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REAL NULLSTELLENSATZ AND $*$-IDEALS IN $*$-ALGEBRAS[*]

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Abstract. For a fixed tuple of square matrices $X = \{X_1, \ldots, X_g\}$ the set $I(X)$ of all noncommutative polynomials p in X and X^* such that $p(X) = 0$ is an ideal in the $*$-*algebra* of all polynomials. This article concerns such zeroes and their corresponding ideals. An algebraic characterization of ideals of the form $I(X)$ is a *real nullstellensatz.* A main result of this article is a strong nullstellensatz for a $*$-ideal of finite codimension in a $*$-algebra. Without the finite codimension assumption, there are examples of such ideals which do not satisfy, very liberally interpreted, any Nullstellensatz.

A polynomial p in noncommuting variables $(x_1, \ldots, x_g, x_1^*, \ldots, x_g^*)$ is called *analytic* if it is a polynomial in the variables x_j only. As shown in this article, $*$-ideals generated by analytic polynomials do satisfy a natural Nullstellensatz and those generated by homogeneous analytic polynomials have a particularly simple description.

Another natural notion of zero of a noncommutative polynomial p is a pair (X, v) such that $p(X)v = 0$; here X is an $n \times n$ matrix tuple and $v \in \mathbb{R}^n$. For fixed (X, v), the set of all such polynomials is a left ideal. The relationship between such zeroes and their left ideals is considerably more developed than is our beginning effort here. This article provides a guide to that literature.

Key words. Real algebraic geometry, Nullstellensatz, Real ideal, Noncommutative polynomial, Free algebraic geometry.

AMS subject classifications. 13J30, 16W10, 47A, 46L89, 16Z05.

1. Introduction. There are three natural notions of a zero of a noncommutative polynomial p: a tuple of (square) matrices X such that

1. $p(X) = 0$;
2. $\det p(X) = 0$; and
3. a pair (X, v) consisting of X and vector v such that $p(X)v = 0$.

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This article begins a systematic study of the first two notions of zero. The more detailed type of zero 3. was first studied about 10 years ago and has developed considerably since then. The set $\{p: \ p(X)v = 0\}$ is a left ideal and articles [3], [4] and [8] gave real Nullstellensatz for left ideals in polynomial algebras as well as in more general contexts. The relationship between the Nullstellensatz in those papers for left ideals and those in this article for $*$-ideals are outlined in Section 5.

Now we turn to the theme of this paper, two sided ideals. A left ideal in the algebra of polynomials in x, x^* is a **$*$-ideal** if $I^* = I$ and it is not hard to see that such an ideal must also be a two-sided ideal. For such $*$-ideals there is a major distinction between those that have finite codimension and those that have not. In the first case we give a very strong real Nullstellensatz whereas in the second case even a very weak version of real Nullstellensatz fails in general. The paper also develops the concomitant general theory of $*$-ideals in general $*$-algebras. In the positive direction, we show, independent of any finiteness hypotheses, that if the $*$-ideal is generated by analytic polynomials, then it automatically satisfies a natural Nullstellensatz; and if it is generated by analytic homogeneous polynomials, then it has an especially simple representation.

We call $\{X\colon \det p(X) = 0\}$ the soft zero set of a polynomial p. The set of polynomials with a prescribed soft zero set is not an ideal, however (under hypotheses) we give a description of these polynomials.

The body of the paper is organized as follows. Section 2 presents basic properties of real ideals, including the Nullstellensatz in the case the real $*$-ideal I has finite codimension. Negative examples which illustrate that, absent additional hypotheses on I, a general Nullstellensatz is problematic are in Section 3. There too is some needed additional theory applicable to general $*$-algebras. The results for $*$-ideals generated by analytic polynomials are in Section 4. The relationship between the Nullstellensatz for left ideals in articles [3] and [4] and those in this article for $*$-ideals are outlined in Section 5. Section 6 contains results for soft zero sets, $\{X\colon \det p(X) = 0\}$.

In the remainder of this introduction, we state our main results precisely, introducing notations and terminology as needed. Let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$ with complex conjugation as involution. Let $\langle x, x^*\rangle$ be the monoid freely generated by $x = (x_1, \dots, x_g)$ and $x^* = (x_1^*, \dots, x_g^*)$, i.e., $\langle x, x^*\rangle$ consists of **words** in the $2g$ noncommuting letters $x_1, \dots, x_g, x_1^*, \dots, x_g^*$ (including the empty word $\emptyset$ which plays the role of the identity 1). Let $\mathbb{F}\langle x, x^*\rangle$ denote the $\mathbb{F}$-algebra freely generated by x, x^*, i.e., the elements of $\mathbb{F}\langle x, x^*\rangle$ are **polynomials** in the noncommuting variables x, x^* with coefficients in $\mathbb{F}$. Equivalently, $\mathbb{F}\langle x, x^*\rangle$ is the **free $*$-algebra** on x. Elements of the free algebra $\mathbb{F}\langle x\rangle$ generated by $x = (x_1, \dots, x_g)$ are known as **analytic polynomials**. A polynomial is **homogeneous** if it is an $\mathbb{F}$ linear combination of words of the same length. While

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the focus of the article is on $\mathbb{F}\langle x, x^*\rangle$ we ultimately give natural extensions to more general algebras.

1.1. Ideals in $\mathbb{F}\langle x, x^*\rangle$. Let $\mathcal{A}$ be a unital associative $\mathbb{F}$-algebra with involution $*$, or **$*$-algebra** for short. A left ideal in the $*$-algebra $\mathcal{A}$ is **real** if $a_1, \dots, a_n \in \mathcal{A}$ and

$$\sum a_j^* a_j \in I + I^*$$

implies $a_j \in I$ for each j. A two-sided ideal is real if it is real as a left ideal. Moreover, as seen in Lemma 2.2(i), a two-sided real ideal is in fact a $*$-ideal.

The **real radical**, denoted $\sqrt[\text{real}]{I}$ of a left ideal I is the intersection of all real left ideals containing I or equivalently, the smallest real left ideal containing I. The real radical of a $*$-ideal is also a $*$-ideal (see Lemma 2.2(ii)).

When $\mathcal{A} = \mathbb{F}\langle x, x^*\rangle$ there is a natural way to generate real $*$-ideals. Given a positive integer n, let $M_n(\mathbb{F})^g$ denote the set of g-tuples $X = (X_1, \dots, X_g)$ of $n \times n$ matrices. Let $M(\mathbb{F})^g$ denote the graded set $(M_n(\mathbb{F})^g)_n$. An element $p \in \mathbb{F}\langle x, x^*\rangle$ is naturally evaluated at X by substituting X_j for x_j and the adjoint X_j^* for x_j^* with the result $p(X)$ being an $n \times n$ matrix. We say that X is a **(hard) zero** of p if $p(X) = 0$.

Given a sequence $S = (S_n)_n$ of subsets S_n of $M_n(\mathbb{F})^g$, define its **hard vanishing set**

$$\mathcal{I}_{\text{hard}}(S) = \{p \in \mathbb{F}\langle x, x^*\rangle : p(X) = 0 \text{ for every } n \text{ and every } X \in S_n\}.$$

It is easily checked that $\mathcal{I}_{\text{hard}}(S)$ is indeed a real $*$-ideal. Moreover, if S is a finite set, then the dimension of $\mathbb{F}\langle x, x^*\rangle / \mathcal{I}_{\text{hard}}(S)$ is finite.

