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COMMON SOLUTIONS OF LINEAR EQUATIONS IN A RING, WITH APPLICATIONS*

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Abstract. This paper gives necessary and sufficient conditions for the existence of a common solution, and two expressions for the general common solution of the equation pair $a_1xb_1 = c1$, $a_2xb_2 = c_2$, via a simpler equation $p_1xp_2 + q_1yq_2$=c, when each element belongs to an associative ring with unit. The paper also considers the same problem in the setting of a strongly $*$-reducing ring. Solutions of the generalized Sylvester equation are also presented. Both the solvability conditions and the expression for the general solution are given in terms of inner inverses. The paper uses the results obtained in the ring setting to give equivalent results for operators between Banach spaces, thus also recovering some of the well known matrix results.

Key words. Equation, System of linear equations, Generalised inverse, Inner inverse, Equation over ring, Operator equation.

AMS subject classifications. 15A24, 47A62, 16B99.

1. Introduction. The necessary and sufficient conditions for the existence of a common solution, and the general common solution of the equation pair $A_1XB_1 = C_1$, $A_2XB_2 = C_2$, were given for matrices over the complex field by Mitra in [5]. Van der Woude derived a set of necessary and sufficient conditions in [10], and Mitra [6] gave an expression for the general common solution, for the same problem over a general field. The system was more recently considered by Navarra et al. for matrices over the complex field [7]. Wang considered the same problem for matrices over regular rings with identity in [13].

Matrix equation $AXB + CYD = E$ was studied over an arbitrary principal ideal domain in [1]. Wang presented the solvability conditions and the general solution for matrices over regular rings in [13] and this solution was revised by Wang et al. in [12]. More recently, Wang and He have studied a more general matrix equation in [11].

In this paper, we study necessary and sufficient conditions for the existence of a common solution of the ring equation pair $a_1xb_1 = c_1, a_2xb_2 = c_2$ and derive two distinct expressions for the general common solution. The results are given in

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terms of generalized inverses, the ring is not necessarily regular and when possible explicit expressions for inner inverses of certain elements are given. The results of Section 3 and Section 5 are new in the setting of an associative ring with a unit. One of the expressions for the general common solution is given in the setting of strongly $*$-reducing rings. This is done in Section 4. Theorem 4.2, Corollary 4.3 and Corollary 4.4 present solutions to the title equation which are new even in the case of matrices over a field.

In Section 5, we apply some of the results obtained to the solution of the equation $axb + cyd = e$ (which we call the generalized Sylvester equation). We highlight the relationship between this last equation, one of its simpler versions, and the title equations.

In Sections 6 and 7, we extend the results obtained earlier for rings to bounded linear operators between Banach or Hilbert spaces. This is achieved by embedding the 'rectangular' operators, via operator matrices, into the ring of operators acting on the direct sum of Banach spaces. Theorems 6.1 and 7.1 are new in the setting of bounded linear operators.

2. Preliminaries. Let $\mathcal{R}$ represent an associative ring. For $x \in \mathcal{R}$, an *inner inverse* of x is an element y such that $xyx = x$; we denote an inner inverse of x by x^-. An element is said *regular* if it possesses an inner inverse. In the case of a matrix $M \in \mathcal{R}^{n\times m}$, M is said to be regular if there exists an element $B \in \mathcal{R}^{m\times n}$ such that $M = MBM$.

THEOREM 2.1. ([2], Theorem 3.1) *Let $a, b, c \in \mathcal{R}$ with a, b regular. Then equation*

$$axb = c \tag{2.1}$$

is consistent in $\mathcal{R}$ if and only if $c = aa^-cb^-b$. If $c = aa^-cbb^-b$, then the general solution of (2.1) is given by

$$x = a^-cb^- + u - a^-aubb^-, \tag{2.2}$$

where $u \in \mathcal{R}$ is arbitrary.

The following theorem can be found in [9] for matrices over the complex field.

THEOREM 2.2. *Let $A \in \mathcal{R}^{m\times n}$, $B \in \mathcal{R}^{k\times l}$, $C \in \mathcal{R}^{m\times l}$ and A, B be regular matrices over $\mathcal{R}$. Then the matrix equation $AXB = C$ is consistent if and only if $AA^-CB^-B = C$. If $AA^-CB^-B = C$, then the general solution of the equation is given by*

$$X = A^-CB^- + U - A^-AUBB^-,$$

where $U \in \mathcal{R}^{n\times k}$.

