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EXTENSION OF BESSEL SEQUENCES TO OBLIQUE DUAL FRAME SEQUENCES AND THE MINIMAL PROJECTION*

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Abstract. An extension of two Bessel sequences to oblique dual frame sequences and its applications to shift-invariant spaces are considered. The best-known situation where this kind of extension is necessary is the construction of a pair of biorthogonal multiresolution analyses, where two generating sets whose shifts are only assumed to be Bessel sequences are given. This extension naturally leads to consideration of the 'minimal projection' extending two closed subspaces. The existence or non-existence of the minimal projection is discussed.

Key words. Frame, Frame sequence, Angle, Projection, Shift-invariant space.

AMS subject classifications. 42C15, 46C99.

1. Introduction. Throughout this article, $\mathcal{H}$ denotes a separable Hilbert space over $\mathbb{R}$ or $\mathbb{C}$, and $\mathbb{I}$ and $\mathbb{J}$ countable index sets. Li and Sun showed that any Bessel sequence can be extended to be a tight frame for $\mathcal{H}$ [18] and found the minimal number of elements to be added to form a frame (the relevant definitions can be found in later sections). Later, Bownik et al. and Christensen et al. showed that a pair of Gabor or shift-invariant Bessel sequences in $L^2(\mathbb{R}^d)$ can be extended to be a pair of Gabor or shift-invariant alternate dual frames for $L^2(\mathbb{R}^d)$ [6, 10]. In this article, we show that any two Bessel sequences can be extended to be a pair of oblique dual frame sequences and give the minimal number of elements to be added to form oblique dual frame sequences. This result generalizes some of the existing ones in the literature ([10, Proposition 2.1], [18, Theorem 3.1]). The extension of two shift-invariant Bessel sequences of $L^2(\mathbb{R}^d)$ is also considered. Unlike the frame extension, our extension naturally leads us to consideration of the 'minimal projection' extending the given Bessel sequences. Arguably, the best-known situation where this kind of extension is necessary is the construction of a pair of biorthogonal frame multiresolution analyses ([5, 15]) starting from two sets of generators such that their shifts are only assumed to be Bessel sequences (see Section 4 for the details). We discuss the existence or non-existence of the minimal projection.

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2. Notations and preliminary results. A sequence $X := \{x_j\}_{j\in\mathbb{J}} \subset \mathcal{H}$ is a *Bessel sequence* if there is non-negative β such that

$$\sum_{j\in\mathbb{J}} |\langle h, x_j\rangle|^2 \le \beta ||h||^2$$

for each $h \in \mathcal{H}$; it is a *frame* for $\mathcal{H}$ if there are positive α and β such that

$$\alpha ||h||^2 \le \sum_{j\in\mathbb{J}} |\langle h, x_j\rangle|^2 \le \beta ||h||^2$$

for each $h \in \mathcal{H}$; it is a *Riesz basis* of $\mathcal{H}$ if X is complete in $\mathcal{H}$ and there are positive α and β such that

$$\alpha \sum_{j\in\mathbb{J}} |a(j)|^2 \le \Big\| \sum_{j\in\mathbb{J}} a(j)\, x_j \Big\|^2 \le \beta \sum_{j\in\mathbb{J}} |a(j)|^2$$

for each $a \in \ell^2(\mathbb{J})$. Finally, X is a *frame (or Riesz) sequence* if it is a frame for (or a Riesz basis of) its closed linear span. These bounds α and β are not unique, but there are optimal bounds. We let α_X and β_X denote the optimal Bessel/frame/Riesz bounds of X. If $\alpha_X = \beta_X$ for a frame X, then X is said to be a *tight frame*. In particular, if $\alpha_X = \beta_X = 1$, then X is called a *Parseval frame*. We state only the very basic facts on these sequences that are directly needed in our discussion, and refer to existing literature for further information [7, 12, 14]. For a Bessel sequence X, we define its *synthesis operator* T_X by

$$T_X : \ell^2(\mathbb{J}) \to \mathcal{H}, \quad T_X a := \sum_{j\in\mathbb{J}} a(j)\, x_j,$$

which is well-known to be bounded. Its adjoint, called the *analysis operator* of X, is

$$T_X^* : \mathcal{H} \to \ell^2(\mathbb{J}), \quad T_X^* h = (\langle h, x_j\rangle)_{j\in\mathbb{J}}\,.$$

Recall that X is a frame sequence if and only if T_X is bounded and has closed range; X is a frame for $\mathcal{H}$ if and only if T_X is bounded and onto. The *frame operator* of a Bessel sequence X is defined to be $S_X := T_X T_X^* \in \mathcal{B}(\mathcal{H})$.

In this article, we are especially concerned with the union of two (vector or scalar) sequences. This sometimes (especially in the abstract setting without any structure) causes minor notational problems. Suppose that we are given two sequences $X := \{x_i\}_{i\in\mathbb{I}}$ and $Y := \{y_j\}_{j\in\mathbb{J}}$ in $\mathcal{H}$. If the index sets $\mathbb{I}$ and $\mathbb{J}$ are disjoint, then we define their union by

$$X \cup Y := \{z_k : z_k = x_i \text{ or } y_j\}\,,$$

which is indexed by $\mathbb{I} \cup \mathbb{J}$. Since we only deal with Bessel sequences, there will not be any rearrangement problem. If $\mathbb{I}$ and $\mathbb{J}$ are not disjoint, we introduce another index

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set $\mathbb{K}$, which is disjoint from $\mathbb{I}$ and whose cardinality is $\mathbb{J}$, and we re-index Y by using $\mathbb{K}$. Then we can define the union of X and Y. In this way, we may freely assume that $\mathbb{I}$ and $\mathbb{J}$ are disjoint whenever this assumption is necessary. Moreover, if we are given two sequences X and Y indexed by the same index set $\mathbb{J}$, then we define the formal index set $\tilde{\mathbb{J}} := \{\tilde{j} : j \in \mathbb{J}\}$ which is disjoint with $\mathbb{J}$ and we re-index Y by $\tilde{\mathbb{J}}$. Then the union of X and Y is indexed by $\mathbb{J} \cup \tilde{\mathbb{J}}$. Now, suppose that we are given two Bessel sequence $X := \{x_i : i \in \mathbb{I}\}$ and $Y := \{y_j : j \in \mathbb{J}\}$ in $\mathcal{H}$, where $\mathbb{I}$ and $\mathbb{J}$ are disjoint. The synthesis operator of $X \cup Y$ is

$$T_{X\cup Y} : \ell^2(\mathbb{I} \cup \mathbb{J}) \to \mathcal{H}, \quad T_{X\cup Y} a = \sum_{i\in\mathbb{I}} a(i)x_i + \sum_{j\in\mathbb{J}} a(j)y_j,$$

and its analysis operator is

$$T^*_{X\cup Y} : \mathcal{H} \to \ell^2(\mathbb{I} \cup \mathbb{J}), \quad T^*_{X\cup Y} f = T^*_X f \cup T^*_Y f,$$