The connection with Nullstellensätze is the following. The **hard variety** $V_{\text{hard}}(I) = (V_{\text{hard}}(I)_n)$ of an ideal I in $\mathbb{F}\langle x, x^*\rangle$ is the sequence

$$V_{\text{hard}}(I)_n = \{X \in M_n(\mathbb{F})^g : p(X) = 0 \text{ for every } p \in I\}.$$

The **hard radical** of I is

$$\sqrt[\text{hard}]{I} = \mathcal{I}_{\text{hard}}(V_{\text{hard}}(I)),$$

which is necessarily a $*$-ideal. Finally, the $*$-ideal has the **Nullstellensatz property** if

$$\sqrt[\text{hard}]{I} = I$$

and I satisfies the **real Nullstellensatz** if

$$\sqrt[\text{hard}]{I} = \sqrt[\text{real}]{I}.$$

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1.2. The case of finite codimension. When the codimension of I in $\mathbb{F}\langle x, x^*\rangle$ is finite, the relation between real Nullstellensatz and real ideals is clean, readily described and essentially a consequence of the existing theory of formally real $*$-algebras.

PROPOSITION 1.1. *Let $I \subseteq \mathbb{F}\langle x, x^*\rangle$ be a two-sided ideal.*

(i) I is finitely generated as a left ideal if and only if either $I = \{0\}$ or I has finite codimension in $\mathbb{F}\langle x, x^*\rangle$.
(ii) If I is a $*$-ideal, then I is real and $0 < \dim(\mathbb{F}\langle x, x^*\rangle/I) < \infty$ if and only if there exists an $n \in \mathbb{N}$ and a $X \in M_n(\mathbb{F})^g$ such that $I = \mathcal{I}_{\mathrm{hard}}(\{X\})$. In particular, if I is an ideal in $\mathbb{F}\langle x, x^*\rangle$ and $0 < \dim(\mathbb{F}\langle x, x^*\rangle/\sqrt[\mathrm{real}]{I}) < \infty$, then $\sqrt[\mathrm{hard}]{I} = \sqrt[\mathrm{real}]{I}$.

REMARK 1.2. Thus, if a $*$-ideal I has finite codimension, then I has the Nullstellensatz property if and only if it is real.

The article contains two proofs of Proposition 1.1. The first is based on a standard applications of the well developed theory of formally real $*$-algebras. Another proof is an easy consequence of the theory developed here in Section 5 to connect Nullstellensatz for left ideals for those for two-sided and $*$-ideals.

1.3. The case of infinite codimension. The **(free) Toeplitz algebra** $\mathcal{T}$ is the quotient of $\mathbb{F}\langle x, x^*\rangle$ by the $*$-ideal I generated by $1-x^*x$. It turns out I is real but $1 - xx^* \in \sqrt[\mathrm{hard}]{I} \setminus I$ and hence I does not satisfy the Nullstellensatz property. Thus, I provides an example which shows that the finite codimension hypothesis is needed in Proposition 1.1 (ii). The details can be found in Section 3. A more elaborate example provided by Proposition 1.8 below shows that there are real ideals I in $\mathbb{F}\langle x, x^*\rangle$ which, even with a most liberal interpretation, do not satisfy a Nullstellensatz.

In spite of these negative examples, the main results of this article show that natural Nullstellensatze hold for ideals generated by analytic polynomials, thus providing optimism that a satisfying general theory may emerge.

COROLLARY 1.3. *If $I \subseteq \mathbb{F}\langle x, x^*\rangle$ is a $*$-ideal generated by analytic polynomials, then*

$$\sqrt[\mathrm{real}]{I} = I.$$

In particular, I is real.

Proof. This corollary is an immediate consequence of Corollary 4.4. $\square$

THEOREM 1.4. *If $I \subsetneq \mathbb{F}\langle x, x^*\rangle$ is a $*$-ideal generated by homogeneous analytic polynomials, then I has the Nullstellensatz property, that is, $\sqrt[\mathrm{hard}]{I} = I$. Moreover, there exists a Hilbert space $\mathcal{H}$ and a tuple of bounded operators X on $\mathcal{H}$ such that $p(X) = 0$ if and only if $p \in I$. Thus, in an operator theoretic sense, $I = \mathcal{I}_{\mathrm{hard}}(X)$.*

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Proof. See the end of §4.3. ◻

For example, in the case of $g = 1$ and a single matrix M, let q denote an analytic annihilating polynomial, $q(M) = 0$, for M. (Note $q^*(M) = 0$.) Let I denote the $*-$ideal generated by the q. Then $V_{\rm hard}(I)$ is $(\{A \in \mathbb{R}^{n\times n} : \ q(A) = 0 \ \})_n$. Since $M \in V_{\rm hard}(I)$ we have that $\mathcal{I}_{\rm hard}(M)$ contains $\sqrt[\rm hard]{I}$. We will show now that $\mathcal{I}_{\rm hard}(M)$ can be strictly larger than $\sqrt[\rm hard]{I}$.

Consider $M = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then q is homogeneous, so Theorem 1.4 applies and says

$$I = \sqrt[\rm hard]{I} = \mathcal{I}_{\rm hard}(V_{\rm hard}(I)) = \{p : \ p(Y) = 0 \text{ for any } Y^2 = 0, \ Y \in \mathbb{R}^{n\times n} \text{ all } n \ \}.$$

Note that $\mathcal{I}_{\rm hard}(M)$ contains $p = 1 - (xx^* + x^*x)$ while I does not. The X in the theorem applied to this case is an infinite direct sum of (order 2) nilpotent matrices.

1.4. General $*$-algebras. Even allowing for a liberal notion of zero, and hence of variety of an ideal, there are real $*$-ideals without the Nullstellensatz property as we soon explain. While the example given here is an ideal in a free $*$-algebra, the natural context for much of the discussion is that of a general (associative) $*$-algebra $\mathcal{A}$ over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$. Its elements will be considered as noncommutative polynomials.

A **$*$-representation** π of $\mathcal{A}$ is a unital $*$-homomorphism from $\mathcal{A}$ to the $*$-algebra of all adjointable operators on some pre-Hilbert space V_π over $\mathbb{F}$. Let $\mathcal{R}$ be the class of all $*$-representations of the $*$-algebra $\mathcal{A}$ and let $\mathcal{C}$ be a fixed subclass of $\mathcal{R}$ whose elements will be considered as (evaluations at) real points. We say that a **real point** $\pi \in \mathcal{C}$ is a **hard zero** of a *polynomial* $a \in \mathcal{A}$ if $\pi(a) = 0$. For a subset T of $\mathcal{C}$, let

$$\mathcal{I}^{\mathcal{C}}_{\rm hard}(T) := \{a \in \mathcal{A} \colon \pi(a) = 0 \text{ for every } \pi \in T\}$$

be its **hard vanishing set**. For a subset $\mathcal{S}$ of $\mathcal{A}$, let

$$V^{\mathcal{C}}_{\rm hard}(\mathcal{S}) := \{\pi \in \mathcal{C} \colon \pi(s) = 0 \text{ for every } s \in \mathcal{S}\}$$

be its **hard variety** and

$$\sqrt[\mathcal{C}-\rm hard]{\mathcal{S}} := \mathcal{I}^{\mathcal{C}}_{\rm hard}(V^{\mathcal{C}}_{\rm hard}(\mathcal{S}))$$

its **hard radical**. If $I(\mathcal{S})$ is the $*$-ideal of $\mathcal{A}$ generated by $\mathcal{S}$, then clearly

$$V^{\mathcal{C}}_{\rm hard}(\mathcal{S}) = V^{\mathcal{C}}_{\rm hard}(I(\mathcal{S})) \quad \text{and} \quad \sqrt[\mathcal{C}-\rm hard]{\mathcal{S}} = \sqrt[\mathcal{C}-\rm hard]{I(\mathcal{S})}.$$

The relation between a $*$-ideal I and its various radicals is summarized by (see Proposition 2.3)

$$I \subseteq \sqrt[\rm real]{I} \subseteq \sqrt[\mathcal{R}-\rm hard]{I} \subseteq \sqrt[\mathcal{C}-\rm hard]{I}.$$

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We say that a $*$-ideal I **satisfies the real Nullstellensatz over** $\mathcal{C}$ if $\sqrt[\mathcal{C}-\mathrm{hard}]{I} = \sqrt[\mathrm{real}]{I}$. We say that I has the **Nullstellensatz property over** $\mathcal{C}$ if $\sqrt[\mathcal{C}-\mathrm{hard}]{I} = I$ (which implies that I is real).