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Proof. We embed each matrix A, B, C into the ring of square matrices over $\mathcal{R}$ of size $N = m + n + k + l$:

$$\boldsymbol{A} := \begin{bmatrix} 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{B} := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{C} := \begin{bmatrix} 0 & 0 & 0 & C \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Each of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ is regular with inner inverses $\boldsymbol{A}^-$, $\boldsymbol{B}^-$, $\boldsymbol{C}^-$, obtained by a suitable embedding of A^-, B^-, C^- into $\mathcal{R}^{N\times N}$ given by

$$(\boldsymbol{A}^-)_{2,1} = A^-, \ (\boldsymbol{B}^-)_{4,3} = B^-, \ (\boldsymbol{C}^-)_{4,1} = C^-,$$

with all other block entries equal to zero. It is not difficult to check that the equation $AXB = C$ is consistent if and only if the equation $\boldsymbol{AXB} = \boldsymbol{C}$ is consistent with some $\boldsymbol{X} \in \mathcal{R}^{N\times N}$, while $\boldsymbol{X}_{2,3} = X$. A direct calculation shows that $\boldsymbol{AA}^-\boldsymbol{CB}^-\boldsymbol{B} = \boldsymbol{C}$ if and only if $AA^-CB^-B = C$. Applying Theorem 2.1, we obtain the statement concerning the consistency of $AXB = C$.

The general solution of $AXB = C$ is given by the $(2, 3)$ entry of the general solution $\boldsymbol{X}$ of $\boldsymbol{AXB} = \boldsymbol{C}$ which involves an arbitrary block matrix $\boldsymbol{U} \in \mathcal{R}^{N\times N}$:

$$X = (\boldsymbol{X})_{2,3} = (\boldsymbol{A}^-\boldsymbol{CB}^- + \boldsymbol{U} - \boldsymbol{A}^-\boldsymbol{AUBB}^-)_{2,3} = A^-CB^- + U - A^-AUBB^-$$

with $U = (\boldsymbol{U})_{2,3} \in \mathcal{R}^{n,k}$. This follows from Theorem 2.1 since matrices $\boldsymbol{A}^-\boldsymbol{CB}^-$ and $\boldsymbol{A}^-\boldsymbol{AUBB}^-$ have all entries except $(2, 3)$ equal to zero; the (2,3) entries are equal to A^-CB^- and $A^-A(\boldsymbol{U})_{2,3}BB^-$, respectively. ∎

The following two lemmas can be obtained as a consequence of [8, Theorem 4] or [2, Lemma 2.2].

LEMMA 2.3. *Let u and v be regular elements of a ring $\mathcal{R}$. Then $s = v(1 - u^-u)$ is regular if and only if $\begin{bmatrix} u \\ v \end{bmatrix}$ is regular. In this case,*

$$\begin{bmatrix} u \\ v \end{bmatrix}^- = \begin{bmatrix} u^- - (1 - u^-u)s^-vu^- & (1 - u^-u)s^- \end{bmatrix}. \tag{2.3}$$

LEMMA 2.4. *Let u and v be regular elements of $\mathcal{R}$. Then $\begin{bmatrix} u & v \end{bmatrix}$ is regular if and only if $t = (1 - uu^-)v$ is regular. In this case,*

$$\begin{bmatrix} u & v \end{bmatrix}^- = \begin{bmatrix} u^- - u^-vt^-(1 - uu^-) \\ t^-(1 - uu^-) \end{bmatrix}.$$

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A direct verification yields the following lemma.

LEMMA 2.5. *Let* $u, v,\ s = v(1-u^-u)$ *and* $t = (1-uu^-)v$ *be regular elements of* $\mathcal{R}$. *Then*

$$g = (1-ss^-)vu^- \quad \textit{and} \quad f = u^-v(1-t^-t)$$

are regular with inner inverses $f^- = v^-u$ *and* $g^- = uv^-$. *Elements* gu *and* $(1-ss^-)v$ *are also regular with inner inverses* $(gu)^- = u^-g^-$ *and* $((1-ss^-)v)^- = v^-$.

The following result will be needed for the discussion of the Sylvester equation.

LEMMA 2.6. *Let* $p_1, p_2, q_1, q_2, c \in \mathcal{R}$, *where* p_1, q_2 *are idempotents,* p_2, q_1 *regular, and* $p_1q_1 = 0 = p_2q_2$. *Then the equation*

$$p_1xp_2 + q_1yq_2 = c \tag{2.4}$$

is solvable if and only if

$$q_1q_1^-cq_2 = c - p_1cp_2^-p_2. \tag{2.5}$$

The general solution of (2.4) *is given by*

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cp_2^- + z_1 - p_1z_1p_2p_2^- \\ q_1^-c + z_2 - q_1^-q_1z_2q_2 \end{bmatrix}, \quad z_1, z_2 \in \mathcal{R}. \tag{2.6}$$

