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A TOTAL POSITIVITY PROPERTY OF THE MARCHENKO-PASTUR LAW*

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Abstract. A property of the Marchenko-Pastur measure related to total positivity is presented. The theoretical results are applied to the accurate computation of the roots of the corresponding orthogonal polynomials, an important issue in the construction of Gaussian quadrature formulas.

Key words. Total positivity, Marchenko-Pastur law, High relative accuracy, Jacobi matrix, Eigenvalue computation.

AMS subject classifications. 65F15, 15B48, 15A23, 15B52.

1. Introduction. Several important properties of Pascal matrices, a relevant example of totally positive matrices, have been included in [5]. In the field of numerical linear algebra, it is known that, if we have an accurate bidiagonal decomposition of a totally positive matrix, the results of Koev [16, 17] show that many computations (linear system solving, least squares computation, eigenvalue and singular value computation) can be performed with high relative accuracy. From now on, given a totally positive matrix A, we will denote by $\mathcal{BD}(A)$ the matrix containing all the nontrivial entries of the bidiagonal decomposition of A [17].

This bidiagonal decomposition is related to the complete Neville elimination of the matrix A, which (when no row exchanges are needed, as it will happen in our case) consists of computing the Neville elimination of A and also of A^T. In the symmetric case, since $A^T = A$ only the Neville elimination of A is needed for obtaining the $\mathcal{BD}(A)$. A detailed explanation can be seen in [17, 22].

In this context, a remarkable property of a Pascal matrix (the fact that its corresponding $\mathcal{BD}(A)$ is the matrix with all the entries equal to 1, i.e., $\mathcal{BD}(A)$ equals `ones(n,n)`) was used by Koev in [17] to perform accurate computations with Pascal matrices. This fact has recently been analyzed in detail and applied to computing

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with high relative accuracy in [1].

On the other hand, in many applications (for example orthogonal polynomial computation [10]) the involved matrices have tridiagonal structure, and the total positivity properties (and related properties) of this type of matrices have recently been studied in [4](see also [26, 7]).

So, similarly to the Pascal case, the first goal of this work has an aesthetic appeal: to find the tridiagonal matrix whose $\mathcal{BD}(A)$ is the tridiagonal matrix with diagonal, sub-diagonal and super-diagonal entries equal to 1, and discover the significance of that matrix in applications. This task is done in Section 2, where we find that this matrix is the Jacobi matrix of the Marchenko-Pastur measure with parameter $c = 1$.

Let us notice that all our Jacobi matrices are not only tridiagonal, but also symmetric. For a tridiagonal symmetric matrix A, the Neville elimination for triangularizing A is the same as Gaussian elimination (without pivoting), and so its bidiagonal decomposition is $A = LDL^T$ (where D is a diagonal matrix and L is a lower bidiagonal matrix with diagonal entries equal to 1). Therefore the matrix $\mathcal{BD}(A)$ is also tridiagonal and symmetric, and has as diagonal entries the diagonal entries of D and as sub-diagonal and super-diagonal entries the sub-diagonal entries of L.

In Section 3, the general case of parameter $c \in (0, \infty)$ is analyzed. The Marchenko-Pastur law plays a very important role in random matrix theory in connection with the empirical spectral distribution of sample covariance matrices of large size [21, 2, 6]. An application of this theory to signal processing can be found in [24]. The Marchenko-Pastur law is also used in free probability theory, being the free analogue of the Poisson measure [14, 3].

Naturally associated to a measure is the problem of numerical computation of integrals with respect to that measure. An important class of quadrature formulas are Gaussian quadrature formulas, whose nodes are the zeros of the corresponding orthogonal polynomials. Section 4 is devoted to this applied aspect of our work and includes numerical experiments illustrating the usefulness of the total positivity properties.

2. The Marchenko-Pastur law of parameter $c = 1$. If we start by considering several tridiagonal $\mathcal{BD}(A)$ (for different orders n) with all the diagonal, sub-diagonal and super-diagonal entries equal to 1, and we compute the eigenvalues of A by using the algorithm `TNEigenvalues` of Koev [16], it can be seen that those eigenvalues are always contained in the open interval $(0, 4)$.

A measure whose corresponding orthogonal polynomials have all their roots in $(0, 4)$ is the Marchenko-Pastur law, so our goal is to prove that our conjecture is right: for a given order n, the Jacobi matrix J of the monic orthogonal polynomials with

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respect to the Marchenko-Pastur measure has as $\mathcal{BD}(J)$ the tridiagonal matrix of order n whose diagonal, sub-diagonal and super-diagonal entries are all equal to 1.