Example 1.5. If $\mathcal{A} = \mathbb{F}\langle x, x^*\rangle$ and Π is the class of all finite-dimensional $*$-representations, then $\sqrt[\Pi-\mathrm{hard}]{I} = \sqrt[\mathrm{hard}]{I}$ for every $*$-ideal I of $\mathcal{A}$. Here we identify every $\pi \in \Pi$ with the g-tuple $(\pi(x_1), \ldots, \pi(x_g)) \in M(\mathbb{F})^g$. Therefore, I satisfies the real Nullstellensatz over Π if and only if it satisfies the (hard) real Nullstellensatz.

Motivated by Example 1.5, we introduce the following abbreviations when $T \subseteq \Pi$ and $\mathcal{S} \subseteq \mathcal{A}$:

$$\mathcal{I}_{\mathrm{hard}}(T) = \mathcal{I}^{\Pi}_{\mathrm{hard}}(T), \quad V_{\mathrm{hard}}(\mathcal{S}) = V^{\Pi}_{\mathrm{hard}}(\mathcal{S}) \quad \text{and} \quad \sqrt[\mathrm{hard}]{\mathcal{S}} = \sqrt[\Pi-\mathrm{hard}]{\mathcal{S}}.$$

Example 1.6. If $\mathcal{A} = \mathbb{F}\langle x, x*\rangle$ and Π_n is the representation class of all n-dimensional *-representations, then the Π_n-hard radical of a two-sided ideal is characterized in [19]. Clearly, this radical must contain all polynomial identities of size n.

Example 1.7. Let $\mathcal{A} = M_n(\mathbb{F}[x])$ be the algebra of all polynomials in commuting variables $x = (x_1, \ldots, x_g)$ with coefficients in $n \times n$ matrices over $\mathbb{F}$. The involution is trivial on variables and it is hermitian transpose on coefficients. Let $\mathcal{E}$ be the class of all n-dimensional $*$-representations (i.e., all evaluations at real points from $\mathbb{F}^g$). By [2, Corollary 18], every $*$-ideal of $\mathcal{A}$ satisfies the real Nullstellensatz over $\mathcal{E}$. The case $n = 1$ corresponds to the classical Real Nullstellensatz [5, 6, 20].

The following proposition, based on an example introduced in [18], shows that a general Nullstellensatz is highly problematic.

Proposition 1.8. *Fix $0 < q < 1$. The $*$-ideal I in the free $*$-algebra $\mathbb{F}\langle a, x, a^*, x^*\rangle$ in the two variables a and x generated by*

$$a^*a - qaa^* \quad \textit{and} \quad xx^* + aa^* - 1$$

satisfies $I = \sqrt[\mathrm{real}]{I}$ but it does not satisfy $I = \sqrt[\mathcal{R}-\mathrm{hard}]{I}$. In other words, I is real but it does not satisfy the Real Nullstellensatz over any representation class.

The proof is based upon results of [18] and some general theory of $*$-algebras developed here. The details are in Section 3

2. Properties of real ideals.

In this section, the basic properties of real ideals, including the proof of Proposition 1.1, are collected.

Lemma 2.1. *If I is a left ideal in the $*$-algebra $\mathcal{A}$, then there is a largest two-sided ideal $Z(I)$ contained in I. Indeed, $Z(I)$ is the kernel of the left regular representation of $\mathcal{A}$ on $\mathcal{A}/I$. Moreover, if I is real, then $Z(I)$ is real.*

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Proof. The kernel Z of the left regular representation is a two-sided ideal contained in I. On the other hand, if $J \subseteq I$ is a two-sided ideal, $\theta \in J$ and $v \in \mathcal{A}$, then $\theta v \in J \subseteq I$ and hence $\pi(\theta) = 0$ and $\theta \in Z$. As an aside, note that

$$Z = \{\vartheta \in I \colon \vartheta p \in I \text{ for every } p \in \mathcal{A}\}.$$

Now suppose I is a real ideal. To see that Z is a real ideal, suppose

$$\sum_{i}^{\text{finite}} p_i^* p_i \in Z + Z^*.$$

Since I is real, each $p_i \in I$. Further, if $a, b \in Z$, then $q(a + b^*)r = (qar) + (r^* b q^*)^* \in Z + Z^*$, for each $q, r \in \mathcal{A}$. Therefore for each $q \in \mathcal{A}$,

$$\sum_{i}^{\text{finite}} q^* p_i^* p_i q \in Z + Z^*,$$

which implies that each $p_i q \in I$. Therefore each $p_i \in Z$. Hence Z is a real ideal. ▯

Lemma 2.2. *Let $I \subseteq \mathcal{A}$ be a two-sided ideal in the $*-$algebra $\mathcal{A}$.*

(i) If I is real, then $I = I^*$.
(ii) The radical $\sqrt[\text{real}]{I}$ is the smallest two-sided real ideal containing I.

Proof. First, if I is real, then for each $\iota \in I$, we have $\iota\iota^* \in I$ since I is two-sided. Thus, $\iota^* \in I$ since I is real. Therefore, $I = I^*$.

As in Lemma 2.1, let $Z(\sqrt[\text{real}]{I})$ be the largest two-sided ideal of $\mathcal{A}$ contained in $\sqrt[\text{real}]{I}$., Since $I \subseteq \sqrt[\text{real}]{I}$, and I is a two-sided ideal, $I \subseteq Z(\sqrt[\text{real}]{I})$. Thus, $Z(\sqrt[\text{real}]{I}) \subseteq \sqrt[\text{real}]{I}$ is a real left ideal containing I. Hence, $Z(\sqrt[\text{real}]{I}) = \sqrt[\text{real}]{I}$. Since all two-sided ideals are also left ideals, and $\sqrt[\text{real}]{I}$ is the smallest real left ideal containing I, it must also be the smallest real two-sided ideal containing I. ▯

Proposition 2.3. *If I is a $*$-ideal of $\mathcal{A}$, then*

$$I \subseteq \sqrt[\text{real}]{I} \subseteq \sqrt[\mathcal{R}-\text{hard}]{I} \subseteq \sqrt[\mathcal{C}-\text{hard}]{I}.$$

Proof. It is clear that the kernel of a $*$-representation is always a real $*$-ideal. In particular, $\mathcal{I}^{\mathcal{C}}_{\text{hard}}(T)$ is a real $*$-ideal for every subset T of $\mathcal{C}$. For $T = V^{\mathcal{C}}_{\text{hard}}(I)$ we get that $\sqrt[\mathcal{C}-\text{hard}]{I}$ is a real $*$-ideal. Since $\sqrt[\mathcal{C}-\text{hard}]{I}$ contains I, it follows that $\sqrt[\text{real}]{I} \subseteq \sqrt[\mathcal{R}-\text{hard}]{I}$. The third inclusion is clear from $\mathcal{C} \subseteq \mathcal{R}$. ▯

2.1. Formally real $*$-algebras. A $*$-algebra $\mathcal{A}$ is **formally real** (or **very proper**) if $a_1, \dots, a_k \in \mathcal{A}$ and $\sum_{i=1}^k a_i^* a_i = 0$ implies $a_1 = \cdots = a_k = 0$. Let

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$\Sigma_{\mathcal{A}}$ denote the set of all finite sums of elements a^*a, $a \in \mathcal{A}$. Alternately, $\mathcal{A}$ is formally real if and only if it is proper (i.e., $a^*a = 0$ implies $a = 0$ for every $a \in \mathcal{A}$) and $-\Sigma_{\mathcal{A}} \cap \Sigma_{\mathcal{A}} = 0$ (i.e., for every $a_1, \ldots, a_k \in \mathcal{A}$ such that $\sum_{i=1}^k a_i^* a_i = 0$ we have that $a_i^* a_i = 0$ for all i).