Proof. Let $d = c - q_1yq_2$ for some $y \in \mathcal{R}$. Consider the equation $p_1xp_2 = d$. By Theorem 2.1 and the assumption $p_1q_1 = 0$, this equation is solvable if and only if $d = p_1cp_2^-p_2$, that is, $q_1yq_2 = c - p_1cp_2^-p_2$. By Theorem 2.1 and the assumption $p_2q_2 = 0$, such y exists if and only if (2.5) holds. The last equation gives rise to a particular solution to (2.4), $\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} cp_2^- & q_1^-c \end{bmatrix}$. The result follows when we observe that the general solution to (2.4) is of the form $\begin{bmatrix} x & y \end{bmatrix} = \begin{bmatrix} x_p & y_p \end{bmatrix} + \begin{bmatrix} x_0 & y_0 \end{bmatrix}$, where the first matrix on the right denotes a particular solutions to the equation and the second denotes the solution to the same equation when $c = 0$. ∎

3. Common solutions of equations $a_1xb_1 = c_1$ **and** $a_2xb_2 = c_2$**.** In this section, we make a blanket assumption that $a_1, a_2, b_1, b_2, c_1, c_2$ are elements of the ring $\mathcal{R}$ with a_1, a_2, b_1, b_2 regular. We consider the common solutions of the equation pair

$$a_1xb_1 = c_1 \quad \text{and} \quad a_2xb_2 = c_2. \tag{3.1}$$

In what follows, we let

$$A := \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad B := \begin{bmatrix} b_1 & b_2 \end{bmatrix}, \quad C = C_{v,w} := \begin{bmatrix} c_1 & v \\ w & c_2 \end{bmatrix}$$

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for some $v, w \in \mathcal{R}$. We assume that A, B, or equivalently

$$s := a_2(1 - a_1^- a_1) \quad \text{and} \quad t := (1 - b_1 b_1^-)b_2,$$

are regular, and set

$$f := b_1^- b_2(1 - t^- t) \quad \text{and} \quad g := (1 - ss^-)a_2 a_1^-.$$

We note that by Lemma 2.5 f, g are regular, $f^- = b_2^- b_1$, $g^- = a_1 a_2^-$, and

$$gg^- = (1 - ss^-)a_2 a_2^-, \quad f^- f = b_2^- b_2(1 - t^- t). \tag{3.2}$$

The following matrices will be useful in future calculations:

$$AA^- = \begin{bmatrix} a_1 a_1^- & 0 \\ g & ss^- \end{bmatrix} \quad \text{and} \quad B^- B = \begin{bmatrix} b_1^- b_1 & f \\ 0 & t^- t \end{bmatrix}.$$

Common solvability of the title equation pair is equivalent to the solvability of $AxB = C$, that is,

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x \begin{bmatrix} b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_1 x b_1 & a_1 x b_2 \\ a_2 x b_1 & a_2 x b_2 \end{bmatrix} = \begin{bmatrix} c_1 & v \\ w & c_2 \end{bmatrix} \tag{3.3}$$

for some v, w.

We note that (3.3) is consistent if and only if $AA^- CB^- B = C$. This last equations is satisfied if and only if

$$\begin{bmatrix} c_1 & v \\ w & c_2 \end{bmatrix} = \begin{bmatrix} a_1 a_1^- c_1 b_1^- b_1 & a_1 a_1^- (c_1 f + vt^- t) \\ (gc_1 + ss^- w)b_1^- b_1 & (gc_1 + ss^- w)f + (gv + ss^- c_2)t^- t \end{bmatrix}. \tag{3.4}$$

The above notation is retained in the rest of this section. The following theorem replaces v, w occuring in (3.4) with alternative parameters m, n.

Theorem 3.1. *Let $c_1 = a_1 a_1^- c_1 b_1^- b_1$. Then equation* (3.3)*, equivalently* (3.1)*, is consistent if and only if there exist m, n such that*

$$ss^- nf + gmt^- t = c_2 - gc_1 f - ss^- c_2 t^- t. \tag{3.5}$$

Proof. Note that under the hypothesis on c_1, and from $a_1 a_1^- c_1 f = c_1 f$, and $gc_1 b_1^- b_1 = gc_1$, to satisfy (3.4) we require that v, w satisfy $v = c_1 f + a_1 a_1^- vt^- t$ and $w = gc_1 + ss^- wb_1^- b_1$. These equalities hold if and only if v, w are of the form $v = c_1 f + a_1 a_1^- mt^- t$, and $w = gc_1 + ss^- nb_1^- b_1$, $m, n \in \mathcal{R}$. To see this, we can

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substitute such v, w into the respective equations and check that the equalities hold. Conversely, when the equalities hold we can set $m = v$ and $n = w$.