Let us denote by $p_k(x)$ (for $k = 0, 1, 2, \ldots$) the monic orthogonal polynomials with respect to the measure $d\lambda$ defined on the real interval $[a, b]$. Then $p_k(x)$ satisfy a three-term recurrence relation:

$$\begin{aligned}
&p_0(x) = 1,\\
&p_1(x) = x - \alpha_0,\\
&p_{k+1}(x) = (x - \alpha_k)p_k(x) - \beta_k p_{k-1}(x), \quad k = 1, 2, \ldots,
\end{aligned}$$

with $\beta_k > 0$.

The Jacobi matrix associated with the measure $d\lambda$ is

$$J_\infty = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & & & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ & \sqrt{\beta_2} & \alpha_2 & \sqrt{\beta_3} & \\ & & \ddots & \ddots & \ddots \\ 0 & & & & \end{bmatrix}. \tag{2.1}$$

The recursion coefficients α_k and β_k can be computed by using

$$\alpha_k = \frac{\Delta'_{k+1}}{\Delta_{k+1}} - \frac{\Delta'_k}{\Delta_k}, \quad k = 0, 1, 2, \ldots; \quad \beta_k = \frac{\Delta_{k+1}\Delta_{k-1}}{\Delta_k^2}, \quad k = 1, 2, \ldots \tag{2.2}$$

where

$$\Delta_0 = 1, \quad \Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}, \quad n = 1, 2, 3, \ldots, \tag{2.3}$$

$$\Delta'_0 = 0, \ \Delta'_1 = \mu_1, \ \Delta'_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}, \quad n = 2, 3, \ldots, \tag{2.4}$$

and $\mu_i = \int_a^b x^i d\lambda$ is the i-th moment with respect to the measure $d\lambda$ [10].

The *Marchenko-Pastur distribution* (with parameter 1) is the probability measure on the interval $[0, 4]$ with density:

$$f(x) = \frac{1}{2\pi x}\sqrt{x(4 - x)}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 30, pp. 106-117, March 2015

Its moments are the *Catalan numbers* [15]:

$$c_n = \binom{2n}{n}\frac{1}{n+1}, \quad n = 0, 1, \ldots$$

THEOREM 2.1. *Let $n \in \mathbb{N}$. The Jacobi matrix J of order $n+1$ of the monic orthogonal polynomials with respect to the Marchenko-Pastur measure is*

$$J = \begin{bmatrix} 1 & 1 & & & 0 \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ 0 & & & 1 & 2 \end{bmatrix}.$$

Proof. As the moments with respect to the Marchenko-Pastur measure are the Catalan numbers, and the determinants of the Hankel matrices whose entries are the Catalan numbers are equal to 1 [19], we obtain that $\Delta_k = 1$ for $k = 1, 2, \ldots, n+1$ (see (2.3)).

Using the formula in page 21 of [11] with $\alpha_j = j$ for $j = 0, \ldots, n-1$ and $\alpha_n = n+1$, we get that $\Delta'_{n+1} = 2n+1$, and therefore $\Delta'_k = 2k-1$ for $k = 2, 3, \ldots, n+1$ (see (2.4)). Let us point out here that this formula can also be found in Theorem 3 of [19], but in this case the roles of i and j in the first product of the formula must be interchanged.

Taking these results and formulas (2.2), (2.3) and (2.4) into account, we derive that $\alpha_0 = 1$, $\alpha_k = 2$ for $k = 1, 2, \ldots, n$, and $\beta_k = 1$ for $k = 1, 2, \ldots, n$, and in consequence, by (2.1), the Jacobi matrix J of order $n+1$ of the monic orthogonal polynomials with respect to the Marchenko-Pastur measure is the one included in the statement of this theorem. ∎

THEOREM 2.2. *Let J be the Jacobi matrix of order n of the monic orthogonal polynomials with respect to the Marchenko-Pastur measure. Then, the matrix $\mathcal{BD}(J)$ containing its bidiagonal decomposition is the tridiagonal matrix of order n whose diagonal, sub-diagonal and super-diagonal entries are all equal to 1.*

Proof. By applying the theoretical results recalled in Section 2 of [22], it is easily seen that the Neville elimination of J (which, as indicated in the Introduction, in the symmetric tridiagonal case is the same as Gaussian elimination) can be performed without row and column exchanges. Therefore, carrying out the Gaussian elimination it is seen the all the nontrivial entries of L and D in $J = LDL^T$ are equal to 1. Consequently, the matrix $\mathcal{BD}(J)$ is the tridiagonal matrix of order n whose diagonal, sub-diagonal and super-diagonal entries are all equal to 1. ∎