Let $I_h = \{a \in I \colon a^* = a\} = I \cap \mathcal{A}_h$ denote the **hermitian** elements of I.

Lemma 2.4. *For a $*$-ideal I of a $*$-algebra $\mathcal{A}$, the following are equivalent.*

1. *I is real; i.e., $I = \sqrt[\text{real}]{I}$.*
2. *Both $(\Sigma_A + I_h) \cap -(\Sigma_A + I_h) = I_h$ and $a^*a \in I$ implies $a \in I$ for every $a \in \mathcal{A}$.*
3. *The quotient $\mathcal{A}/I$ is formally real.*

Remark 2.5. There is an iterative description of $\sqrt[\text{real}]{I}$ along the lines of that for the real radical of a left ideal as described in [4, Section 5].

2.2. Proof of Proposition 1.1. The length of the longest word in a noncommutative polynomial $f \in \mathbb{F}\langle x, x^*\rangle$ is the **degree** of f and is denoted by $\deg(f)$. The set of all words of degree at most k is $\langle x, x^*\rangle_k$, and $\mathbb{F}\langle x, x^*\rangle_k$ is the vector space of all noncommutative polynomials of degree at most k.

Given a subspace W of $\mathbb{F}\langle x, x^*\rangle$, let

$$W_d = \{w \in W \colon \deg(w) \le d\} = W \cap \mathbb{F}\langle x, x^*\rangle_d,$$

denote the elements of W of degree at most d. Likewise, let

$$W_d^{\text{hom}} := \{w \in W \colon w = 0 \text{ or } w \text{ is homogeneous of degree } d\},$$

denote the homogeneous of degree d elements of W.

Proof of Proposition 1.1 (i). If $\mathcal{A}$ is a finitely-generated algebra, and $I \subseteq \mathcal{A}$ is an ideal such that the dimension of $\mathcal{A}/I$ is finite, then I is finitely generated as a left ideal by [11, Lemma 3]. The special case $\mathcal{A} = \mathbb{F}\langle x, x^*\rangle$ is all that is required for this proof.

Next, suppose that I is finitely generated as a left ideal, but $I \neq 0$. There exists some degree d such that I is generated by some nonzero polynomials of degree bounded by d. By [4][Proposition 2.20] there exists a subspace V of $\mathbb{F}\langle x, x^*\rangle$ such that

$$\mathbb{F}\langle x, x^*\rangle = I \oplus V_d^{\text{hom}} \oplus V_{d-1}.$$

Let $\iota \in I \setminus \{0\}$. If $V_d^{\text{hom}} \neq \{0\}$, then let $\nu \in V_d^{\text{hom}} \setminus \{0\}$. Then $\iota\nu \in \mathbb{F}\langle x, x^*\rangle V_d^{\text{hom}}$ and $\iota\nu \in I$, so $\iota\nu = 0$, which is a contradiction. Therefore $V_d^{\text{hom}} = \{0\}$. It follows that

$$\dim(\mathbb{F}\langle x, x^*\rangle / I) = \dim(V_{d-1}) < \infty,$$

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since V_{d-1} is a subspace of the finite dimensional space $\mathbb{F}\langle x, x^*\rangle_{d-1}$. $\square$

Remark 2.6. It is not true in general that if $\mathcal{A}$ is a finitely-generated $*-$algebra and I is a two-sided ideal which is also a finitely-generated left ideal, then $\dim(\mathcal{A}/I) < \infty$. As an example, consider the (commutative) polynomial ring $\mathbb{F}[t_1, \ldots, t_g]$ in two or more variables g with the trivial involution. Since this algebra is commutative, there is no distinction between left and two-sided ideals. In particular, the ideal I generated by t_1 is finitely generated, but $\mathbb{F}[t_1, \ldots, t_g]/I = \mathbb{F}[t_2, \ldots, t_g]$ is not finite dimensional.

We now turn our attention to proving Proposition 1.1(ii). Proposition 2.7 summarizes the structure theory of finite-dimensional formally-real $*$-algebras. Remark 2.8 explains the history of this result.

Proposition 2.7. *If $\mathcal{A}$ is a finite-dimensional formally real $*$-algebra, then $\mathcal{A}$ is a $*$-algebra direct sum of formally real simple $*$-algebras. Moreover:*

1. *If $\mathbb{F} = \mathbb{C}$, then every finite-dimensional formally real simple $*$-algebra is of the form $M_n(\mathbb{C})$ where $n \in \mathbb{N}$ and the involution is conjugate transpose.*
2. *If $\mathbb{F} = \mathbb{R}$, then every finite-dimensional formally real simple $*$-algebra is of the form $M_n(\mathbb{R})$ or $M_n(\mathbb{C})$ or $M_n(\mathbb{H})$ where $n \in \mathbb{N}$ and the involution is conjugate transpose. Here $\mathbb{H}$ denotes the quaternions.*

In particular, $\mathcal{A}$ has a faithful finite-dimensional $*$-representation.

Proof. Let $\mathcal{A}$ be finite dimensional and formally real. By [14, Theorem 2.2], $\mathcal{A}$ is semisimple. By [9, Chapter 0], every semisimple algebra with involution is a $*$-algebra direct sum of simple $*$-algebras and every simple $*$-algebra is one of the following types (where D is a division algebra with involution): (i) $M_n(D) \otimes M_n(D)^{\mathrm{op}}$ with exchange involution, (ii) $M_n(D)$ with conjugate transpose involution or (iii) $M_{2n}(D)$ with symplectic involution. The exchange and the symplectic involution are clearly not formally real. (They are not even proper.) Therefore every formally real simple $*$-algebra is of type (ii). Finally we use the Frobenious theorem which says that $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are the only finite-dimensional division algebras over $\mathbb{R}$. It remains to show that the only formally real involution on $\mathbb{H}$ is the standard involution. By the Noether-Skolem theorem, every involution $\#$ on $\mathbb{H}$ is of the form $x^{\#} = hx^*h^{-1}$ where $*$ is the standard involution and $h \in D$. Now $x^{\#\#} = x$ implies that h^*h^{-1} is in $Z(\mathbb{H}) = \mathbb{R}$. Since $h^{**} = h$, we get $h^* = \pm h$. If $h^* = h$, then $h \in \mathbb{R}$ and so $x^{\#} = hx^*h^{-1} = x^*$ for every $x \in \mathbb{H}$. If $h^* = -h$, then $h = \alpha i + \beta j + \gamma k$ for some $\alpha, \beta, \gamma \in \mathbb{R}$. It follows that $ihi+jhj+khk = (-\alpha i+\beta j+\gamma k)+(\alpha i-\beta j+\gamma k)+(\alpha i+\beta j-\gamma k) = \alpha i+\beta j+\gamma k = h$. Multiplying through by h^{-1} we get $1 + ii^{\#} + jj^{\#} + kk^{\#} = 0$. Therefore, $\#$ is equal to $*$ in the first case while it is is not formally real in the second case. $\square$

Remark 2.8. Recall that a $*$-algebra $\mathcal{A}$ is **proper** if $a^*a \neq 0$ for every nonzero

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$a \in \mathcal{A}$. The following generalization of part (1) of Proposition 2.7 is well-known; see e.g. [16, Theorem 9.7.22]: If $\mathbb{F} = \mathbb{C}$, then every finite-dimensional proper simple $*$-algebra is of the form $M_n(\mathbb{C})$ where $n \in \mathbb{N}$ and the involution is conjugate transpose. This is also clear from our proof of part (1) above.