The remaining condition on c_2 of Equation (3.4), rewritten using the new expressions for v, w, gives (3.5). □

Theorem 3.2. *Let* $a_1a_1^-c_1b_1^-b_1 = c_1$ *and* $a_2a_2^-c_2b_2^-b_2 = c_2$*. Then equations* (3.1) *have a common solution if and only if*

$$(1 - ss^-)(c_2 - gc_1f)(1 - t^-t) = 0. \tag{3.6}$$

Proof. We note that by Lemma 2.5, f, g of equation (3.5) are regular. Setting $p_1 := ss^-$, $p_2 := f$, $q_1 := g$, $q_2 := t^-t$, we can verify that $p_1q_1 = 0$ and $p_2q_2 = 0$, for idempotents p_1, q_2. Hence, we can apply Lemma 2.6 with $c = c_2 - gc_1f - ss^-c_2t^-t$. For the choice $f^- = b_2^-b_1$ and $g^- = a_1a_2^-$ we can verify that $t^-tf^-f = 0 = gg^-ss^-$. It follows that (3.5) is solvable if and only if

$$gg^-c_2t^-t = (c_2 - gc_1f) - ss^-c_2t^-t - ss^-c_2f^-f.$$

Applying (3.2) and the identity $a_2a_2^-c_2b_2^-b_2 = c_2$, we simplify this equation to

$$(c_2 - gc_1f) - c_2t^-t - ss^-c_2(1 - t^-t) = 0.$$

Equation (3.6) then follows when we note that $ss^-g = 0 = ft^-t$. □

A special case of Theorem 3.2 recovers the following standard result which can be found for example in [2, Theorem 3.11].

Corollary 3.3. *Let* a, b *be regular elements of the ring* $\mathcal{R}$*. Then equations* $ax = c$ *and* $xb = d$ *have a common solution if and only if* $aa^-c = c$, $db^-b = d$ *and* $ad = cb$.

Our aim was to solve (3.3) for some x with no prior condition on v, w. When the equation is solvable for some x, and therefore for some choice of v, w, it is solvable for possibly multiple choices of v, w (unless the pair a_i, b_i are invertible). Each solution x generates a pair v_x, w_x. From the discussion in the proof of Theorem 3.1, when (3.3) is solvable, Lemma 2.6 gives the general solutions m, n and hence v, w:

$$\begin{aligned} v &= c_1f + g^-(1 - ss^-)c_2t^-t + (a_1a_1^- - g^-g)z_2t^-t, \\ w &= gc_1 + ss^-c_2(1 - t^-t)f^- + ss^-z_1(b_1^-b_1 - ff^-), \end{aligned} \tag{3.7}$$

z_1, $z_2 \in \mathcal{R}$ and f^-, g^- as given on the line preceding (3.2).

The preceding discussion gives us a way of deriving the general solution of (3.1). A common solution x of equations (3.1) is a solution to (3.3) for a particular choice

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v, w (namely $v = a_1xb_2$ and $w = a_2xb_1$). Conversely, a solution of (3.3) given by $x = A^-C_{v,w}B^- + z - A^-AzBB^-$ is a solution of (3.1).

Theorem 3.4. *If the equations in* (3.1) *have a common solution, then their general common solution is given by*

$$\begin{aligned} x = &[a_1^- c_1 - (1 - a_1^- a_1)s^-(a_2a_1^-c_1 - w)]b_1^-[1 - b_2t^-(1 - b_1b_1^-)] \\ &+ [(1 - (1 - a_1^-a_1)s^-a_2)a_1^-v + (1 - a_1^-a_1)s^-c_2]t^-(1 - b_1b_1^-) \\ &+ z - (a_1^-a_1 + (1 - a_1^-a_1)s^-s)z(b_1b_1^- + tt^-(1 - b_1b_1^-)), \end{aligned} \tag{3.8}$$

with v, w given by (3.7)*, and z_1, z_2 and z arbitrary elements of $\mathcal{R}$.*

Proof. The general solution of (3.1) is given by the general solution of $AxB = C$, $x = A^-CB^- + z - A^-AzBB^-$, with v, w as in (3.7) and z arbitrary. Expanding the last equation gives (3.8). ∎

Theorem 3.2 and Theorem 3.4 are new in the case of an associative ring with a unit. These are also given in a similar form for matrices over a regular ring in [13, Theorem 2.4]. The method of this paper is distinct from that of [13]. Moreover, the ring is not assumed regular and explicit inner inverses are provided for certain elements that appear in the expressions for the general solution, and all the results are presented in terms of inner rather than reflexive inner inverses.