Electronic Journal of Linear Algebra ISSN 1081-3810
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REMARK 2.3. The results of [17] and [9] and the fact that $\mathcal{BD}(J)$ is a tridiagonal matrix with all its diagonal, sub-diagonal and super-diagonal entries positive imply J is a *totally positive* matrix. Note that J is not a *strictly totally positive* matrix, but a *totally nonnegative* matrix in the terminology of [17]. Although, according to a result of page 100 of [26] (see also Section 2 of [4]), the fact that J is positive definite (as derived from the proof of Theorem 2.2) and it has all its off-diagonal entries nonnegative implies J is totally positive, our emphasis on the construction of $\mathcal{BD}(J)$ is important for the applications (see Section 4).

3. The general case of parameter $c \in (0, \infty)$. Following the presentation in [2], we recall the basic result concerning the Marchenko-Pastur law.

The eigenvalues $s_j^{(n)}$ (for $j = 1, \ldots, n$) of the sample covariance matrix $\frac{1}{n} X_n X_n^T$, where X_n is a $p \times n$ matrix whose entries are independent and identically distributed with mean 0 and variance 1, and $\frac{p}{n} \to c$ when $n \to \infty$, satisfy

$$\frac{1}{p} \#\{s_j^{(n)} : s_j^{(n)} < x\} \to F(x),$$

almost surely, where $F'(x) = f(x)$ with

$$f(x) = \begin{cases} \frac{1}{2\pi x c} \sqrt{(x-a)(b-x)}, & a < x < b, \\ 0, & \text{otherwise}, \end{cases}$$

where $0 < c \leq 1$, $a = (\sqrt{c}-1)^2$ and $b = (\sqrt{c}+1)^2$. When $c > 1$, there is an additional Dirac measure at $x = 0$ of mass $1 - \frac{1}{c}$.

The *Marchenko-Pastur law* with parameter c is the probability measure on the interval $[a, b] = [(\sqrt{c} - 1)^2, (\sqrt{c} + 1)^2]$ which has the above density function $f(x)$.

A useful presentation which illustrates the above result by using the MATLAB function `randn(p,n)`, and histograms (which fill the area under the curve $f(x)$) of the eigenvalues of the sample covariance matrix is given in [6].

The kth moment of the Marchenko-Pastur law is

$$\mu_k(c) = \sum_{r=1}^{k} \frac{1}{r} \binom{k}{r-1} \binom{k-1}{r-1} c^{r-1}, \quad k = 1, 2, \ldots$$

while for $k = 0$ we have $\mu_0 = 1$ for $0 < c \leq 1$ and $\mu_0 = \frac{1}{c}$ for $c > 1$ [15].

The fact that $\mu_0 = \frac{1}{c}$ for $c > 1$ explains the role of the Dirac measure at $x = 0$. For example, if $c = 2$ half of the eigenvalues are zero (and so the mass is equal to $1 - \frac{1}{2}$) and the area under the curve $f(x)$ is equal to $\mu_0 = \frac{1}{2}$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 30, pp. 106-117, March 2015

As we see, the moment μ_k is a polynomial in c whose coefficients are the *Narayana numbers* ([25], sequence A001263):

$$T(k,r) = \frac{1}{r}\binom{k}{r-1}\binom{k-1}{r-1}.$$

EXAMPLE 3.1. As we have said above, for $c = 1$ the moments are the Catalan numbers ([25], sequence A000108):

$$1, 1, 2, 5, 14, 42, 132, 429, \ldots$$

EXAMPLE 3.2. For $c = 2$ the moments are the *little Schröder numbers* ([25], sequence A001003) (as we have seen, the first one must be changed from 1 to $\frac{1}{2}$, since $\mu_0 = \frac{1}{2}$) [27]:

$$\frac{1}{2}, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, \ldots$$

The Jacobi matrix in this general case follows from the three-term recurrence relation for the monic orthogonal polynomials associated to the Marchenko-Pastur measure given in Section 6 of [13]. However, it must be observed that in [13] no distinction is made for the cases $0 < c \le 1$ (in that case $\alpha_0 = \frac{\mu_1}{\mu_0} = 1$) and $c > 1$ (now we have $\alpha_0 = \frac{\mu_1}{\mu_0} = c$). Taking this into account we have the result below:

THEOREM 3.3. *Let $n \in \mathbb{N}$. The Jacobi matrix J_l of order n of the monic orthogonal polynomials with respect to the Marchenko-Pastur measure with parameter $0 < c \le 1$ and the Jacobi matrix J_g of order n of the monic orthogonal polynomials with respect to the Marchenko-Pastur measure with parameter $c > 1$ are:*

$$J_l = \begin{bmatrix} 1 & \sqrt{c} & & & 0 \\ \sqrt{c} & 1+c & \sqrt{c} & & \\ & \ddots & \ddots & \ddots & \\ & & \sqrt{c} & 1+c & \sqrt{c} \\ 0 & & & \sqrt{c} & 1+c \end{bmatrix}, \quad J_g = \begin{bmatrix} c & \sqrt{c} & & & 0 \\ \sqrt{c} & 1+c & \sqrt{c} & & \\ & \ddots & \ddots & \ddots & \\ & & \sqrt{c} & 1+c & \sqrt{c} \\ 0 & & & \sqrt{c} & 1+c \end{bmatrix}.$$

Finally, the next theorem shows the matrices representing the bidiagonal decomposition of J_l and J_g. Since both matrices $\mathcal{BD}(J_l)$ and $\mathcal{BD}(J_g)$ are tridiagonal matrices with all their diagonal, sub-diagonal and super-diagonal entries positive we have that both J_l and J_g are totally positive matrices.

THEOREM 3.4. *Let J_l be the Jacobi matrix of order n of the monic orthogonal polynomials with respect to the Marchenko-Pastur measure with parameter $0 < c \le 1$*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 106-117, March 2015

and J_g be the Jacobi matrix of order n of the monic orthogonal polynomials with respect to the Marchenko-Pastur measure with parameter $c > 1$. Then, the matrices containing their bidiagonal decompositions are:

$$\mathcal{BD}(J_l) = \begin{bmatrix} 1 & \sqrt{c} & & & 0 \\ \sqrt{c} & 1 & \sqrt{c} & & \\ & \ddots & \ddots & \ddots & \\ & & \sqrt{c} & 1 & \sqrt{c} \\ 0 & & & \sqrt{c} & 1 \end{bmatrix}, \quad \mathcal{BD}(J_g) = \begin{bmatrix} c & \frac{1}{\sqrt{c}} & & & 0 \\ \frac{1}{\sqrt{c}} & c & \frac{1}{\sqrt{c}} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{\sqrt{c}} & c & \frac{1}{\sqrt{c}} \\ 0 & & & \frac{1}{\sqrt{c}} & c \end{bmatrix}.$$

Proof. Analogously to the proof of Theorem 2.2, the result is easily seen by performing the Neville elimination (which in this case is the same as Gaussian elimination) on the matrices J_l and J_g. ☐

The fact that the Jacobi matrices we are considering are totally positive allows us to compute their eigenvalues with high relative accuracy by using the algorithms of Koev [16, 17, 18] starting from their bidiagonal decompositions, which we have given explicitly in Theorems 2.2 and 3.4.

In addition, using the results of [12] we see that all the Jacobi matrices we are considering are *oscillatory*, and so the eigenvectors of those matrices also possess additional theoretical properties, such as the fact that any eigenvector corresponding to the jth largest eigenvalue has exactly $j-1$ sign changes among its components (see Chapter II of [8]).

4. Orthogonal polynomials and Gauss quadrature rules. The three-term recurrence relation of the monic orthogonal polynomials corresponding to the Jacobi matrices in Theorem 3.3 follows:

$$\begin{aligned} & p_0(x) = 1, \\ & p_1(x) = x - 1, \\ & p_{k+1}(x) = (x-(1+c))p_k(x) - cp_{k-1}(x), \quad k = 1, 2, \ldots \end{aligned} \tag{4.1}$$

in the case $c \le 1$, and

$$\begin{aligned} & p_0(x) = 1, \\ & p_1(x) = x - c, \\ & p_{k+1}(x) = (x-(1+c))p_k(x) - cp_{k-1}(x), \quad k = 1, 2, \ldots \end{aligned} \tag{4.2}$$

in the case $c > 1$.