Part (2) is a slight generalization of the structure theorem for positive involutions on finite-dimensional real $*$-algebras. The history of this result is explained in [12, Section 2] and [15]. Recall that an involution $*$ on a finite-dimensional real algebra $\mathcal{A}$ is **positive** if $\operatorname{tr}(a^*a) > 0$ for every nonzero $a \in \mathcal{A}$. Clearly, every positive involution is formally real but the converse is false; see [1].

Part (2) does not generalize to proper involutions because there are proper involutions on $\mathbb{H}$ which are not standard. For example, a short computation shows that the involution defined by $i^* = i, j^* = j$ (and so $k^* = -k$) is proper but it is not formally real. □

We are now able to prove the following generalization of Proposition 1.1 (ii).

Corollary 2.9. *If I is a $*$-ideal of a $*$-algebra $\mathcal{A}$, then I is real with* $\dim(\mathcal{A}/I) < \infty$ *if and only if I is the kernel of some finite-dimensional $*$-representation. In particular, if I is a $*$-ideal of a $*$-algebra $\mathcal{A}$ such that* $\dim(\mathcal{A}/\sqrt[\mathrm{real}]{I}) < \infty$, *then* $\sqrt[\mathrm{hard}]{I} = \sqrt[\mathrm{real}]{I}$.

Proof. First, if I is a real $*$-ideal with $\dim(\mathcal{A}/I) < \infty$, then $\mathcal{A}/I$ is a finite-dimensional formally real $*$-algebra. Therefore, Proposition 2.7 implies that $\mathcal{A}/I$ has a finite-dimensional faithful $*$-representation, which implies that $I = \sqrt[\mathrm{hard}]{I}$.

Conversely, if $I = \ker \pi$ for some finite-dimensional $*$-representation π, then $\mathcal{A}/I \cong \operatorname{im}\pi$ implies $\dim(\mathcal{A}/I) < \infty$. Further, if

$$\sum_{i}^{\text{finite}} p_i^* p_i \in \ker \pi,$$

then $\sum_i \pi(p_i)^* \pi(p_i) = 0$, which implies that each $\pi(p_i) = 0$, or equivalently, each $p_i \in \ker \pi$. Therefore I is real. □

An alternative proof of Proposition 1.1 (ii) (which does not generalize to arbitrary $*$-algebras) will be given in Section 5.

3. No general real Nullstellensatz for free $*$-algebras. This cautionary section gives the details of the Toeplitz algebra mentioned at the outset of Subsection 1.3. It also contains a discussion of the Weyl algebra which has no bounded representations and the proof of Proposition 1.8. The additional theory of $*$-algebras used in the proof of Proposition 1.8 is developed in Subsection 3.1. These examples, taken together, support the general theme that Nullstellensatz for various representation

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classes impose serious restrictions on a $*$-algebra.

EXAMPLE 3.1. Let $\mathcal{A} = \mathbb{F}\langle x, x^*\rangle$ denote the free $*$-algebra in one variable x and let I be the $*$-ideal of $\mathcal{A}$ generated by $1 - x^*x$. The algebra $\mathcal{A}/I$ is called **the Toeplitz algebra**. Let $\mathbf{X}$ denote the shift operator on $\ell^2(\mathbb{N})$ and let $\pi\colon \mathcal{A} \to L(\ell^2(\mathbb{N}))$ denote the (bounded) $*$-representation of evaluation at $\mathbf{X}$. As is shown below, $I = \ker \pi$. It follows that I is a real ideal and it satisfies the Real Nullstellensatz over $\mathcal{C} = \{\pi\}$. (Hence it also satisfies the Real Nullstellensatz over the class of all bounded $*$-representations.)

Clearly $\mathbf{X}^*\mathbf{X} = \mathbf{I}$, so $I \subseteq \ker \pi$. Conversely, let p be an element of $\mathcal{A}$ and let $\sum_{i=0}^m \sum_{j=0}^n c_{ij} x^i (x^*)^j$ be its canonical form modulo I. Suppose that $\pi(p) = 0$. It follows that for every integer k,

$$0 = (\sum_{i=0}^m \sum_{j=0}^n c_{ij} X^i (X^*)^j) e_k = \sum_{j=0}^k \sum_{i=0}^m c_{ij} e_{k-j+i}$$

where $e_0, e_1, e_2, \ldots$ is the standard basis of $\ell^2(\mathbb{N})$. For $k = 0$ we get that $c_{i0} = 0$ for every i. For $k = 1$ we deduce that $c_{i1} = 0$ for every i and so on. Hence $I = \ker \pi$.

On the other hand, I does not satisfy the real Nullstellensatz over the class Π of all finite-dimensional $*$-representations (i.e., evaluations on same size square matrices over $\mathbb{F}$.) Namely, we will show that the element $1 - xx^*$ belongs to $\sqrt[\mathrm{hard}]{I}$ but it does not belong to I. We already know that $I = \sqrt[\mathrm{real}]{I}$. For a square matrix $\mathbf{Y}$, the relation $\mathbf{Y}^T\mathbf{Y} = \mathbf{I}$ is equivalent to $\mathbf{Y}\mathbf{Y}^T = \mathbf{I}$, hence $1 - xx^* \in \sqrt[\mathrm{hard}]{I}$. Since $(1 - \mathbf{X}\mathbf{X}^*)e_0 = e_0 \neq 0$, it follows that $1 - xx^* \notin I$. $\square$

The following is also standard.

EXAMPLE 3.2. Let I be the $*$-ideal in $\mathcal{A} = \mathbb{F}\langle a, a^*\rangle$ generated by $aa^* - a^*a - 1$. The algebra $\mathcal{A}/I$ is called **the Weyl algebra**. It has a faithful $*$-representation π_0, see [25, Example2]), but it does not have any bounded $*$-representations, see [21, Theorem 13.6]. Hence I satisfies the Real Nullstellensatz over $\mathcal{C} = \{\pi_0\}$ but it does not satisfy the Real Nullstellensatz over the class of all bounded $*$-representations of $\mathcal{A}$.

For nice generalizations of Examples 3.1 and 3.2, see [22] and [24].

The more elaborate negative example of Proposition 1.8 requires some additional theory of $*$-algebras which is developed in Subsection 3.1. The proof of the proposition follows in Subsection 3.1.

3.1. $*$-Semisimple $*$-algebras. A $*$-algebra which has a faithful $*$-representation is called **$*$-semisimple**, see [23, Definition 6.4.1]. Every $*$-semisimple $*$-algebra is formally real. Namely, for every pre-Hilbert space V, the $*$-algebra of

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all adjointable linear operators on V is formally real. Hence, the kernel of every $*$-representation is a real $*$-ideal.

Lemma 3.3. *For a $*$-ideal I of a $*$-algebra $\mathcal{A}$, the following are equivalent.*

1. $I = \sqrt[\mathcal{R}-\mathrm{hard}]{I}$,
2. $I = \mathcal{I}^{\mathcal{R}}_{\mathrm{hard}}(T)$ *for a subset* T *of* $\mathcal{R}$,
3. I *is the kernel of some $*$-representation,*
4. $\mathcal{A}/I$ *is $*$-semisimple.*

Proof. Clearly, (1) implies (2). Conversely, (2) implies (1) because

$$V^{\mathcal{R}}_{\mathrm{hard}}(\mathcal{I}^{\mathcal{R}}_{\mathrm{hard}}(T)) \supseteq T$$

and $\mathcal{I}$ is inclusion-reversing. The implication (3) implies (2) results from

$$\ker \pi = \mathcal{I}^{\mathcal{R}}_{\mathrm{hard}}(\{\pi\}).$$

To show that (2) implies (3) take for π the direct sum of all $*$-representations from T. The equivalence of (3) and (4) is clear. □

Lemma 3.3 will sometimes be used in combination with the following:

Lemma 3.4. *Let I be a $*$-ideal of a $*$-algebra $\mathcal{A}$. If there exists a positive hermitian linear functional L on $\mathcal{A}$ such that*

$$I = \{a \in \mathcal{A} \colon L(a^*a) = 0\},$$

then the $*$-algebra $\mathcal{A}/I$ is $*$-semisimple.