4. General solution in $*$-reducing rings. In this section, $\mathcal{R}$ denotes a ring with involution $*$. A ring $\mathcal{R}$ is said to have the *Gelfand-Naimark* property if $1 + a^*a$ is invertible for each $a \in \mathcal{R}$. An element $a \in \mathcal{R}$ is said $*$-*cancellable* if for every $x \in \mathcal{R}$, $a^*ax = 0$ implies $ax = 0$ and $xaa^* = 0$ implies $xa = 0$. If $a^*a = 0$ implies $a = 0$, then $\mathcal{R}$ is said $*$-*reducing*. Elements of a $*$-reducing ring are $*$-cancellable. The matrix ring $\mathcal{R}^{n\times n}$ is $*$-reducing if for every $n \in \mathbb{N}$, $\sum_{i=1}^n a_i^*a_i = 0$ implies $a_i = 0$ for each i. A ring $\mathcal{R}$ satisfying this property is called a *strongly $*$-reducing* ring. For such rings, the set $\mathcal{R}^{n\times m}$ is also $*$-reducing.

For the definition of the Moore-Penrose inverse in a ring with involution and for some of its properties, see [4]. The Moore-Penrose inverse of $a \in \mathcal{R}$ will be denoted by $a^\dagger$. Moore-Penrose invertibility of $a \in \mathcal{R}$ is equivalent to a being $*$-cancellable and both a^*a and aa^* being regular. The following lemma is given in [2, Lemma 2.8]; alternatively see [4] or [3, Proposition 1]. Lemma 4.1 is valid for rectangular matrices over $\mathcal{R}$.

Lemma 4.1. *Let $a \in \mathcal{R}$ be $*$-cancellable. If a^*a is regular , then a is regular with an inner inverse of a given by*

$$a^- = (a^*a)^-a^*.$$

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If aa^* is regular, $a^+ = a^*(aa^*)^-$ is also an inner inverse of a. If a^*a is Moore-Penrose invertible, then so is a with $a^\dagger = (a^*a)^\dagger a^* = a^*(aa^*)^\dagger$.

In this section, we assume that $\mathcal{R}$ is strongly $*$-reducing and let

$$e := a_1^*a_1 + a_2^*a_2 \quad \text{and} \quad l := b_1b_1^* + b_2b_2^*$$

each be regular. Then by Lemma 4.1, we can choose inner inverses of A and B

$$A^+ := \begin{bmatrix} e^+a_1^* & e^+a_2^* \end{bmatrix} \quad \text{and} \quad B^+ := \begin{bmatrix} b_1^*l^+ \\ b_2^*l^+ \end{bmatrix}. \tag{4.1}$$

In this case, we have $A^+A = e^+e$ and $BB^+ = ll^+$ and the following theorem.

Theorem 4.2. *Let conditions of Theorem 3.2 be satisfied and let moreover $\mathcal{R}$ be a strongly $*$-reducing ring. Let e, l be regular. Then the general common solution of the title equations is given by*

$$\begin{aligned} x = {} & (a_1^*a_1 + a_2^*a_2)^+[a_1^*c_1b_1^* + a_2^*c_2b_2^* + a_2^*wb_1^* + a_1^*vb_2^*](b_1b_1^* + b_2b_2^*)^+ \\ & + z - (a_1^*a_1 + a_2^*a_2)^+(a_1^*a_1 + a_2^*a_2)z(b_1b_1^* + b_2b_2^*)(b_1b_1^* + b_2b_2^*)^+, \end{aligned} \tag{4.2}$$

with $z \in \mathcal{R}$ and v, w as in (3.7).

Proof. As proof of Theorem 3.4 with inner inverses of A, B as given in (4.1). ☐

Corollary 4.3. *Let $a, b \in \mathcal{R}$ be regular and let $aa^-c = c$, $db^-b = d$ and $ad = cb$. Then the general common solution of $ax = c$, $xb = d$ is given by*

$$\begin{aligned} x = {} & (1 + a^*a)^-[a^*c + db^* + a^-c + (1 - a^-a)db^- + a^*cbb^*](1 + bb^*)^- \\ & + (1 + a^*a)^-(1 - a^-a)z(1 - bb^-)(1 + bb^*)^-, \end{aligned}$$

with $z \in \mathcal{R}$.

Corollary 4.4. *Let $a, b \in \mathcal{R}$ be regular and let $aa^-c = c$, $db^-b = d$ and $ad = cb$ and moreover, let $\mathcal{R}$ have the Gelfand-Naimark property. Then the general common solution of $ax = c$, $xb = d$ is given by*

$$\begin{aligned} x = {} & (1 + a^*a)^{-1}[a^*c + db^* + a^-c + (1 - a^-a)db^- + a^*cbb^*](1 + bb^*)^{-1} \\ & + (1 - a^-a)z(1 - bb^-), \end{aligned} \tag{4.3}$$

with $z \in \mathcal{R}$.

