v = 1_l + d^\dagger bhf(d^\dagger b)^*$ *are invertible, where* $g = 1_k - cc^\dagger$ *and* $h = 1_n - c^\dagger c$*. In this case,*

$$\begin{pmatrix} a & 0 \\ b & d \end{pmatrix}^\dagger = \begin{pmatrix} (1 - hf(d^\dagger b)^*v^{-1}d^\dagger b)\sigma & \gamma \\ -\rho ba^\dagger u^{-1} & \rho(1_l - ba^\dagger u^{-1}(ba^\dagger)^*eg) \end{pmatrix}, \tag{4.16}$$

where $\rho = v^{-1}d^\dagger(1_l - bc^\dagger)$*,* $\sigma = (1_k - c^\dagger b)a^\dagger u^{-1}$ *and*

$$\gamma = c^\dagger + hf(d^\dagger b)^*\rho(1_l - ba^\dagger u^{-1}(ba^\dagger)^*eg) + \sigma(ba^\dagger)^*eg. \tag{4.17}$$

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Proof. We can apply Theorem 4.4 with $a^{(1,2,3)} = a^\dagger$, $d^{(1,3)} = d^\dagger$, and $c^{(1,3)} = c^\dagger$ to obtain that $T^{(1,3)}$ exists iff $u = 1_m + (ba^\dagger)^* egba^\dagger$ is invertible and, using that $d^\dagger e = 0$, a $\{1,3\}$-inverse of T is of the form

$$T^{(1,3)} = \begin{pmatrix} 1_n & 0 \\ -d^\dagger b & 1_l \end{pmatrix} \begin{pmatrix} \sigma & \sigma(ba^\dagger)^* eg + fc^\dagger \\ 0 & d^\dagger \end{pmatrix} = \widetilde{Q}X,$$

where $\sigma = (1 - c^\dagger b)a^\dagger u^{-1}$. Similarly, we apply Theorem 4.5 to derive that $T^{(1,4)}$ exists iff $v = 1_l + d^\dagger bhf(d^\dagger b)^*$ is invertible, and a $\{1,4\}$-inverse of T is of the form

$$T^{(1,4)} = \begin{pmatrix} a^\dagger & hf(d^\dagger b)^* \rho + c^\dagger e \\ 0 & \rho \end{pmatrix} \begin{pmatrix} 1_m & 0 \\ -ba^\dagger & 1_k \end{pmatrix} = Y\widetilde{P}$$

where $\rho = v^{-1} d^\dagger (1_l - bc^\dagger)$.

We now compute $T^\dagger = T^{(1,4)} T T^{(1,3)} = Y\widetilde{P}T\widetilde{Q}X$. One sees that

$$\widetilde{P}T\widetilde{Q} = \begin{pmatrix} a & 0 \\ bf - dd^\dagger b & d \end{pmatrix}.$$

Using that $\rho(bf - dd^\dagger b) = -v^{-1} d^\dagger b(a^\dagger a + c^\dagger c)$ and $\rho d = v^{-1} d^\dagger d$, we have

$$Y\widetilde{P}T\widetilde{Q} = \begin{pmatrix} (1 - hf(d^\dagger b)^* v^{-1} d^\dagger b)(a^\dagger a + c^\dagger c) & hf(d^\dagger b)^* v^{-1} d^\dagger d \\ -v^{-1} d^\dagger b(a^\dagger a + c^\dagger c) & v^{-1} d^\dagger d \end{pmatrix}.$$

Using $(a^\dagger a + c^\dagger c)\sigma = \sigma$ we obtain

$$T^\dagger = Y\widetilde{P}T\widetilde{Q}X = \begin{pmatrix} (1 - hf(d^\dagger b)^* v^{-1} d^\dagger b)\sigma & \gamma \\ -v^{-1} d^\dagger b\sigma & -v^{-1} d^\dagger b\sigma(ba^\dagger)^* eg + \rho \end{pmatrix},$$

where $\gamma = (1 + hf(d^\dagger b)^* v^{-1} d^\dagger b)(\sigma(ba^\dagger)^* eg + c^\dagger) + hf(d^\dagger b)^* v^{-1} d^\dagger$.

Finally, the formula in this theorem is proved by taking into account that $v^{-1} d^\dagger b\sigma = \rho ba^\dagger u^{-1}$. $\square$

We recall that a ring $\mathcal{R}$ with involution has the Gelfand-Naimark property (GN-property) if $1 + x^* x$ is invertible for all $x \in \mathcal{R}$.

We can rewrite u and v in Theorem 4.7 as $u = 1_m + (egba^\dagger)^*(egba^\dagger)$ and $v = 1_l + d^\dagger bhf(d^\dagger bhf)^*$. On account of this, we obtain the next corollary.

Corollary 4.8. *Let $\mathcal{R}$ be a ring with involution such that it has the GN-property. Consider $T = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ with $a, b, d \in \mathcal{R}$. Let e, f, and c as in (4.15). If $a^\dagger$, $d^\dagger$ and $c^\dagger$ exists then $T^\dagger$ exists, and it is given by (4.16)–(4.17).*

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We find, after some computations, that our formula (4.16)–(4.17) is the same as the expression given by (10)–(19) in [14, Section 2.2] for the Moore-Penrose inverse of a 2×2 lower triangular matrix. Here, we conclude that Corollary 4.8 generalizes the main result of [13, Section 2.2] to more general conditions that those assumed therein.

COROLLARY 4.9. *Let* $T = \begin{pmatrix} a & 0 \\ b & d \end{pmatrix}$ *be such that* $a^\dagger$ *exists and* d *is an invertible matrix of order* $k \times k$. *Then* $T^\dagger$ *exists if and only if* $v = 1_k + d^{-1}bf(d^{-1}b)^*$ *is invertible. In this case*

$$\begin{pmatrix} a & 0 \\ b & d \end{pmatrix}^\dagger = \begin{pmatrix} (1_n - f(d^{-1}b)^* v^{-1} d^{-1} b) a^\dagger & f(d^{-1}b)^* v^{-1} d^{-1} \\ -v^{-1} d^{-1} b a^\dagger & v^{-1} d^{-1} \end{pmatrix},$$

Proof. Follows from previous theorem with $d^\dagger = d^{-1}$. Then $e = 0$, $c = 0$, $g = 1_k$, and $h = 1_n$ and (4.16) reduces to the formula in this corollary. ∎

For the special case of a companion matrix of the form

$$\begin{pmatrix} 0 & a \\ 1_k & \mathbf{b} \end{pmatrix} = \begin{pmatrix} a & 0 \\ \mathbf{b} & 1_k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1_k & 0 \end{pmatrix} = TU,$$

where $a \in \mathcal{R}$ is Moore-Penrose invertible and $\mathbf{b} \in \mathcal{M}_{k\times 1}(\mathcal{R})$, using that $(TU)^\dagger = U^* T^\dagger$ we obtain from previous corollary that $(TU)^\dagger$ exists iff $v = 1_k + \mathbf{b}(1 - a^\dagger a)\mathbf{b}^*$ is invertible. In this case,

$$(TU)^\dagger = \begin{pmatrix} -v^{-1}\mathbf{b}a^\dagger & v^{-1} \\ (1 - f\mathbf{b}^* v^{-1} \mathbf{b}) a^\dagger & f\mathbf{b}^* v^{-1} \end{pmatrix},$$

which is the result in [19, Theorem 2.1].

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