Proof. Note that for a left ideal I of $\mathcal{A}$, the following are equivalent:

1. There exists a positive hermitian linear functional L on $\mathcal{A}$ such that $I = \{a \in \mathcal{A} \colon L(a^*a) = 0\}$.
2. There exists an inner product $\langle \cdot, \cdot \rangle$ on the vector space $\mathcal{A}/I$ such that

$$\langle [xy], [z] \rangle = \langle [y], [x^*z] \rangle$$

   for every $x, y, z \in \mathcal{A}$.
3. There exists an inner product on the $*$-algebra $\mathcal{A}/I$ such that the left regular representation of $\mathcal{A}$ on the pre-Hilbert space $\mathcal{A}/I$ is a $*$-representation.

Namely, to show that (1) implies (2), take $\langle [y], [z] \rangle := L(z^*y)$ and to show that (2) implies (1), take $L(x) := \langle [x], [1] \rangle$. Clearly, (3) just rephrases (2). Finally (3) implies the claim because, by Lemma 2.1, the kernel of the left regular representation of $\mathcal{A}$ on $\mathcal{A}/I$ is equal to the largest two-sided ideal contained in I. □

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Lemma 3.3 can be generalized as follows. Let $\mathcal{C}$ be a representation class of a $*$-algebra $\mathcal{A}$. We say that $\mathcal{A}$ is **$\mathcal{C}$-semisimple** if $\bigcap_{\pi\in\mathcal{C}} \ker\pi = \{0\}$. Clearly, $\mathcal{A}$ is $\mathcal{R}$-semisimple iff it is $*$-semisimple. For every $*$-ideal I of $\mathcal{A}$ we have that $I = \sqrt[\mathcal{C}-\mathrm{hard}]{I}$ iff $I = \mathcal{I}^{\mathcal{C}}_{\mathrm{hard}}(T)$ for some $T \subseteq \mathcal{C}$ iff $\mathcal{A}/I$ is $\mathcal{C}'$-semisimple where $\mathcal{C}'$ is obtained canonically from $\mathcal{C}$.

Proof of Proposition 1.8. In [18], it was shown that $\mathcal{A}/I$ is formally real but not $*$-semisimple. An alternative simpler proof of this fact is given below. Lemmas 2.4 and 3.3 now imply that

$$\sqrt[\mathrm{real}]{I} = I \neq \sqrt[\mathcal{R}-\mathrm{hard}]{I}.$$

In particular, I is real. Since $\sqrt[\mathcal{R}-\mathrm{hard}]{I} \subseteq \sqrt[\mathcal{C}-\mathrm{hard}]{I}$ for every representation class $\mathcal{C}$, we have

$$\sqrt[\mathrm{real}]{I} \neq \sqrt[\mathcal{C}-\mathrm{hard}]{I}.$$

So I does not satisfy the real Nullstellensatz over any representation class $\mathcal{C}$. $\square$

Proposition 3.5 ([18]). *The $*$-algebra $\mathcal{A}/I$ from Proposition 1.8 is formally real but is not $*$-semisimple.*

Proof. To prove that $\mathcal{A}/I$ is not $*$-semisimple we define a relation $\leq$ on $\mathcal{A}/I$ by $u \leq v$ if and only if $v - u \in \Sigma_{\mathcal{A}/I}$. By the second relation $aa^* \leq 1$, hence $a^*a \leq q$ by the first relation. Suppose that $a^*a \leq k$ for some $k \in \mathbb{R}$. Since $k(k - aa^*) = (k - aa^*)^2 + a(k - a^*a)a^*$, it follows that $aa^* \leq k$. Hence $a^*a \leq qk$ by the first relation. By induction $a^*a \leq q^m$ for every m. It follows that a is in the kernel of every $*$-representation of $\mathcal{A}/I$.

To prove that $\mathcal{A}/I$ is formally real we use the relations $aa^* = 1 - xx^*$ and $a^*a = q(1 - xx^*)$ to reduce each element of $\mathcal{A}/I$ to its **canonical form** whose monomials do not contain a^*a or aa^* as subwords. Such monomials are linearly independent, so the canonical form is unique. Suppose that

$$\sum_{i=1}^{n} p_i p_i^* = 0$$

for some nonzero $p_1, \ldots, p_n \in \mathcal{A}/I$. Write each p_i in its canonical form as

$$p_i = \sum_{j=1}^{m_i} z_{ij}(a^*)^j + \sum_{k=1}^{m_i} w_{ik} a^k + t_i$$

where neither of the monomials in z_{ij}, w_{kj}, t_i has either a or a^* as its last letter. Let d be the minimum of degrees of all monomials that appear in z_{ij}, w_{kj}, t_i. (Note that degree is well-defined for canonical forms.) Write $z_{ij} = z'_{ij} + z''_{ij}$, $w_{ik} = w'_{ik} + w''_{ik}$,

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and $t_i = t_i' + t_i''$, where all monomials in z_{ij}', w_{ik}', and t_i' have degree d while those in z_{ij}'', w_{ik}'', and t_i'' have degrees $> d$. Since

$$(a^*)^m a^m = q^m - \sum_{l=0}^{m-1} q^{m-l}(a^*)^l xx^* a^l \quad \text{and} \quad a^m(a^*)^m = 1 - \sum_{l=0}^{m-1} a^l xx^*(a^*)^l$$

for each m, the canonical form of $\sum_{i=1}^n p_i p_i^*$ is equal to $s + o$ where

$$s := \sum_{i,j} q^j z_{ij}'(z_{ij}')^* + \sum_{i,k} w_{ik}'(w_{ik}')^* + \sum_i t_i'(t_i')^* \tag{3.1}$$

consists of monomials of degree $2d$ and o consists of monomials of degree $> 2d$. By the uniqueness of the canonical form, we have that

$$s = 0. \tag{3.2}$$

Let us order the monomials that appear in z_{ij}', w_{ik}', t_i' lexicographically and let m be the first of them. Therefore, we can rewrite (3.1) and (3.2) as

$$0 = \sum_l (\alpha_l m + r_l)(\alpha_l m + r_l)^* \tag{3.3}$$

with m before each monomial of every r_l. Since $y_1 := \sum_l (\alpha_l m)(\alpha_l m + r_l)^*$ and $y_2 := \sum_l r_l(\alpha_l m + r_l)^*$ have disjoint monomials and $y_1 + y_2 = 0$ by (3.3), it follows that $y_1 = y_2 = 0$. Canceling m in $y_1^* = 0$ gives $\sum_i \bar{\alpha}_l(\alpha_l m + r_l) = 0$. Consequently, $\sum_l \bar{\alpha}_l \alpha_l = 0$, a contradiction with $p_i \neq 0$. $\square$

On the positive side, every $*$-ideal generated by unshrinkable words (i.e., words not decomposable as $d^* du$ or $ud^* d$ where d, u are words and d is nonempty) is real and satisfies the Real Nullstellensatz over $\mathcal{R}$, see [3, 17, 18]. It is not known yet if such ideals satisfy the Real Nullstellensatz over the class $\mathcal{B}$. Many more positive examples are given in Section 4 of [18].