Proof. Follows from Corollary 4.3 when we observe that $(1 + a^*a)(1 - a^-a) = (1 - a^-a)$ and $(1 - bb^-)(1 + bb^*) = (1 - bb^-)$. ☐

Remark 4.5. As far as the author is aware, Equation (4.3) gives a new explicit representation of the solution of $ax = c$, $xb = d$, even in the case of matrices over a field.

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An application of Theorem 3.4, or equivalently Theorem 4.2 yields the general hermitian solution of the ring equation $axb = c$. The general hermitian solution of $axb = c$ is given by $\frac{1}{2}(x + x^*)$ where x is the common solution of the equation pair $axb = c$ and $b^*xa^* = c^*$.

5. Applications to the equation $axb + cyd = e$. In this section, we consider the general Sylvester equation

$$axb + cyd = e, \tag{5.1}$$

with a, b, c, d fixed regular elements, and unknowns x, y. Equation (5.1) is solvable if and only if $aa^-(e - cyd)b^-b + cyd = e$, for some y; or equivalently if and only if the following pair of equations have a common solution y:

$$(1 - aa^-)cyd = (1 - aa^-)e \quad \text{and} \quad cyd(1 - b^-b) = e(1 - b^-b). \tag{5.2}$$

Remark 5.1. From Section 3, we know that the solvability of a system of two equations of this type, hence of (5.1) reduces to the simpler Sylvester equation of the type (2.4).

To apply Theorem 3.2 it is necessary to assume that the elements

$$a_1 = (1 - aa^-)c; \quad b_2 = d(1 - b^-b); \quad s = c(1 - a_1^- a_1) \tag{5.3}$$

are regular. Following the notation of the previous section, we set

$$f = d^-d(1 - b^-b) \quad \text{and} \quad g = (1 - ss^-)c((1 - aa^-)c)^-, \tag{5.4}$$

and recall that f and g are regular, and that $t = 0$. The solvability conditions and the general solution of (5.1) are then as follows.

Theorem 5.2. *Let a, b, c, d, and a_1, b_2, s be regular. Then* (5.1) *is solvable if and only if*

$$\begin{gathered}(1 - ss^-)[e(1 - b^-b) - g(1 - aa^-)ef] = 0,\\ (1 - aa^-)c((1 - aa^-)c)^-(1 - aa^-)ed^-d = (1 - aa^-)e\\ cc^-e(1 - b^-b)(d(1 - b^-b))^-d(1 - b^-b) = e(1 - b^-b),\end{gathered}$$

where f and g are as in (5.4).

Proof. Follows from the above discussion and Theorem 3.2. ◻

An application of Theorem 3.4 gives the general solution y of (5.2) and hence of equation (5.1). The general solution for x is then expressed in terms of the general

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solution y.

THEOREM 5.3. *Let a, b, c, d and a_1, b_2, s be regular. Then the general solution of the consistent equation* (5.1) *is given by*

$$\begin{aligned} y &= [a_1^-(1-aa^-)e - (1-a_1^-a_1)s^-(ca_1^-(1-aa^-)e-w)]d^- \\ &\quad + z - [a_1^-a_1 + (1-a_1^-a_1)s^-s]zdd^-, \\ x &= a^-(e-cyd)b^- + u - a^-aubb^-, \end{aligned}$$

where

$$w = g(1-aa^-)e + ss^-e(1-b^-b)f^- + ss^-p(d^-d - ff^-), \quad z, u, p \in \mathcal{R}.$$

Proof. Apply Equation (3.8), noting that here $t = 0$ and setting $t^- = 0$. □

REMARK 5.4. Theorem 5.2 and Theorem 5.3 are essentially as given in [13], without the assumption that the ring is regular. In the case of a $*$-reducing ring, we can apply (4.2) to obtain an alternative solution of (5.1).