where the last term is the union of two scalar sequences. Note that, by definition,

$$\|T^*_{X\cup Y} f\|^2 = \|T^*_X f\|^2 + \|T^*_Y f\|^2 .$$

Moreover,

$$S_{X\cup Y} = S_X + S_Y. \tag{2.1}$$

More generally, if F and G are sequences in $\mathcal{H}$ indexed by $\mathbb{I}$ and X and Y are sequences in $\mathcal{H}$ index by $\mathbb{J}$, then the *mixed frame operator (or mixed dual Gramian)*, $T_{F\cup G}T^*_{X\cup Y} : \mathcal{H} \to \mathcal{H}$, of $F \cup X$ and $G \cup Y$ is, by definition,

$$T_{F\cup X}T^*_{G\cup Y} = T_F T^*_G + T_X T^*_Y. \tag{2.2}$$

An operator $P \in \mathcal{B}(\mathcal{H})$ is a *projection (or oblique projection)* if $P^2 = P$. We write $\mathcal{A} \le \mathcal{H}$ if $\mathcal{A}$ is a closed subspace of $\mathcal{H}$ and $\mathcal{A} < \mathcal{H}$ if $\mathcal{A}$ is a proper closed subspace of $\mathcal{H}$. Suppose that $\mathcal{A}, \mathcal{B} \le \mathcal{H}$. Recall that $\mathcal{A}$ and $\mathcal{B}$ are said to be *complementary spaces* if $\mathcal{A} + \mathcal{B} = \mathcal{H}$ and $\mathcal{A}$ and $\mathcal{B}$ have trivial intersection. In this case, we write $\mathcal{H} = \mathcal{A} \dotplus \mathcal{B}$. If P is a projection, then $\operatorname{ran} P$ and $\ker P$ are complementary spaces. Conversely, if $\mathcal{A}$ and $\mathcal{B}$ are complementary spaces, then there is a projection $P_{\mathcal{A},\mathcal{B}}$ such that $\operatorname{ran} P_{\mathcal{A},\mathcal{B}} = \mathcal{A}$ and $\ker P_{\mathcal{A},\mathcal{B}} = \mathcal{B}$ ([8, Section II.3]). We let $P_{\mathcal{A}} := P_{\mathcal{A},\mathcal{A}^\perp}$ denote the orthogonal projection onto $\mathcal{A}$.

Suppose that $\mathcal{U}$ and $\mathcal{V}$ are two closed subspaces of $\mathcal{H}$. We define ([20, Eqs. (28) and (38)])

$$S(\mathcal{U},\mathcal{V}) := \begin{cases} \sup\left\{\frac{\|P_{\mathcal{V}}u\|}{\|u\|} : u \in \mathcal{U}\setminus\{0\}\right\} & \text{if } \mathcal{U} \neq \{0\}, \\ 0 & \text{if } \mathcal{U} = \{0\}, \end{cases}$$

$$R(\mathcal{U},\mathcal{V}) := \begin{cases} \inf\left\{\frac{\|P_{\mathcal{V}}u\|}{\|u\|} : u \in \mathcal{U}\setminus\{0\}\right\} & \text{if } \mathcal{U} \neq \{0\}, \\ 1 & \text{if } \mathcal{U} = \{0\}. \end{cases}$$

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Note that, for $u \in \mathcal{U}$, $||P_{\mathcal{V}}u|| \geq R(\mathcal{U},\mathcal{V})||u||$. A bounded operator T is said to be *bounded below* if there is $\varepsilon > 0$ such that $||Tx|| \geq \varepsilon||x||$ for each x. Recall that T is bounded below if and only if T^* is onto. We recall that the following conditions are equivalent:

- $R(\mathcal{U},\mathcal{V}) >$ and $R(\mathcal{V},\mathcal{U}) > 0$;
- $R(\mathcal{U},\mathcal{V}) = R(\mathcal{V},\mathcal{U}) > 0$;
- $\mathcal{U}$ and $\mathcal{V}^\perp$ are complementary spaces;
- $P_{\mathcal{V}} : \mathcal{U} \to \mathcal{V}$ is bounded below and onto.

If any of the above conditions is satisfied, we say that $\mathcal{U}$ and $\mathcal{V}$ satisfy the *angle condition*. We also recall that $S(\mathcal{U},\mathcal{V}) < 1$ if and only if $\mathcal{U}+\mathcal{V}$ is closed and $\mathcal{U}$ and $\mathcal{V}$ have trivial intersection. See [2, 11, 13, 17, 19, 20] for further results on $S(\mathcal{U},\mathcal{V})$ and $R(\mathcal{U},\mathcal{V})$.

Now, suppose that X and Y are Bessel sequences indexed by $\mathbb{J}$. We say that they are *oblique dual frame sequences* [9, 13] if X and Y are frame sequences and

$$x = \sum_{j\in\mathbb{J}} \langle x, y_j\rangle\, x_j = (T_X T_Y^*|_{\overline{\operatorname{span}}\, X})\, x, \quad \forall\, x \in \overline{\operatorname{span}}\, X.$$

In this case, $\operatorname{ran} T_X$ and $\operatorname{ran} T_Y$ satisfy the angle condition, and so do $\operatorname{ran} T_X^*$ and $\operatorname{ran} T_Y^*$. We say that X and Y are *oblique dual frame sequences with respect to* $\mathcal{U}$ *and* $\mathcal{V}$ if they are oblique dual frame sequences and $\operatorname{ran} T_X = \mathcal{U}$ and $\operatorname{ran} T_Y = \mathcal{V}$. The following is [16, Proposition 2.8] (see also [1]).

Lemma 2.1 ([16]). *Suppose that X and Y are Bessel sequences. Then the following conditions are equivalent:*

- *X and Y are oblique dual frame sequences;*
- *X and Y are frame sequences and $T_X T_Y^* = P_{\overline{\operatorname{span}}\, X, (\overline{\operatorname{span}}\, Y)^\perp}$;*
- *X is a frame sequence and $T_X T_Y^* T_X = T_X$ and $T_Y^* T_X T_Y^* = T_Y^*$.*

And the following is Theorem 1.4 of [13].

Lemma 2.2 ([13]). *If $\mathcal{U}$ and $\mathcal{V}$ are closed subspaces satisfying the angle condition and X is a frame for $\mathcal{U}$, then there is Y such that X and Y are oblique dual frame sequences with respect to $\mathcal{U}$ and $\mathcal{V}$.*

For $k \in \mathbb{Z}^d$, define the *shift-operator* $T_k : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by $(T_k f)(x) := f(x-k)$. A closed subspace $\mathcal{U}$ of $\mathcal{H}$ is a *shift-invariant space* if $\mathcal{U}$ is invariant under T_k for each $k \in \mathbb{Z}^d$. If $F \subset L^2(\mathbb{R}^d)$, then we let

$$E(F) := \left\{T_k f : f \in F, k \in \mathbb{Z}^d\right\},$$

and call it the collection of the *shifts* of F. In the theory of shift-invariant spaces, it is often convenient to permit the elements of F to be repeated. That is, $E(\{f,f\}) \neq$

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$E(\{f\})$ even though their closed spans are equal. We finally recall that, by the Bownik decomposition [4, Theorem 3.3], for any shift-invariant space $\mathcal{U}$, there is a countable $\Xi \subset L^2(\mathbb{R}^d)$ such that $E(\Xi)$ is a Parseval frame for $\mathcal{U}$. See [4] for the basic facts on the theory of shift-invariant spaces. The following is Theorem 1.5 of [13].