A careful reading of Section 6 of [13] (see also Section 4 of [20]) shows that, if $\{U_n(t)\}_n$ are the Chebyshev polynomials of the second kind (in $[-1, 1]$), the following

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 106-117, March 2015

relationship between the orthogonal polynomials corresponding to the Marchenko-Pastur measure and the polynomials $U_n\left(\frac{x-c-1}{2\sqrt{c}}\right)$ (which are the Chebyshev polynomials of the second kind shifted to the interval $[(\sqrt{c}-1)^2, (\sqrt{c}+1)^2]$) holds:

$$p_n(x) = (\sqrt{c})^n U_n\left(\frac{x-c-1}{2\sqrt{c}}\right) + (\sqrt{c})^{n-1} U_{n-1}\left(\frac{x-c-1}{2\sqrt{c}}\right) \tag{4.3}$$

in the case $c > 1$.

This relation can be proved by using the recurrence relation (4.2) and the recurrence relation for the Chebyshev polynomials of the second kind:

$$\begin{aligned}
&U_0(t) = 1,\\
&U_1(t) = 2t,\\
&U_{n+1}(t) = 2tU_n(t) - U_{n-1}(t), \quad n = 1, 2, \ldots
\end{aligned}$$

Analogously, by using (4.1), we can obtain the corresponding relationship for the case $c \le 1$ (not given in [13, 20]):

$$p_n(x) = (\sqrt{c})^n U_n\left(\frac{x-c-1}{2\sqrt{c}}\right) + (\sqrt{c})^{n+1} U_{n-1}\left(\frac{x-c-1}{2\sqrt{c}}\right). \tag{4.4}$$

Taking $c = 1$ in (4.4), we get

$$p_n(x) = U_n\left(\frac{x}{2} - 1\right) + U_{n-1}\left(\frac{x}{2} - 1\right).$$

Now, let us consider the Chebyshev polynomials of the fourth kind (in [-1,1]), given by the following recurrence relation:

$$\begin{aligned}
&W_0(t) = 1,\\
&W_1(t) = 2t + 1,\\
&W_{n+1}(t) = 2tW_n(t) - W_{n-1}(t), \quad n = 1, 2, \ldots
\end{aligned}$$

From the recurrence relations it is easily seen (see Chapter 1 of [23]) that

$$W_n(t) = U_n(t) + U_{n-1}(t).$$

Consequently, the following relation between the orthogonal polynomials corresponding to the Marchenko-Pastur measure for $c = 1$ and the Chebyshev polynomials of the fourth kind shifted to $[0, 4]$ follows:

$$p_n(x) = W_n\left(\frac{x}{2} - 1\right).$$

It is also known (see Section 2.2 of [23]) that the roots of $W_n(t)$ are (in decreasing order)

$$t_k = \cos\frac{2k\pi}{2n+1}, \quad k = 1, \ldots, n,$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 106-117, March 2015

and so the roots of $p_n(x)$ are

$$x_k = 2\left(1 + \cos\frac{2k\pi}{2n+1}\right), \quad k = 1, \ldots, n. \tag{4.5}$$

If one needs to compute numerically the integral of a continuous function f with respect to the Marchenko-Pastur measure, a usual choice is to construct Gauss quadrature formulas. As it is well known [10], the nodes of those formulas are the zeros of the corresponding orthogonal polynomials. More precisely, the nodes of a n-point Gauss quadrature formula are the roots of the orthogonal polynomial of degree n, and those roots are the eigenvalues of the corresponding Jacobi matrix of order n.

The numerical experiments included in this section illustrate the relevance of the total positivity of the Jacobi matrix for computing its eigenvalues with high relative accuracy. As seen in [22], this property is more important when computing with matrices with high condition numbers. This is, for instance, the case of the Marchenko-Pastur distribution with $c = 1$, where there are positive eigenvalues very close to zero.

Example 4.1. Let J_1 be the Jacobi matrix of the orthogonal polynomials of the Marchenko-Pastur distribution with $c = 1$. We will compute the eigenvalues of the three matrices J_1 of orders 10, 100 and 500 by means of our approach, which computes the eigenvalues from $\mathcal{BD}(J_1)$ using the algorithm `TNEigenvalues` [16], by using the command `eig` from Matlab and by using formula (4.5) in Matlab. The algorithm `TNEigenvalues` computes accurate eigenvalues of a totally positive matrix by using its bidiagonal factorization, and its implementation in Matlab can be taken from [18].

It is important to observe that, as we have seen, the (exact) matrix $\mathcal{BD}(J_1)$ is explicitly known, and so it is not necessary to compute it starting from J_1. This is also true for the general case and it is a crucial fact to guarantee high relative accuracy in the computation of the eigenvalues, since in general the elimination process to compute the bidiagonal decomposition can introduce inaccuracies.

