Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 30, pp. 80-84, February 2015

ELA

A NORM INEQUALITY FOR PAIRS OF COMMUTING POSITIVE SEMIDEFINITE MATRICES*

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Abstract. For $i = 1, \ldots, k$, let A_i and B_i be positive semidefinite matrices such that, for each i, A_i commutes with B_i. It is shown that, for any unitarily invariant norm,

$$\left|\left|\left| \sum_{i=1}^{k} A_i B_i \right|\right|\right| \leq \left|\left|\left| \Big(\sum_{i=1}^{k} A_i\Big) \Big(\sum_{i=1}^{k} B_i\Big) \right|\right|\right|.$$

The $k = 2$ case was recently conjectured by Hayajneh and Kittaneh and proven by them for the trace norm and the Hilbert-Schmidt norm. A simple application of this norm inequality answers a question of Bourin in the affirmative.

Key words. Matrix Inequality, Unitarily Invariant Norm, Positive semidefinite matrix.

AMS subject classifications. 15A60.

1. Preliminaries. In this paper, we denote the vectors of eigenvalues and singular values of a matrix A by $\lambda(A)$ and $\sigma(A)$, respectively. We adhere to the convention to sort singular values, and eigenvalues as well whenever they are real, in non-increasing order. In general, for a real vector x, we will write $x^\downarrow$ for the vector with the same components as x but sorted in non-increasing order.

For real n-dimensional vectors x and y, we say that x is *weakly majorised* by y, denoted $x \prec_w y$, if and only if for $k = 1, \ldots, n$, $\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow$. We say that x is *majorised* by y, denoted $x \prec y$, if and only if $x \prec_w y$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. If, moreover, x and y are non-negative, we say that x is weakly *log-majorised* by y, denoted $x \prec_{w,\log} y$, if and only if for $k = 1, \ldots, n$, $\prod_{i=1}^k x_i^\downarrow \leq \prod_{i=1}^k y_i^\downarrow$.

According to Weyl's Majorant Theorem ([1] Theorem II.3.6, or [4], Theorem 2.4), the vector of singular values of any matrix log-majorises the vector of the absolute values of its eigenvalues: $|\lambda(A)| \prec_{\log} \sigma(A)$. As $x \prec_{w,\log} y$ implies $x^r \prec_w y^r$ for any $r > 0$, Weyl's Majorant Theorem can in slightly weaker form be stated as

$$|\lambda(A)|^r \prec_w \sigma^r(A), \text{ for all } r > 0. \tag{1.1}$$

*Received by the editors on December 7, 2014. Accepted for publication on January 15, 2015. Handling Editor: Roger A. Horn.

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The sum of the k largest singular values of a matrix defines a norm, known as the k-th Ky Fan norm. The convexity of the Ky Fan norms can be expressed as a majorisation relation: for any p such that $0 \le p \le 1$,

$$\sigma(pA + (1-p)B) \prec_w p\sigma(A) + (1-p)\sigma(B).$$

When A and B are positive semidefinite, their singular values coincide with their eigenvalues and we have

$$\lambda(pA + (1-p)B) \prec p\lambda(A) + (1-p)\lambda(B). \tag{1.2}$$

For positive semidefinite matrices A and B, the eigenvalues of AB are real and non-negative. Furthermore $\lambda(AB) \prec_{\log} \lambda(A) \circ \lambda(B)$ ([4] eq. (2.4)). Hence, we also have

$$\lambda(AB) \prec_w \lambda(A) \circ \lambda(B). \tag{1.3}$$

2. A majorisation relation for singular values. We start with a rather technical result concerning a majorisation relation for singular values. For any matrix A, we denote by $\mathrm{diag}(A)$ the matrix obtained from A by setting all its off-diagonal elements equal to zero.

LEMMA 2.1. *Let S be an $n \times m$ complex matrix, and let L and M be diagonal, positive semidefinite $m \times m$ matrices. Then*

$$\sigma(SL\,\mathrm{diag}(S^*S)MS^*) \prec_w \sigma((S(LM)^{1/2}S^*)^2) \prec_w \sigma(SLS^*SMS^*). \tag{2.1}$$

Proof. Let us begin with the first majorisation inequality. Since L, M, and $\mathrm{diag}(S^*S)$ are diagonal, they commute, and we can write

$$SL\,\mathrm{diag}(S^*S)MS^* = S(LM)^{1/2}\,\mathrm{diag}(S^*S)(LM)^{1/2}S^*.$$

This is a positive semidefinite matrix, hence its singular values are equal to its eigenvalues. The same is true for $(S(LM)^{1/2}S^*)^2$. Let us introduce $X = S(LM)^{1/4}$. Then we have to show that

$$\lambda(X\,\mathrm{diag}(X^*X)X^*) \prec \lambda(XX^*XX^*).$$

In terms of the matrix $T = X^*X \ge 0$, this is equivalent to

$$\lambda(T\,\mathrm{diag}(T)) \prec \lambda(T^2).$$

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ELA

Now note that there exist some number m of unitary matrices U_j such that $\mathrm{diag}(T) = \sum_{j=1}^m (U_j T U_j^*)/m$. Exploiting inequalities (1.2) and (1.3) in turn, we obtain