LEMMA 2.3 ([13]). *Suppose that $\mathcal{U}$ and $\mathcal{V}$ are shift-invariant spaces of $L^2(\mathbb{R}^d)$ satisfying the angle condition and that $E(\Phi)$ is a frame for $\mathcal{U}$ for some countable $\Phi \subset L^2(\mathbb{R}^d)$. Then there is $\Psi \subset L^2(\mathbb{R}^d)$ whose cardinality is that of Φ such that $E(\Phi)$ and $E(\Psi)$ are oblique dual frame sequences with respect to $\mathcal{U}$ and $\mathcal{V}$.*

3. Extension of two Bessel sequences to oblique dual frame sequences. We show that two Bessel sequences can always be extended to be two oblique dual frame sequences. This extends some of the results in [6, 10, 18]. Suppose that we are given two Bessel sequences F and G in $\mathcal{H}$. Then $\mathcal{U} := \mathcal{H}$ and $\mathcal{V} := \mathcal{H}$ satisfy the angle condition with $F \subset \mathcal{U}$ and $G \subset \mathcal{V}$.

THEOREM 3.1. *Let F and G be Bessel sequences in $\mathcal{H}$ indexed by $\mathbb{I}$. Suppose there are closed subspaces $\mathcal{U} \supset F$ and $\mathcal{V} \supset G$ satisfying the angle condition. Then there are Bessel sequences X and Y indexed by $\mathbb{J}$ such that $F \cup X$ and $G \cup Y$ are oblique dual frame sequences with respect to $\mathcal{U}$ and $\mathcal{V}$.*

Proof. Define

$$T := P_{\mathcal{U},\mathcal{V}^\perp} - T_F T_G^* \in \mathcal{B}(\mathcal{H}). \tag{3.1}$$

Let X be any frame for $\mathcal{U}$. Then, by Lemmas 2.2 and 2.1, there is a frame B for $\mathcal{V}$ such that

$$T_X T_B^* = P_{\mathcal{U},\mathcal{V}^\perp}. \tag{3.2}$$

Let

$$Y := T^*(B), \tag{3.3}$$

which is easily seen to be a Bessel sequence. In particular, $T_Y = T^* T_B$. Hence, $T_Y^* = T_B^* T$. Note that

$$\operatorname{ran} T_Y \subset \operatorname{ran} T^* = \operatorname{ran}\left(P_{\mathcal{V},\mathcal{U}^\perp} - T_G T_F^*\right) \subset \mathcal{V} + \operatorname{ran} T_G \subset \mathcal{V}. \tag{3.4}$$

Since $\operatorname{ran} T \subset \mathcal{U}$, $T = P_{\mathcal{U},\mathcal{V}^\perp} T = T_X T_B^* T$. Therefore,

$$\begin{aligned} P_{\mathcal{U},\mathcal{V}^\perp} &= T_F T_G^* + T = T_F T_G^* + (T_X T_B^*)\, T = T_F T_G^* + T_X\, (T_B^* T) \\ &= T_F T_G^* + T_X T_Y^* = T_{F\cup X} T_{G\cup Y}^*, \end{aligned} \tag{3.5}$$

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where the last equality holds by (2.2). In particular, $\mathcal{U} = \operatorname{ran} P_{\mathcal{U},\mathcal{V}^\perp} \subset \operatorname{ran} T_{F\cup X}$. Since $F \cup X \subset \mathcal{U}$, $\operatorname{ran} T_{F\cup X} \subset \mathcal{U}$. Therefore $\operatorname{ran} T_{F\cup X} = \mathcal{U}$. This shows that $F \cup X$ is a frame for $\mathcal{U}$. On the other hand, by taking the adjoint of both sides of (3.5), we have $P_{\mathcal{V},\mathcal{U}^\perp} = T_{G\cup Y} T^*_{F\cup X}$. In particular, $\mathcal{V} = \operatorname{ran} P_{\mathcal{V},\mathcal{U}^\perp} \subset \operatorname{ran} T_{G\cup Y}$. Note that, by (3.4), we have

$$\operatorname{ran} T_{G\cup Y} \subset \operatorname{ran} T_G + \operatorname{ran} T_Y \subset \mathcal{V} + \mathcal{V} = \mathcal{V}.$$

Therefore $\operatorname{ran} T_{G\cup Y} = \mathcal{V}$. This shows that $G \cup Y$ is a frame for $\mathcal{V}$. It remains to show that $F \cup X$ and $G \cup Y$ are oblique duals of each other. (3.5) implies that

$$(T_{F\cup X} T^*_{G\cup Y})\, T_{F\cup X} = P_{\mathcal{U},\mathcal{V}^\perp} T_{F\cup X} = T_{F\cup X}.$$

Since $\operatorname{ran} T_G \subset \mathcal{V}$, $\ker T^*_G = (\operatorname{ran} T_G)^\perp \supset \mathcal{V}^\perp$. If $h \in \mathcal{V}^\perp \subset \ker T^*_G$, then

$$\begin{aligned} T^*_{G\cup Y} h &= T^*_G h \cup T^*_Y h = 0_{\ell^2(\mathbb{I})} \cup T^*_B T h = 0_{\ell^2(\mathbb{I})} \cup \left(T^*_B \left(P_{\mathcal{U},\mathcal{V}^\perp} - T_F T^*_G\right) h\right) \\ &= 0_{\ell^2(\mathbb{I})} \cup \left(T^*_B P_{\mathcal{U},\mathcal{V}^\perp} h - T_F T^*_G h\right) = 0_{\ell^2(\mathbb{I})} \cup 0_{\ell^2(\mathbb{J})} = 0_{\ell^2(\mathbb{I}\cup\mathbb{J})}. \end{aligned}$$

Hence, $\mathcal{V}^\perp \subset \ker T^*_{G\cup Y}$. Since $\operatorname{ran}(I - P_{\mathcal{U},\mathcal{V}^\perp}) = \ker P_{\mathcal{U},\mathcal{V}^\perp} = \mathcal{V}^\perp$, it follows that $T^*_{G\cup Y}(I - P_{\mathcal{U},\mathcal{V}^\perp}) = 0$. Therefore,

$$T^*_{G\cup Y}\left(T_{F\cup X} T^*_{G\cup Y}\right) = T^*_{G\cup Y} P_{\mathcal{U},\mathcal{V}^\perp} = T^*_{G\cup Y} P_{\mathcal{U},\mathcal{V}^\perp} + T^*_{G\cup Y}\left(I - P_{\mathcal{U},\mathcal{V}^\perp}\right) = T^*_{G\cup Y}.$$

Hence, $F \cup X$ and $G \cup Y$ are oblique dual frame sequences by Lemma 2.1. ∎

Note that in the construction in Theorem 3.1 we can take X to be any frame for $\mathcal{U}$. We now consider the minimal cardinality of X and Y.