6. Applications to common solutions of operator equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$.

THEOREM 6.1. *Let E, F, G, D, N, M be Banach spaces. Let $A_1 \in \mathcal{B}(F, E)$, $A_2 \in \mathcal{B}(F, N)$, $B_1 \in \mathcal{B}(D, G)$, $B_2 \in \mathcal{B}(M, G)$ and*

$$T := (I_G - B_1B_1^-)B_2 \quad \textit{and} \quad S := A_2(I_F - A_1^-A_1)$$

be all regular. Moreover, let $A_1A_1^-C_1B_1^-B_1 = C_1$ and $A_2A_2^-C_2B_2^-B_2 = C_2$. Then the equations

$$A_1XB_1 = C_1 \quad \textit{and} \quad A_2XB_2 = C_2 \tag{6.1}$$

have a common solution if and only if

$$(I_N - SS^-)C_2(I_M - T^-T) = (I_N - SS^-)A_2A_1^-C_1B_1^-B_2(I_M - T^-T). \tag{6.2}$$

In this case, the general common solution is given by

$$\begin{aligned} X =& (A_1^-C_1 - (I_F - A_1^-A_1)S^-(A_2A_1^-C_1 - W))B_1^-(I_G - B_2T^-(I_G - B_1B_1^-)) \\ &+ ((I_F - (I_F - A_1^-A_1)S^-A_2)A_1^-V + (I_F - A_1^-A_1)S^-C_2)T^-(I_G - B_1B_1^-) \\ &+ Z - (A_1^-A_1 + (I_F - A_1^-A_1)S^-S)Z(B_1B_1^- + TT^-(I_G - B_1B_1^-)), \end{aligned} \tag{6.3}$$

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where

$$V = C_1B_1^-B_2(I_M - T^-T) + A_1A_2^-(I_N - SS^-)C_2T^-T + A_1A_1^-QT^-T - A_1A_2^-(I_N - SS^-)A_2A_1^-QT^-T, \tag{6.4}$$

$$W = (I_N - SS^-)A_2A_1^-C_1 + SS^-C_2(I_M - T^-T)B_2^-B_1 + SS^-PB_1^-B_1 - SS^-PB_1^-B_2(I_M - T^-T)B_2^-B_1, \tag{6.5}$$

with P, Q, Z arbitrary elements of $\mathcal{B}(D,N)$, $\mathcal{B}(M,E)$ and $\mathcal{B}(G,F)$, respectively.

Proof. We embed each of A_i, B_i, C_i into the ring of square operator matrices acting on the sum of Banach spaces. The embedding of an operator A acting from the space in the i_A-th position to the space in the j_A-th position of $K := E \oplus F \oplus G \oplus D \oplus N \oplus M$ is the 6×6 operator matrix $\boldsymbol{A} \in \mathcal{B}(K)$ determined by

$$\boldsymbol{A} = (\boldsymbol{A})_{i,j} = \begin{cases} A & \text{if } (i,j) = (j_A, i_A), \\ 0 & \text{otherwise.} \end{cases}$$

If the operator A is regular, the operator matrix $\boldsymbol{A}$ is regular. Using the above described embedding, we define $\boldsymbol{A_1}, \boldsymbol{B_1}, \boldsymbol{C_1}, \boldsymbol{A_2}, \boldsymbol{B_2}, \boldsymbol{C_2} \in \mathcal{B}(K)$ as the 6×6 operator matrices satisfying

$$(\boldsymbol{A}_1)_{1,2} = A_1, \quad (\boldsymbol{A}_2)_{5,2} = A_2, \quad (\boldsymbol{B}_1)_{3,4} = B_1, \quad (\boldsymbol{B}_2)_{3,6} = B_2,$$
$$(\boldsymbol{C}_1)_{1,4} = C_1, \quad (\boldsymbol{C}_2)_{5,6} = C_2,$$

with all other entries equal to 0.

We apply the results from the previous section, observing that (6.1) have a common solution if and only if the equations

$$\boldsymbol{A_1 Y B_1} = \boldsymbol{C_1} \quad \text{and} \quad \boldsymbol{A_2 Y B_2} = \boldsymbol{C_2} \tag{6.6}$$

in the ring $\mathcal{B}(K)$ have a common solution $\boldsymbol{Y}$.

To apply Theorem 3.2 and Theorem 3.4 we choose inner inverses A_i^-, B_i^- of A_i, B_i, and then define $\boldsymbol{A_i}, \boldsymbol{B_i}$ by embedding them into $\mathcal{B}(K)$ to satisfy

$$(\boldsymbol{A}_1^-)_{2,1} = A_1^-, \quad (\boldsymbol{A}_2^-)_{2,5} = A_2^-, \quad (\boldsymbol{B}_1^-)_{4,3} = B_1^-, \quad (\boldsymbol{B}_2^-)_{6,3} = B_2^-,$$

with all other entries equal to 0. The general solution of (6.1) is then given by the (2,3)-entry of the matrix $\boldsymbol{Y}$, where $\boldsymbol{Y}$ is the general solution of (6.6). ∎

The following corollary recovers a well known result; see for example [2, Theorem 4.5].