The relative errors obtained in these computations (err_1, err_2 and err_3 respectively) are presented in Table 4.1. The condition numbers of the matrices are also included in Table 4.1.

As traditional algorithms for computing eigenvalues of ill-conditioned totally positive matrices only compute the largest eigenvalues with guaranteed relative accuracy, and the tiny eigenvalues may be computed with no relative accuracy at all (see [16]), only the relative errors obtained in the computations of the smallest eigenvalue of the matrices J_1 are showed in Table 1.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 30, pp. 106-117, March 2015

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The relative error of each computed eigenvalue is obtained by using the eigenvalues calculated in MATLAB by means of `eig` by using variable precision arithmetic (`vpa`).

n	$\kappa_2(J_1)$	err_1	err_2	err_3
10	1.8e+2	4.3e-17	1.4e-15	1.9e-15
100	1.6e+4	3.6e-16	1.8e-12	3.2e-13
500	4.1e+5	7.6e-16	5.6e-11	1.0e-11

TABLE 4.1
Relative errors in Example 4.1.

The results appearing in Table 4.1 indicate that, while the relative accuracy with which the command `eig` from MATLAB computes the smallest eigenvalue of the matrix J_1 decreases as the condition number $\kappa_2(J_1)$ of these matrices increases, our approach computes the smallest eigenvalues of the three matrices with high relative accuracy. Let us observe that in the case of $n = 500$, while the relative error obtained in the computation of the smallest eigenvalue by means of `eig` from MATLAB is $5.6e - 11$ (which is related to the fact that $\kappa_2(J_1) = 4.1e + 5$), the relative error obtained when using our approach is $7.6e - 16$

As for the use of formula (4.5), when it is used in floating point arithmetic it provides a perfect example of cancellation: for k close to n and n large, $\cos \frac{2k\pi}{2n+1}$ is very close to -1, and so several significant digits are lost.

Now we include two numerical experiments illustrating the good behaviour of our approach when computing the eigenvalues of the Jacobi matrix corresponding to the Marchenko-Pastur measure in the cases $c \leq 1$ and $c > 1$.

EXAMPLE 4.2. Let $J_{0.97}$ be the Jacobi matrix of the orthogonal polynomials of the Marchenko-Pastur distribution with $c = 0.97$. We will compute the eigenvalues of the matrices $J_{0.97}$ of orders 10, 100 and 500 by means our approach and by using the command `eig` from MATLAB. The condition numbers of the three matrices $J_{0.97}$ and the relative errors resulting in the computations of their smallest eigenvalues (err_1 and err_2 respectively) are presented in Table 4.2.

n	$\kappa_2(J_{0.97})$	err_1	err_2
10	1.5e+2	1.4e-16	1.1e-14
100	5.7e+3	7.8e-16	4.7e-13
500	1.5e+4	3.2e-15	3.5e-13

TABLE 4.2
Relative errors in Example 4.2.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 30, pp. 106-117, March 2015

Let us observe that, as in the previous example, while the relative accuracy with which the command `eig` from MATLAB computes the smallest eigenvalue of the Jacobi matrix $J_{0.97}$ decreases as the condition number $\kappa_2(J_{0.97})$ of these matrices increases, our approach computes the smallest eigenvalues of the three matrices with high relative accuracy.

EXAMPLE 4.3. Let $J_{1.02}$ be the Jacobi matrix of the orthogonal polynomials of the Marchenko-Pastur distribution with $c = 1.02$. As in Example 4.2, we will compute the eigenvalues of the matrices $J_{1.02}$ of orders 10, 100 and 500 by means our approach and by using the command `eig` from MATLAB. The condition numbers of the three matrices $J_{1.02}$ and the relative errors resulting in the computations of their smallest eigenvalues (err_1 and err_2 respectively) are presented in Table 4.3.

n	$\kappa_2(J_{1.02})$	err_1	err_2
10	1.6e+2	4.6e-16	2.1e-15
100	7.9e+3	8.1e-16	1.6e-13
500	3.2e+4	5.5e-17	9.2e-13

TABLE 4.3
Relative errors in Example 4.3.

The results included in this table show that our approach and the command `eig` from MATLAB behave as in the other two examples included in this section. While the relative accuracy with which the command `eig` from MATLAB computes the smallest eigenvalue of the Jacobi matrix $J_{1.02}$ decreases as the condition number $\kappa_2(J_{1.02})$ of these matrices increases, our approach computes the smallest eigenvalues of the three matrices with high relative accuracy.

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ELA

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