THEOREM 3.2. *Suppose that F and G are Bessel sequences in $\mathcal{H}$ indexed by $\mathbb{I}$. Suppose there are closed subspaces $\mathcal{U} \supset F$ and $\mathcal{V} \supset G$ satisfying the angle condition. If there are Bessel sequences X and Y indexed by $\mathbb{J}$ such that $F \cup X$ and $G \cup Y$ are oblique dual frame sequences with respect to $\mathcal{U}$ and $\mathcal{V}$, then*

$$\dim \operatorname{ran}\left(P_{\mathcal{U},\mathcal{V}^\perp} - T_F T^*_G\right) \le \operatorname{card} X (= \operatorname{card} \mathbb{J}). \tag{3.6}$$

Moreover, the equality in (3.6) *can be achieved.*

Proof. Since $T_X T^*_Y = P_{\mathcal{U},\mathcal{V}^\perp} - T_F T^*_G$ if $F \cup X$ and $G \cup Y$ are oblique dual frame sequences with respect to $\mathcal{U}$ and $\mathcal{V}$, we have

$$\dim \operatorname{ran}\left(P_{\mathcal{U},\mathcal{V}^\perp} - T_F T^*_G\right) \le \dim \operatorname{ran} T_X T^*_Y \le \dim \operatorname{ran} T_X \le \operatorname{card} X = \operatorname{card} \mathbb{J}.$$

Hence, (3.6) holds. We now show that the equality in (3.6) can be achieved. Suppose first that the left-hand side of (3.6) is ∞. Then, there are X and Y such that $F \cup X$ and $G \cup Y$ are oblique dual frame sequences with respect to $\mathcal{U}$ and $\mathcal{V}$ by Theorem 3.1. Now, (3.6) implies that $\operatorname{card} X = \infty$. Suppose, on the other hand, that

$$N := \dim \operatorname{ran}\left(P_{\mathcal{U},\mathcal{V}^\perp} - T_F T^*_G\right) < \infty.$$

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Let $T := P_{\mathcal{U},\mathcal{V}^\perp} - T_F T_G^* \in \mathcal{B}(\mathcal{H})$. Since T has finite dimensional range, it has closed range. Choose an orthonormal basis $X := \{x_1, x_2, \ldots, x_N\}$ of $\operatorname{ran} T \subset \mathcal{U}$. Define $Y := \{y_1, y_2, \ldots, y_N\}$, where $y_j := T^* x_j$. Since $\operatorname{ran} T^* \subset \mathcal{V}$, $Y \subset \mathcal{V}$. Moreover, for any $h \in \mathcal{H}$,

$$T_X T_Y^* h = \sum_{j=1}^{N} \langle h, T^* x_j \rangle \, x_j = \sum_{j=1}^{N} \langle Th, x_j \rangle \, x_j = Th.$$

Hence, (3.5) holds. The same argument after (3.5) in Theorem 3.1 shows that $F \cup X$ and $G \cup Y$ are oblique dual frame sequences with respect to $\mathcal{U}$ and $\mathcal{V}$. ∎

Note that X in the proof of Theorem 3.2 is not a frame for $\mathcal{U}$ if $\operatorname{ran}(P_{\mathcal{U},\mathcal{V}^\perp} - T_F T_G^*)$ is a proper subspace of $\mathcal{U}$.

The proof of the following lemma is well-known and straightforward.

Lemma 3.3. *Suppose that Φ and Ψ are countable subsets of $L^2(\mathbb{R}^d)$ with the same cardinality. Then, for each $k \in \mathbb{Z}^d$, T_k commutes with $T_{E(\Phi)} T_{E(\Psi)}^*$.*

We show that two Bessel sequences of shifts in $L^2(\mathbb{R}^d)$ can be extended to be oblique dual frame sequences of shifts.

Corollary 3.4. *Let Φ and Ψ be countable subsets of $L^2(\mathbb{R}^d)$ with the same cardinality. Suppose that $\mathcal{U}(\supset \Phi)$ and $\mathcal{V}(\supset \Psi)$ are shift-invariant spaces of $L^2(\mathbb{R}^d)$ satisfying the angle condition and that $E(\Phi)$ and $E(\Psi)$ are Bessel sequences. Then there are countable $\tilde{\Phi}, \tilde{\Psi} \subset L^2(\mathbb{R}^d)$ with the same cardinality such that $E(\Phi) \cup E(\tilde{\Phi})$ and $E(\Psi) \cup E(\tilde{\Psi})$ are oblique dual frame sequences with respect to $\mathcal{U}$ and $\mathcal{V}$.*

Proof. There is $X := E(\Xi)$ such that $E(\Xi)$ is a Parseval frame for $\mathcal{U}$. By Lemma 2.3, there is Θ whose cardinality is that of Ξ such that X and $B := E(\Theta)$ are oblique dual frame sequences with respect to $\mathcal{U}$ and $\mathcal{V}$. Lemma 3.3 and (3.2) imply that T in (3.1) commutes with shifts. Then Y in (3.3) consists of shifts. The proof is complete by following the argument after (3.3) in the proof of Theorem 3.1. ∎

Recall that an invariant space of T is a *reducing space* of T if it is also invariant by T^*. Hence, a shift-invariant space is a reducing space of T_k for each $k \in \mathbb{Z}^d$, which implies that its orthogonal complement is also a shift-invariant space. We now show that a complementary space of a shift-invariant space is not necessarily shift-invariant. We use the following form of the Fourier transform: for $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$,

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} \, dx.$$

It is extended to be a unitary operator on $L^2(\mathbb{R}^d)$ by the Plancherel theorem.