COROLLARY 6.2. *Let E, F, G, M be Banach spaces. Let $A \in \mathcal{B}(F,E)$ and $B \in \mathcal{B}(M,G)$ be regular and let $C \in \mathcal{B}(G,E)$, $D \in \mathcal{B}(M,F)$. Then the equations $AX = C$*

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and $XB = C$ *have a common solution* $X \in \mathcal{B}(G,F)$ *if and only if* $AA^-C = C$, $DB^-B = D$ *and* $AD = CB$. *In this case, the general common solution is given by*

$$X = A^-C + (I - A^-A)DB^- + (I - A^-A)P(I - BB^-), \tag{6.7}$$

with $P \in \mathcal{B}(G,F)$ *arbitrary.*

Proof. The result follows on application of Theorem 6.1. We have $S = (I - A^-A)$ and $T = 0$ both regular and we choose $S^- = S$ and $T^- = 0$. From (6.2), equations $AX = C$ and $XB = D$ have a common solution if and only if $AA^-C = C$, $DB^-B = D$ and $A^-AD = A^-CB$. If $AA^-C = C$, $DB^-B = D$, then $A^-AD = A^-CB$ if and only if with $AD = CB$. Equation (6.7) then follows from (6.3). □

COROLLARY 6.3. *Let* A_1, A_2, B_1, B_2 *be matrices over* $\mathcal{R}$ *of appropriate sizes and let*

$$A_1A_1^-C_1B_1^-B_1 = C_1, \quad \text{and} \quad A_2A_2^-C_2B_2^-B_2 = C_2.$$

Let $T = (I - B_1B_1^-)B_2$ *and* $S = A_2(I - A_1^-A_1)$. *Then equations*

$$A_1XB_1 = C_1 \quad \text{and} \quad A_2XB_2 = C_2$$

have a common solution if and only if

$$(I - SS^-)C_2(I - T^-T) = (I - SS^-)A_2A_1^-C_1B_1^-B_2(I - T^-T).$$

In this case, the general common solution is given by (6.3).

Proof. The proof of Theorem 6.1 was algebraic and can be applied here with each matrix considered as an operator between appropriate spaces. □

7. Applications to the operator equation $AXB + CYD = E$**.** In [13, Theorem 3.1], the general common solution of matrix equations over regular rings $A_1XB_1 = C_1, A_2XB_2 = C_2$ is used to derive the general solution for the Sylvester equation $AXB + CYD = E$ for matrices over regular rings. The solution derived by Wang in [13] is missing some important terms and was revised by Wang et al. in [12]. We will give a complete solution below for bounded linear operators over Banach spaces. The result is valid for matrices over rings, since the proof is algebraic and each matrix can be seen as an operator between appropriate spaces. Let

$$A_1 := (I - AA^-)C \quad \text{and} \quad B_2 := D(I - B^-B). \tag{7.1}$$

Then we have the following.

THEOREM 7.1. *Let* $A \in \mathcal{B}(F,M), B \in \mathcal{B}(G,K)$, $D \in \mathcal{B}(G,N)$ *and* $C \in \mathcal{B}(H,M)$ *be all regular. Let* A_1, B_2 *as defined in* (7.1) *be regular. Let also* $S := C(I - A_1^-A_1)$

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be regular. Then the equation $AXB + CYD = E$ is solvable for some $X \in \mathcal{B}(K, F)$ and $Y \in \mathcal{B}(N, H)$ if and only if

$$\begin{gathered} A_1 A_1^-(I - AA^-)ED^-D = (I - AA^-)E, \\ CC^-E(I - B^-B)B_2^- = E(I - B^-B), \quad \text{and} \\ (I - SS^-)[E - CA_1^-(I - AA^-)ED^-D](I - B^-B) = 0. \end{gathered}$$

In this case, the general solution of the equation is given by

$$\begin{aligned} Y &= (A_1^-(I - AA^-)E - (I - A_1^-A_1)S^-CA_1^-(I - AA^-)E - W)D^- \\ &\quad + Z - (A_1^-A_1 + (I - A_1^-A_1)S^-S)ZDD^-, \\ X &= A^-(E - CYD)B^- + U - A^-AUBB^-, \end{aligned}$$

where

$$W = (I - SS^-)CA_1^-(I - AA^-)E + SS^-(E + PD^-D)(I - B^-B)B_2^-D$$

and P, U, Z are arbitrary operators in $\mathcal{B}(G, M)$, $\mathcal{B}(G, F)$, and $\mathcal{B}(N, H)$, respectively.

Proof. The operator equation is solvable if and only if $AXB = E - CYD$, hence if and only if $(I - AA^-)CYD = (I - AA^-)E$ and $CYD(I - B^-B) = E(I - B^-B)$. The result then follows directly from Theorem 6.1 noting that $T = 0$ and setting $T^- = 0$. □

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