$$
\begin{aligned}
\lambda(T\,\mathrm{diag}(T)) &= \lambda(T^{1/2}\,\mathrm{diag}(T) T^{1/2}) \\
&= \lambda\left(T^{1/2} \ \sum_{j=1}^m \frac{1}{m}(U_j T U_j^*) \ T^{1/2} \right) \\
&\prec \sum_{j=1}^m \frac{1}{m} \lambda(T^{1/2} U_j T U_j^* T^{1/2}) \\
&= \sum_{j=1}^m \frac{1}{m} \lambda(T U_j T U_j^*) \\
&\prec_w \sum_{j=1}^m \frac{1}{m} \lambda(T)\lambda(U_j T U_j^*) \\
&= \sum_{j=1}^m \frac{1}{m} \lambda^2(T) = \lambda(T^2),
\end{aligned}
$$

which proves the first inequality of (2.1).

For the second inequality, note that, since $(LM)^{1/2}$ and S^*S are both positive semidefinite, their product has real, non-negative eigenvalues. Thus,

$$
\lambda^2((LM)^{1/2} S^* S) = |\lambda(L^{1/2} S^* S M^{1/2})|^2 \prec_w \sigma^2(L^{1/2} S^* S M^{1/2}),
$$

by Weyl's Majorant Theorem (eq. (1.1) with $r = 2$). This implies that

$$
\begin{aligned}
\sigma((S(LM)^{1/2} S^*)^2) &= \lambda((LM)^{1/2} S^* S (LM)^{1/2} S^* S) \\
&= \lambda^2((LM)^{1/2} S^* S) \\
&\prec_w \sigma^2(L^{1/2} S^* S M^{1/2}) \\
&= \lambda^2((M^{1/2} S^* S L S^* S M^{1/2})^{1/2}) \\
&= \lambda(M^{1/2} S^* S L S^* S M^{1/2}) \\
&= \lambda(S L S^* S M S^*) \\
&= |\lambda(S L S^* S M S^*)| \\
&\prec_w \sigma(S L S^* S M S^*),
\end{aligned}
$$

where in the last line we again exploit Weyl's Majorant Theorem (eq. (1.1) with $r = 1$). This proves the second inequality of (2.1). $\square$

3. Main result.

We can now state and prove the main result of this paper.

THEOREM 3.1. *For $i = 1, \ldots, k$, let A_i and B_i be positive semidefinite $d \times d$ matrices such that, for each i, A_i commutes with B_i. Then for all unitarily invariant*

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norms,

$$|||\sum_{i=1}^k A_i B_i ||| \le ||| \big(\sum_{i=1}^k A_i^{1/2} B_i^{1/2}\big)^2 ||| \le ||| \big(\sum_{i=1}^k A_i\big) \big(\sum_{i=1}^k B_i\big) ||| . \tag{3.1}$$

Proof. Let A_i and B_i have eigenvalue decompositions

$$A_i = U_i L_i U_i^*, \quad B_i = U_i M_i U_i^*,$$

where the U_i are unitary matrices, and L_i and M_i are positive semidefinite diagonal matrices. Let

$$L = \bigoplus_{i=1}^k L_i, \quad M = \bigoplus_{i=1}^k M_i, \quad S = (U_1|U_2|\cdots|U_k).$$

Then

$$\sum_{i=1}^k A_i = SLS^*, \quad \sum_{i=1}^k B_i = SMS^*, \quad \sum_{i=1}^k A_i B_i = SLMS^*.$$

In addition, the diagonal elements of S^*S are 1 since all columns of S are normalised. Hence, $\mathrm{diag}(S^*S) = I$. By Lemma 2.1, we then have

$$\sigma\left(\sum_{i=1}^k A_i B_i\right) \prec_w \sigma\left(\big(\sum_{i=1}^k A_i^{1/2} B_i^{1/2}\big)^2\right) \prec_w \sigma\left(\big(\sum_{i=1}^k A_i\big)\ \big(\sum_{i=1}^k B_i\big)\right)$$

which is equivalent to (3.1). □

The case $k = 2$ is an inequality recently conjectured by Hayajneh and Kittaneh (Conjecture 1.2 in [3]) and proven by them for the trace norm and the Hilbert-Schmidt norm.

A simple consequence of Theorem 3.1 is that for any set of k positive semidefinite matrices A_i, all positive functions f and g, and all unitarily invariant norms,

$$|||\sum_{i=1}^k f(A_i) g(A_i) ||| \le ||| \big(\sum_{i=1}^k f(A_i)\big) \big(\sum_{i=1}^k g(A_i)\big) ||| . \tag{3.2}$$

Setting $k = 2$, $f(x) = x^p$ and $g(x) = x^q$ yields the inequality

$$|||A^{p+q} + B^{p+q}||| \le |||(A^p + B^p)(A^q + B^q)|||, \tag{3.3}$$

which was conjectured by Bourin [2].

Acknowledgment. We acknowledge support by an Odysseus grant from the Flemish FWO.

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ELA

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