Proposition 3.5. *There is a shift-invariant space $\mathcal{U} \leq L^2(\mathbb{R}^d)$ such that one of the complementary spaces of $\mathcal{U}$ is not shift-invariant.*

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Proof. Suppose it is not the case. Then, any projection P whose range is shift-invariant necessarily commutes with T_k for each $k \in \mathbb{Z}^d$. Suppose that $\mathcal{U}$ is a shift-invariant space. Recall that, for any $A \in \mathcal{B}(\mathcal{U}^\perp, \mathcal{U})$, the following operator matrix with respect to the decomposition $L^2(\mathbb{R}^d) = \mathcal{U} \oplus \mathcal{U}^\perp$

$$\begin{bmatrix} I_{\mathcal{U}} & A \\ 0 & 0 \end{bmatrix}$$

defines a projection P whose range is $\mathcal{U}$. Clearly, the matrix representation of T_k with respect to the same decomposition is

$$\begin{bmatrix} T_k & 0 \\ 0 & T_k \end{bmatrix}$$

since $\mathcal{U}^\perp$ is also shift-invariant. Since P and T_k commute, so do A and T_k. It suffices to construct a shift-invariant space $\mathcal{U}$ of $L^2(\mathbb{R})$ and $A \in \mathcal{B}(\mathcal{U}^\perp, \mathcal{U})$ such that A and T_k do not commute for some $k \in \mathbb{Z}$. For $j \in \mathbb{Z}$, let

$$\varphi_j := (\chi_{\mathbb{T}+j})^\vee,$$

where $\vee$ denotes the inverse Fourier transform and $\mathbb{T} := [-1/2, 1/2)$. Define

$$\mathcal{U} := \mathcal{S}(\{\varphi_j : j \in 2\mathbb{Z}\}).$$

It is easy to see that

$$\mathcal{U}^\perp = \mathcal{S}(\{\varphi_j : j \in 2\mathbb{Z}+1\}).$$

Note that $E(\{\varphi_j : j \in \mathbb{Z}\})$ and $E(\{\varphi_j : j \in 2\mathbb{Z}+1\})$ are orthonormal bases of $\mathcal{U}$ and $\mathcal{U}^\perp$, respectively. Define a linear map $A : \mathcal{U}^\perp \to \mathcal{U}$ by extending linearly the following map:

$$\begin{cases} AT_k\varphi_{2j+1} & := & T_k\varphi_{2j}, \quad j \in \mathbb{Z} \setminus \{0\}, k \in \mathbb{Z}, \\ AT_k\varphi_1 & := & T_k\varphi_0, \quad k \in \mathbb{Z} \setminus \{0,1\}, \\ A\varphi_1 & := & T_1\varphi_0, \\ AT_1\varphi_1 & := & \varphi_0. \end{cases}$$

This is a unitary map by definition. On the other hand,

$$T_1A\varphi_1 = T_1T_1\varphi_0 = T_2\varphi_0 \neq \varphi_0 = AT_1\varphi_1.$$

Hence, A and T_1 do not commute. $\square$

4. The minimal projection problem. We first address the motivation of this problem. Let $D \in \mathcal{B}(L^2(\mathbb{R}^d))$ be the unitary dyadic dilation operator such that $(Df)(x) := 2^{d/2}f(2x)$. Suppose we are given two finite sequences Φ and $\tilde{\Phi}$ such that $E(\Phi)$ is a frame for V_0 and $E(\tilde{\Phi})$ is a frame for $\tilde{V}_0$. Suppose also that

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- $E(\Phi)$ and $E(\tilde{\Phi})$ are oblique dual frame sequences,
- $V_0 \subset D(V_0)$ and $\tilde{V}_0 \subset D(\tilde{V}_0)$,
- $\cap_{n\in\mathbb{Z}} D^n(V_0) = \{0\} = \cap_{n\in\mathbb{Z}} D^n(\tilde{V}_0)$,
- $\cup_{n\in\mathbb{Z}} D^n(V_0)$ and $\cup_{n\in\mathbb{Z}} D^n(\tilde{V}_0)$ are dense in $L^2(\mathbb{R}^d)$.

Then, we say that $\{D^n(V_0)\}_{n\in\mathbb{Z}}$ and $\{D^n(\tilde{V}_0)\}_{n\in\mathbb{Z}}$ are a pair of biorthogonal frame multiresolution analyses [5, 15]. If we are given a pair, then, with luck, we may construct a pair of oblique dual wavelet frames. Now, suppose we are given two finite sequences A and B of $L^2(\mathbb{R}^d)$ such that $E(A)$ and $E(B)$ are Bessel sequences. If $\mathcal{U}$ and $\mathcal{V}$ are two finitely generated shift-invariant spaces satisfying the angle condition such that $A \subset \mathcal{U}$ and $B \subset \mathcal{V}$, then by Corollary 3.4, we can construct $V_0 = \mathcal{U}$ and $\tilde{V}_0 = \mathcal{V}$ satisfying the first condition of a pair of biorthogonal frame multiresolution analyses. The last two conditions are not hard to satisfy [3, Theorems 4.3 and 4.9]. (The second one is not that easy. It is related with the 'refinement equation'.) For computational purposes, we hope that the lengths of $\mathcal{U}$ and $\mathcal{V}$ (that is, the minimal cardinality of generating sequences) are as small as possible. So if we are given another such finitely generated shift-invariant spaces $\mathcal{A}$ and $\mathcal{B}$ such that $A \subset \mathcal{A} \subset \mathcal{U}$ and $B \subset \mathcal{B} \subset \mathcal{V}$, then we prefer the pair $\mathcal{A}$ and $\mathcal{B}$ to the pair $\mathcal{U}$ and $\mathcal{V}$. Now, consider the projections $P := P_{\mathcal{A},\mathcal{B}^\perp}$ and $Q := P_{\mathcal{U},\mathcal{V}^\perp}$. Then we have $\mathcal{A} = \operatorname{ran} P \subset \operatorname{ran} Q = \mathcal{U}$ and $\mathcal{B} = \operatorname{ran} P^* \subset \operatorname{ran} Q^* = \mathcal{V}$. So we may ask whether there are minimal such pair of spaces. This leads us to consider the minimal projection problem (Problem 4.1)

For two projections P and Q in $\mathcal{B}(\mathcal{H})$, define $P \prec Q$ if $\operatorname{ran} P \subset \operatorname{ran} Q$ and $\operatorname{ran} P^* \subset \operatorname{ran} Q^*$. Consider a projection P which is not an orthogonal projection. Then $P \prec I_{\mathcal{H}}$ and $1 = \|I_{\mathcal{H}}\| < \|P\|$ by [8, Proposition II.3.2]. Suppose that $\{e_1, e_2, e_3\}$ is an orthonormal subset of $\mathcal{H}$. Let $\mathcal{U} := \operatorname{span}\{e_1, e_2\}$, $\mathcal{V} := \operatorname{span}\{e_1, 2^{-1/2}(e_2 + e_3)\}$ and $\mathcal{W} := \operatorname{span}\{e_1\}$. It is routine to see that $\mathcal{U} \dotplus \mathcal{V}^\perp = \mathcal{H}$. Hence, $P := P_{\mathcal{U},\mathcal{V}^\perp}$ is a projection such that $P_{\mathcal{W}} \prec P$. In this case, $1 = \|P_{\mathcal{W}}\| < \|P\|$. So there are projections such that $P \prec Q$ with $||P|| < ||Q||$ and there are projections $R \prec S$ with $||S|| < ||R||$.

Problem 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be two closed subspaces of $\mathcal{H}$. Find a projection P (temporarily called the *minimal projection extending $\mathcal{A}$ and $\mathcal{B}$*) satisfying the following conditions:

(1) $\mathcal{A} \subset \operatorname{ran} P$;
(2) $\mathcal{B} \subset \operatorname{ran} P^*$;
(3) If Q is another projection satisfying (1) and (2), then $P \prec Q$.