4. Ideals generated by analytic polynomials. A classical commutative polynomial of several complex variables is analytic if it depends only on $z = (z_1, \ldots, z_g)$ (and not on $\overline{z}$). By analogy, a polynomial in $\mathbb{F}\langle x, x^*\rangle$ is **analytic** if it does not contain any x_j^* variables. For example, $p = x_1 x_2 + x_1$ is analytic and $p = x_1^* x_2$ is not. Let $\mathbb{F}\langle x\rangle$ denote the analytic polynomials in $\mathbb{F}\langle x, x^*\rangle$.

Let $I(P)$ denote the $*$-ideal generated by a collection P (not necessarily finite) of analytic polynomials. Recall that $I(P)$ has the **(hard) Nullstellensatz property** if

$$\sqrt[\text{hard}]{I(P)} = I(P).$$

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The ideal $I_p = I(\{p\})$ generated by the single polynomial $p = x_1x_2 - x_2x_1 + 1$ does not have the hard Nullstellensatz property. Indeed, $p(X)$ is never equal to 0 since $\mathrm{tr}(p(X)) = \mathrm{tr}(1) > 0$. Thus, $\sqrt[\text{hard}]{I_p} = \mathbb{F}\langle x, x^*\rangle$. However, $I_p \neq \mathbb{F}\langle x, x^*\rangle$ (the Gröbner basis for I is $\{p\}$, and so $1 \notin I_p$). On the other hand, we do not know of a polynomial p for which $p(X) = 0$ has a solution but $\sqrt[\text{hard}]{I_p} \neq I_p$.

In this section, we give conditions on P which imply that $I(P)$ has the hard Nullstellensatz property. For instance, Theorem 4.8 says if $I(P)$ is homogeneous, then it has the Nullstellensatz property. We note that left ideals with analytic generators have a good Nullstellensatz (see [8]) using zeros of the type as described in Section 5.

We start by introducing Gröbner basis machinery which is required to prove Theorem 4.8.

4.1. Non-commutative Gröbner bases. A **monomial order** $\prec$ on $\langle x, x^*\rangle$ is a total order on the elements of $\langle x, x^*\rangle$ with the following properties.

1. $\prec$ is a well ordering, that is, each nonempty subset of $\langle x, x^*\rangle$ has a minimal element.
2. If $a, b \in \langle x, x^*\rangle$ with $a \prec b$, and $c \in \langle x, x^*\rangle$, then $ca \prec cb$ and $ac \prec bc$.

Given a monomial order $\prec$, every nonzero polynomial $p \in \mathbb{F}\langle x, x^*\rangle$ can be written as $p = \sum_{i=1}^{s} c_i t_i$ where $c_1, \ldots, c_s$ are nonzero elements of $\mathbb{F}$ and $t_1 \succ \cdots \succ t_s$ belong to $\langle x, x^*\rangle$. In this case, $T(p) := t_1$ is the **leading monomial** of p, according to $\prec$, and $\mathrm{lc}(p) := c_1$ is the **leading coefficient** of p, according to $\prec$. We say that p is **monic** if $\mathrm{lc}(p) = 1$. Given two words $a, b \in \langle x, x^*\rangle$, we say a **divides** b if $b = cad$ for some $c, d \in \langle x, x^*\rangle$.

Given a two-sided ideal $I \subseteq \mathbb{F}\langle x, x^*\rangle$, a **reduced Gröbner basis** for I is a set $G \subseteq I$ with the following properties:

1. for each $f \in I$ there exists $g \in G$ such that $T(g)$ divides $T(f)$.
2. each element of G is monic, and
3. if g_1 and g_2 are distinct elements of G, then $T(g_1)$ does not divide any term of g_2.

By [13, Proposition 1.1], every two-sided ideal of $\mathbb{F}\langle x, x^*\rangle$ has a unique reduced Gröbner basis.

Let $\langle x\rangle \subseteq \langle x, x^*\rangle$ be the set of **analytic monomials** in $\langle x, x^*\rangle$, and let $\mathbb{F}\langle x\rangle \subseteq \mathbb{F}\langle x, x^*\rangle$ be the set of **analytic polynomials** in $\mathbb{F}\langle x, x^*\rangle$.

Proposition 4.1. *Let $\prec$ is a monomial order on $\mathbb{F}\langle x, x^*\rangle$. Let $I \subseteq \mathbb{F}\langle x, x^*\rangle$ be a $*$-ideal generated by some nonzero analytic polynomials. Then the reduced Gröbner basis for I is of the form $G \cup H^*$, where G and H consist entirely of analytic polyno-*

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mials.

Proof. Let G be the reduced Gröbner basis for $I \cap \mathbb{F}\langle x\rangle$ and let H^* be the reduced Gröbner basis for $I^* \cap \mathbb{F}\langle x\rangle^*$. Since I is generated by some analytic polynomials as a $*$-ideal, it is generated by some analytic and some antianalytic polynomials as a two-sided ideal. Therefore $G \cup H^*$ generates I as a two-sided ideal in $\mathbb{F}\langle x, x^*\rangle$. It is clear that $G \cup H^*$ satisfies conditions (2) and (3) of being a reduced Gröbner basis for I. To prove that $G \cup H^*$ satisfies condition (1) we need the following:

Claim 1. Every element of I is a linear combination of polynomials of the form

$$p_1 p_2 \cdots p_k, \tag{4.1}$$

where p_i alternate between analytic and antianalytic polynomials and each p_i is either

(a) a nonconstant monomial which is not the leading monomial of an element of I; or

(b) a polynomial asb, where $a, b \in \langle x, x^*\rangle$ and $s \in G \cup H^*$.

Moreover, we can assume that at least one p_i in each product is of the second form.

Since $G \cup H^*$ generates I as a two-sided ideal, each $f \in I$ is of the form

$$f = \sum_{j,\text{finite}} a_j g_j b_j + \sum_{j',\text{finite}} c_{j'} h^*_{j'} d_{j'}$$

where $g_j \in G$, $h_{j'} \in H$ and $a_j, b_j, c_{j'}, d_{j'} \in \mathbb{F}\langle x, x^*\rangle$. Now, every element of $\mathbb{F}\langle x, x^*\rangle$ is clearly a linear combination of products $r_1 \cdots r_m$ where r_i alternate between analytic and antianalytic monomials. If r_i is analytic then, by [13, Theorem 1.1, assertion 1], we have that $r_i = \iota_i + \omega_i$ for some $\iota_i \in I \cap \mathbb{F}\langle x\rangle$ and some ω_i which is a linear combination of analytic monomials that are not the leading monomial of some element of $I \cap \mathbb{F}\langle x\rangle$. Further, $\iota_i = \sum_{i',\ \text{finite}} a_{i'} g_{i'} b_{i'}$ for some $g_{i'} \in G$ and $a_{i'}, b_{i'} \in \mathbb{F}\langle x\rangle$. A similar argument prevails if r_i is antianalytic thus completing the proof of Claim 1.

Claim 2. If $p_1 \cdots p_k$ and $q_1 \cdots q_\ell$ are of the form (4.1) and

$$T(p_1 \cdots p_k) = T(q_1 \cdots q_\ell),$$

then $q_1 \cdots q_k - p_1 \cdots p_k$ is a linear combination of polynomials of the form (4.1).

The assumption implies that $T(p_1) \cdots T(p_k) = T(q_1) \cdots T(q_\ell)$. It follows that $k = \ell$ and $T(p_i) = T(q_i)$ for each i. Moreover, for each i:

(a) either $T(p_i) = p_i = q_i$; or

(b) p_i and q_i are both analytic and in the ideal generated by G; or

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(c) p_i and q_i are both antianalytic and in the ideal generated by H^*.

Consequently, $q_1 \cdots q_k - p_1 \cdots p_k = q_1 \cdots q_k - (p_1 - q_1 + q_1) \cdots (p_k - q_k + q_k) = \bar{q}$ where $T(\bar{q}) \prec T(q_1 \cdots q_k)$ and $\bar{q}$ is a linear combination of polynomials of the form (4.1), proving Claim 2.