Trivially, the minimal projection is unique if it exists. On the other hand, the maximal projection is trivial since it is always $I_{\mathcal{H}}$. If $\mathcal{A} = \mathcal{B}$, the the minimal projection is $P_{\mathcal{A}}$. Anyway, if we define $\mathcal{C} := \overline{\operatorname{span}}\,(\mathcal{A} \cup \mathcal{B})$, then $P_{\mathcal{C}}$ is a projection satisfying (1) and (2).

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We show that, in many cases, there does not exist the minimal projection. Which means that often there is no 'minimal' solution of the extension problem. The following lemma is a restatement of [13, Lemma 2.2].

LEMMA 4.2 ([13]). *Let $\mathcal{C} < \mathcal{H}$ with an orthonormal basis $\{c_j : j \in \mathbb{J}\}$ and $f \in \mathcal{C}^\perp$ with $||f|| = 1$. We assume that $1 \in \mathbb{J}$. Define*

$$\mathcal{C}_1 := \overline{\operatorname{span}}\left(\left\{\frac{1}{\sqrt{2}}(c_1 + f)\right\} \bigcup \{c_k : k \neq 1\}\right).$$

Then $R(\mathcal{C}, \mathcal{C}_1) = R(\mathcal{C}_1, \mathcal{C}) \geq 2^{-1/2} > 0$.

Trivially, $\mathcal{C} \cap \mathcal{C}_1 < \mathcal{C}_1$. Now, let $\mathcal{A}, \mathcal{B} \leq \mathcal{H}$. If both $R(\mathcal{A}, \mathcal{B})$ and $R(\mathcal{B}, \mathcal{A})$ are positive, then $P_{\mathcal{A},\mathcal{B}^\perp}$ is a projection. It is trivially the minimal projection extending $\mathcal{A}$ and $\mathcal{B}$. The remaining cases to be considered are: only one is 0; both are 0. Moreover, if one of $\mathcal{A}$ or $\mathcal{B}$ is $\mathcal{H}$, then $I_{\mathcal{H}}$ is the minimal projection. We now consider the first non-trivial case.

THEOREM 4.3. *Suppose that $\mathcal{H}$ is a separable Hilbert space, $\mathcal{B} < \mathcal{H}$ and*

$$0 = R(\mathcal{B}, \mathcal{A}) < R(\mathcal{A}, \mathcal{B}). \tag{4.1}$$

Suppose also that $\overline{\mathcal{A} + \mathcal{B}} < \mathcal{H}$. Then the minimal projection extending $\mathcal{A}$ and $\mathcal{B}$ does not exist.

Proof. If $\mathcal{A} = \mathcal{H}$ then $R(\mathcal{B}, \mathcal{A}) = R(\mathcal{B}, \mathcal{H}) = 1$. Hence, $\mathcal{A} < \mathcal{H}$. We have, by [2, Lemma 3.2],

$$\mathcal{B} = P_{\mathcal{B}}(\mathcal{A}) \oplus (\mathcal{B} \ominus \mathcal{A}) \quad \text{and} \quad \mathcal{C} := \mathcal{B} \ominus \mathcal{A} \neq \{0\}. \tag{4.2}$$

Moreover, $P_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ is bounded below by $R(\mathcal{A}, \mathcal{B})$ (in particular, $P_{\mathcal{B}}(\mathcal{A})$ is closed) but is not onto by [2, Lemma 3.2].

We first show that there is a projection extending $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{U} := \mathcal{A} \oplus \mathcal{C}$. Consider

$$P_{\mathcal{B}} : \mathcal{U}\ (= \mathcal{A} \oplus \mathcal{C}) \to \mathcal{B}\ (= P_{\mathcal{B}}(\mathcal{A}) \oplus \mathcal{C}). \tag{4.3}$$

For $u = a + b \in \mathcal{A} \oplus \mathcal{C}$,

$$P_{\mathcal{B}}(u) = P_{\mathcal{B}}(a + b) = P_{\mathcal{B}}a + P_{\mathcal{B}}b = P_{\mathcal{B}}a + b.$$

(4.2) implies that the map (4.3) is onto. On the other hand,

$$\begin{aligned}
\|P_{\mathcal{B}}(a + b)\|^2 &= \|P_{\mathcal{B}}a + b\|^2 = \|P_{\mathcal{B}}a\|^2 + \|b\|^2 \quad \text{(by (4.2))} \\
&\geq R(\mathcal{A}, \mathcal{B})^2 \|a\|^2 + \|b\|^2 \geq R(\mathcal{A}, \mathcal{B})^2 \left(\|a\|^2 + \|b\|^2\right) \\
&= R(\mathcal{A}, \mathcal{B})^2 \|a + b\|^2 .
\end{aligned}$$

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Hence, the map in (4.3) is bounded below. Therefore $\mathcal{U}$ and $\mathcal{B}$ satisfy the angle condition. This shows that $P_{\mathcal{U},\mathcal{B}^\perp}$ is a projection extending $\mathcal{A}$ and $\mathcal{B}$.

Since $\overline{\mathcal{A}+\mathcal{B}} < \mathcal{H}$, there is f with unit norm that is orthogonal to both $\mathcal{A}$ and $\mathcal{B}$. Let $\{c_j\}_{j\in\mathbb{J}}$ be an orthonormal basis of $\mathcal{C}$. Define $\mathcal{C}_1$ as in Lemma 4.2 and let $\mathcal{U}_1 := \mathcal{A}\oplus\mathcal{C}_1$. Recall that $P_{\mathcal{C}} : \mathcal{C}_1 \to \mathcal{C}$ is bounded below and onto. Consider the following map:

$$P_{\mathcal{B}} : \mathcal{A}\oplus\mathcal{C}_1 \ (=\mathcal{U}_1) \to P_{\mathcal{B}}(\mathcal{A})\oplus\mathcal{C} \ (=\mathcal{B}). \tag{4.4}$$

Let $d\in\mathcal{C}_1$ and $\{P_{\mathcal{B}}a_i\}_{i\in\mathbb{I}}$ be an orthonormal basis of $P_{\mathcal{B}}(\mathcal{A})$, where $a_i\in\mathcal{A}$. Recall that $\{2^{-1/2}(c_1+f)\}\cup\{c_j\}_{j\in\mathbb{J},j\neq 1}$ is an orthonormal basis of $\mathcal{C}_1$. Hence,

$$d = \left\langle d, \frac{1}{\sqrt{2}}(c_1+f)\right\rangle \frac{1}{\sqrt{2}}(c_1+f) + \sum_{j\in\mathbb{J},j\neq 1}\langle d, c_j\rangle\, c_j.$$

Since $f\perp\mathcal{B}$ and $c_j\in\mathcal{B}$ for each $j\in\mathbb{J}$, $P_{\mathcal{B}}d\in\mathcal{C}\perp\mathcal{A}$. Hence, $\langle d, P_{\mathcal{B}}a_i\rangle = \langle P_{\mathcal{B}}d, a_i\rangle = 0$ for each $i\in\mathbb{I}$. Therefore, $P_{\mathcal{B}}d = P_{\mathcal{C}}d$ for any $d\in\mathcal{C}_1$. In particular, for $a+d\in\mathcal{A}\oplus\mathcal{C}_1$, $P_{\mathcal{B}}(a+d) = P_{\mathcal{B}}a + P_{\mathcal{C}}d \in P_{\mathcal{B}}(\mathcal{A})\oplus\mathcal{C}$. It is now routine to see that the map in (4.4) is bounded below and onto since the maps $P_{\mathcal{B}} : \mathcal{A}\to P_{\mathcal{B}}(\mathcal{A})$ and $P_{\mathcal{C}} : \mathcal{C}_1\to\mathcal{C}$ are bounded below and onto. Therefore, $P_{\mathcal{U}_1,\mathcal{B}^\perp}$ is another projection extending $\mathcal{A}$ and $\mathcal{B}$. If Q is the minimal projection extending $\mathcal{A}$ and $\mathcal{B}$, then $\mathcal{A}\subset\operatorname{ran}Q$ and $\mathcal{B}\subset\operatorname{ran}Q^*$ and $Q\prec P_{\mathcal{U},\mathcal{B}^\perp}$ and $Q\prec P_{\mathcal{U}_1,\mathcal{B}^\perp}$. Hence,

$$\begin{aligned}&\mathcal{A}\subset\operatorname{ran}Q\subset\mathcal{U}\cap\mathcal{U}_1 = \mathcal{A}\oplus(\mathcal{C}\cap\mathcal{C}_1) < \mathcal{A}\oplus\mathcal{C}_1 = \mathcal{U}_1 \quad\text{and}\\ &\mathcal{B}\subset\operatorname{ran}Q^*\subset\mathcal{B}.\end{aligned}$$

In particular, $\ker Q = \mathcal{B}^\perp$, and hence, a closed subspace of $\mathcal{A}\oplus(\mathcal{C}\cap\mathcal{C}_1)$ and $\mathcal{B}^\perp$ are complementary spaces. If they are complementary spaces, then the map

$$P_{\mathcal{B}} : \mathcal{A}\oplus(\mathcal{C}\cap\mathcal{C}_1)\to P_{\mathcal{B}}\oplus\mathcal{C}\,(=\mathcal{B}) \tag{4.5}$$

is onto. Since the map (4.4) is bijective and $\mathcal{C}\cap\mathcal{C}_1$ is a proper closed subspace of $\mathcal{C}_1$, (4.5) is not onto. Therefore, the minimal projection extending $\mathcal{A}$ and $\mathcal{B}$ does not exist. ∎

Note that Theorem 4.3 includes the case that $\mathcal{A}<\mathcal{B}<\mathcal{H}$. It does not exclude the case that one (or both) of $\mathcal{A}$ or $\mathcal{B}$ is finite dimensional. We now show that (4.1) does not imply that $\mathcal{A}+\mathcal{B}$ is closed.

Example 4.4. We construct two closed subspaces $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{H}$ satisfying (4.1) such that $\mathcal{A}+\mathcal{B}$ is not closed and $\overline{\mathcal{A}+\mathcal{B}}<\mathcal{H}$. Let

$$\{x_n\}_{n\in\mathbb{N}}\cup\{y_n\}_{n\in\mathbb{N}}\cup\{z_n\}_{n\in\mathbb{N}}\cup\{w\}$$

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be an orthonormal basis of $\mathcal{H}$. For $n \in \mathbb{N}$ define

$$a_n := \sqrt{1 - \frac{1}{n+1}} x_n + \sqrt{\frac{1}{n+1}}\, y_n.$$

Now, let

$$\mathcal{A} := \overline{\operatorname{span}}\, \{a_n\}_{n\in\mathbb{N}} \quad \text{and} \quad \mathcal{B} := \overline{\operatorname{span}}\, (\{x_n\}_{n\in\mathbb{N}} \cup \{z_n\}_{n\in\mathbb{N}}).$$

Since z_1 is in $\mathcal{B}$ and orthogonal to $\mathcal{A}$, $R(\mathcal{B}, \mathcal{A}) = 0$. On the other hand, for $a \in \mathcal{A}$, there is a square integrable sequence of scalars $(\alpha_n)_{n\in\mathbb{N}}$ such that $a = \sum_{n\in\mathbb{N}} \alpha_n a_n$. Then

$$P_{\mathcal{B}} a = \sum_{m\in\mathbb{N}} \langle a, x_m \rangle\, x_m + \sum_{n\in\mathbb{N}} \langle a, z_n \rangle\, z_n = \sum_{m\in\mathbb{N}} \alpha_m \left(1 - \frac{1}{\sqrt{m+1}}\right) x_m.$$

Therefore,

$$\|P_{\mathcal{B}} a\|^2 \geq \frac{1}{2} \sum_{m\in\mathbb{N}} |\alpha_m|^2 = \frac{1}{2} \|a\|^2.$$

This shows that $R(\mathcal{A}, \mathcal{B}) \geq 2^{-1/2}$. Note that $\mathcal{A} \cap \mathcal{B}$ is trivial. Moreover,

$$\|P_{\mathcal{B}} a_n\| = 1 - \frac{1}{n+1} \to 1.$$

Hence, $S(\mathcal{A}, \mathcal{B}) = 1$. This shows that $\mathcal{A} + \mathcal{B}$ is not closed. Moreover, $w \notin \overline{\mathcal{A} + \mathcal{B}}$. Note that $P_{\mathcal{B}}(\mathcal{A}) = \overline{\operatorname{span}}\, \{x_n\}_{n\in\mathbb{N}}$ and $\mathcal{B} \ominus \mathcal{A} = \overline{\operatorname{span}}\, \{z_n\}_{n\in\mathbb{N}}$. Therefore, (4.2) holds.

Example 4.5. We finally consider the following case:

$$0 = R(\mathcal{B}, \mathcal{A}) = R(\mathcal{A}, \mathcal{B}). \tag{4.6}$$

Let $Q := P_{\mathcal{A}} : \mathcal{B} \to \mathcal{A}$ and $R := P_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$. Note that Q and R are adjoints of each other. If $\mathcal{A} \perp \mathcal{B}$, then they satisfy (4.6). Moreover, $Q = 0 = R$. It is easy to see that $P_{\mathcal{A}\oplus\mathcal{B}}$ is the minimal projection extending $\mathcal{A}$ and $\mathcal{B}$.

Also, if $\mathcal{A} := \operatorname{span}\{e_1, e_2\}$ and $\mathcal{B} := \operatorname{span}\{e_1, e_3\}$, where $\{e_1, e_2, e_3, \ldots\}$ is an orthonormal basis of $\mathcal{H}$, then $\mathcal{A}$ and $\mathcal{B}$ satisfy (4.6). In this case, neither Q nor R are ont-to-one. Moreover, $\operatorname{ran} Q$ is a proper subspace of $\mathcal{A}$ and $\operatorname{ran} R$ is a proper subspace of $\mathcal{B}$. Suppose that $P_{\tilde{\mathcal{A}}, \tilde{\mathcal{B}}^\perp}$ is a projection extending $\mathcal{A}$ and $\mathcal{B}$. Since $\mathcal{B} \leq \tilde{\mathcal{B}}$, $\tilde{\mathcal{B}}^\perp \leq \mathcal{B}^\perp = \overline{\operatorname{span}}\, \{e_2, e_4, e_5, \ldots\}$. Since $\mathcal{A} \subset \tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}} \dotplus \tilde{\mathcal{B}}^\perp = \mathcal{H}$, $e_3 \in \tilde{\mathcal{A}}$. Similarly, $e_2 \in \tilde{\mathcal{B}}$. Hence, $\{e_1, e_2, e_3\} \subset \tilde{\mathcal{A}} \cap \tilde{\mathcal{B}}$. This shows that $P_{\mathcal{C}}$ is the minimal projection extending $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{C} := \operatorname{span}\{e_1, e_2, e_3\}$.

A less trivial example is the following one, which is a variation of [16, Lemma 2.3]. Let $\{a_n\}_{n\in\mathbb{N}} \cup \{c_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Define

$$\mathcal{A} := \overline{\operatorname{span}}\, \{a_n : n \in \mathbb{N}\} \quad \text{and} \quad \mathcal{B} := \overline{\operatorname{span}}\, \{b_n : n \in \mathbb{N}\}, \quad \text{where}$$

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$$b_n := \sqrt{\frac{1}{n+1}}\, a_n + \sqrt{1 - \frac{1}{n+1}}\, c_n, \quad n \in \mathbb{N}. \tag{4.7}$$

Then,

$$P_{\mathcal{A}} b_n = \sqrt{\frac{1}{n+1}}\, a_n \to 0 \quad \text{and} \quad P_{\mathcal{B}} a_n = \sqrt{\frac{1}{n+1}}\, b_n \to 0.$$

Since neither Q nor R are bounded below, (4.6) is satisfied. Since Q and R are adjoints of each other, neither Q nor R are onto. The above calculation shows that $\operatorname{ran} Q$ is dense in $\mathcal{A}$ and $\operatorname{ran} R$ is dense in $\mathcal{B}$. It is routine to see that both Q and R are one-to-one. Now, for $n \in \mathbb{N}$, define

$$\begin{aligned} \mathcal{A}_n &:= \overline{\operatorname{span}}\,(\{a_1, a_2, \ldots\} \cup \{b_n, b_{n+1}, \ldots\}) \supset \mathcal{A} \quad \text{and} \\ \mathcal{B}_n &:= \overline{\operatorname{span}}\,(\{b_1, b_2, \ldots\} \cup \{a_n, a_{n+1}, \ldots\}) \supset \mathcal{B}. \end{aligned}$$

Suppose we show that $P_{\mathcal{A}_n \mathcal{B}_n^\perp}$ is a projection extending $\mathcal{A}$ and $\mathcal{B}$ for each $n \in \mathbb{N}$. If S is the minimal projection extending $\mathcal{A}$ and $\mathcal{B}$, then

$$\mathcal{A} \subset \operatorname{ran} S \subset \bigcap_{n \in \mathbb{N}} \mathcal{A}_n = \mathcal{A} \quad \text{and} \quad \mathcal{B} \subset \operatorname{ran} S^* \subset \bigcap_{n \in \mathbb{N}} \mathcal{B}_n = \mathcal{B}$$

This implies that $\mathcal{H} = \mathcal{A} \dotplus \mathcal{B}^\perp$, which is a contradiction. So it suffices to show that $P_{\mathcal{A}_n \mathcal{B}_n^\perp}$ is a projection extending $\mathcal{A}$ and $\mathcal{B}$ for each $n \in \mathbb{N}$. By (4.7) we see that the closed linear span of

$$X_n := \{a_1, \ldots, a_{n-1}\} \cup \{a_n, a_{n+1}, \ldots\} \cup \{c_n, c_{n+1}, \ldots\}$$

is $\mathcal{A}_n$. Actually, X_n is an orthonormal basis of $\mathcal{A}_n$. On the other hand, (4.7) also implies that the closed linear span of

$$Y_n := \{b_1, \ldots, b_{n-1}\} \cup \{a_n, a_{n+1}, \ldots\} \cup \{c_n, c_{n+1}, \ldots\}$$

is $\mathcal{B}_n$. Let α_n and β_n be the optimal Riesz bounds of the finite Riesz sequence $\{b_1, \ldots, b_{n-1}\}$. We can directly see that Y_n is a Riesz basis of $\mathcal{B}_n$ with the Riesz bounds $\alpha_{Y_n} = \min\{1, \alpha_n\}$ and $\beta_{Y_n} := \max\{1, \beta_n\}$. We now show that the bounded linear map

$$Q : \mathcal{B}_n \to \mathcal{A}_n, \quad x \mapsto P_{\mathcal{A}_n} x, \tag{4.8}$$

is bounded below and onto, which implies that $\mathcal{H} = \mathcal{A}_n \dotplus \mathcal{B}_n^\perp$. Since Y_n is a Riesz basis of $\mathcal{B}_n$, for a fixed $x \in \mathcal{B}_n$, there are square integrable unique scalars $\beta_1, \ldots, \beta_{n-1}$, $\alpha_n, \alpha_{n+1}, \ldots, \gamma_n, \gamma_{n+1}, \ldots$ such that

$$x = \sum_{j=1}^{n-1} \beta_j\, b_j + \sum_{k=n}^{\infty} \alpha_k\, a_k + \sum_{l=n}^{\infty} \gamma_l\, c_l.$$

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Moreover,

$$\alpha_{Y_n} \|x\|^2 \leq \sum_{j=1}^{n-1} |\beta_j|^2 + \sum_{k=n}^{\infty} |\alpha_k|^2 + \sum_{l=n}^{\infty} |\gamma_l|^2 \leq \beta_{Y_n} \|x\|^2 .$$

Since X_n is an orthonormal basis of $\mathcal{A}_n$,

$$Qx = \sum_{j=1}^{n-1} \frac{\beta_j}{\sqrt{j+1}} a_j + \sum_{k=n}^{\infty} \alpha_k a_k + \sum_{l=n}^{\infty} \gamma_l c_l. \tag{4.9}$$

Hence,

$$\|Qx\|^2 = \sum_{j=1}^{n-1} \frac{|\beta_j|^2}{j+1} + \sum_{k=n}^{\infty} |\alpha_k|^2 + \sum_{l=n}^{\infty} |\gamma_l|^2 \geq \frac{\alpha_{Y_n}}{n} \|x\|^2 .$$

This shows that (4.8) is bounded below. It is easy to see that(4.8) is onto by using (4.9). Hence, there is no minimal projection in this case.

The above examples show that nothing general can be said about the case (4.6). Hence, we need extra conditions to deny or guarantee the existence of the minimal projection in this case.

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