To complete the proof of the proposition, let $f \in I$ be given. By Claim 1,

$$f = \sum_{k=1}^{n} c_k z_k$$

where $c_k \in \mathbb{F}$ and z_k are of the form (4.1). It can be assumed that there is an m such that $T(z_1) = \cdots = T(z_m) \succ T(z_{m+1}), \ldots, T(z_n)$. For each $k = 2, \ldots, m$, Claim 2 implies $z_k - z_1$ is a linear combination of polynomials of the form (4.1) and $T(z_k - z_1) \prec T(z_1)$. It follows that

$$f = (\sum_{k=1}^{m} c_k) z_1 + \bar{z}$$

where $T(\bar{z}) \prec T(z_1)$. Now, if $\sum_{k=1}^{m} c_k \neq 0$, then $T(f) = T(z_1)$ is divisible by the leading coefficient of an element of $G \cup H^*$. If $\sum_{k=1}^{m} c_k = 0$, we continue by induction. Therefore the leading monomial of every element of I is in the ideal generated by the leading monomials of elements of $G \cup H^*$. ☐

4.2. $*$ Ideals with analytic generators. A **graded order** $\prec$ on $\langle x, x^* \rangle$ is a monomial order such that $a \prec b$ if $\deg a < \deg b$. (Recall that $\deg a$ is the number of letters in the word a).

Lemma 4.2. *Let $I \subseteq \mathbb{F}\langle x, x^* \rangle$ be a two-sided ideal. If $\prec$ be a graded order on $\langle x, x^* \rangle$ and if W is the space spanned by all monomials which are not the leading monomial of an element of I, then $\mathbb{F}\langle x, x^* \rangle = I \oplus W$. Further, if $p = \iota + \omega \in \mathbb{F}\langle x, x^* \rangle$, with $\iota \in I$ and $\omega \in W$, then* $\deg(p) = \max\{\deg(\iota), \deg(\omega)\}$.

Proof. The first part is true for every monomial order; see assertion (1) in [13, Theorem 1.1].

Next, suppose $p = \iota + \omega$, where $\iota \in I$ and $\omega \in W$. The only way that $\deg(p) \neq \max\{\deg(\iota), \deg(\omega)\}$ is if the highest degree terms of ι and ω cancel each other out. In this case $T(\omega) = T(-\iota)$ since $\prec$ is a graded monomial order. On the other hand, $T(\omega)$ is not the leading monomial of an element from I; a contradiction. ☐

Proposition 4.3 and Corollary 4.4 are the main results of this subsection.

Proposition 4.3. *Let $I \subseteq \mathbb{F}\langle x, x^* \rangle$ be a $*$-ideal generated by analytic polynomi-*

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als. There exists a positive hermitian linear functional L such that

$$I = \{a \in \mathbb{F}\langle x, x^*\rangle \colon L(a^*a) = 0\}. \tag{4.2}$$

Hence, the $*$-algebra $\mathbb{F}\langle x, x^*\rangle/I$ is $*$-semisimple.

Corollary 4.4. *If $I \subseteq \mathbb{F}\langle x, x^*\rangle$ is a $*$-ideal generated by analytic polynomials, then*

$$\sqrt[\text{real}]{I} = \sqrt[\mathcal{R}-\text{hard}]{I} = I.$$

Proof of Corollary 4.4. Lemma 3.3 and Proposition 4.3 imply $\sqrt[\mathcal{R}-\text{hard}]{I} = I$. By Proposition 2.3 we have $\sqrt[\text{real}]{I} \subseteq \sqrt[\mathcal{R}-\text{hard}]{I}$ and of course $I \subseteq \sqrt[\text{real}]{I}$, so we get the conclusion. □

Proof of Proposition 4.3. Let $\prec$ be a graded monomial order on $\langle x, x^*\rangle$ so that $a \prec b$ if $\deg a < \deg b$. Let W be the space spanned by all monomials which are not the leading monomial of an element of I so that, by Lemma 4.2, $\mathbb{F}\langle x, x^*\rangle = I \oplus W$. We will construct a positive hermitian linear functional L on $\mathbb{F}\langle x, x^*\rangle$ such that

$$I = \{a \in \mathbb{F}\langle x, x^*\rangle \colon L(a^*a) = 0\} \tag{4.3}$$

as follows. Set $\tilde{L}(I) = \{0\}$. For each monomial $m \in W$, set

$$\tilde{L}(m) = \begin{cases} 0 & \text{if } m \text{ is not a square} \\ c_d & \text{if } m \text{ is a square of degree } 2d \end{cases}$$

where each $c_d > 0$ is a constant to be chosen inductively. Finally, we define $L(a)$ to be

$$L(a) := \frac{1}{2}\tilde{L}(a) + \frac{1}{2}\tilde{L}(a^*)^*$$

so that L is hermitian. Note that if $\iota \in I$, then $\iota^* \in I$, so $L(\iota) = 0$.

We will need the following:

Claim. Let $m_1, m_2 \in W$ be monomials of degree $\leq d$. If $m_1 \neq m_2$, then $L(m_1^* m_2)$ depends only on $c_1, \ldots, c_{d-1}$.

It suffices to show that $\tilde{L}(m_1^* m_2)$ depends only on $c_1, \ldots, c_{d-1}$.

If $m_1^* m_2 \in W$, either $\tilde{L}(m_1^* m_2) = 0$ or $\tilde{L}(m_1^* m_2) = c_e$ for some $e < d$ because $m_1 \neq m_2$ implies that $m_1^* m_2$ is not a square of degree $2d$.

If $m_1^* m_2 \notin W$, we can decompose $m_1^* m_2$ as $\iota + \omega$, where $\iota \in I$, $T(\iota) = m_1^* m_2$, and $\omega \in W$, and $\deg(\iota), \deg(\omega) \leq \deg(m_1^* m_2)$.

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If $\deg(m_1^* m_2) < 2d$, then ω is spanned by monomials u which either are not squares, in which case $\tilde{L}(u) = 0$, or which are squares, in which case $\tilde{L}(u)$ depends only on $c_0, \ldots, c_{d-1}$ because $\deg u < 2d$.

If $\deg(m_1^* m_2) = 2d$, then $\deg m_1 = \deg m_2 = d$. Let m_1 and m_2 be of the form

$$m_1 = u_1 \cdots u_k \quad \text{and} \quad m_2 = v_1 \cdots v_\ell,$$

where the u_i and v_j alternate between being nonempty analytic and antianalytic words. Since $m_1, m_2 \in W$, each u_i and each v_j is not the leading monomial of an element of I. On the other hand, since $m_1^* m_2 \notin W$,

$$m_1^* m_2 = u_k^* \cdots u_1^* v_1 \cdots v_\ell$$

is the leading monomial of some $p \in I$. Let $G \cup H^*$ be the reduced Gröbner basis of I given by Proposition 4.1. By property (1) of reduced Gröbner bases, $u_k^* \cdots u_1^* v_1 \cdots v_\ell = T(p)$ is divisible by the leading monomial of some $q \in G \cup H^*$. Note that $T(q)$ is either analytic or antianalytic but it divides neither of the words u_i and v_j. The only way this can happen is that $T(q)$ divides $u_1^* v_1$ and so $u_1^* v_1$ is either analytic or antianalytic. Let us decompose $u_1^* v_1 = \iota_1 + \omega_1$, where $\iota_1 \in I$, $\omega_1 \in W$, and both ι_1 and ω_1 are (anti)analytic if $u_1^* v_1$ is (anti)analytic. Also, by Lemma 4.2, $\deg(\iota_1), \deg(\omega_1) \leq \deg(u_1^* v_1)$. Therefore

$$\tilde{L}(m_1^* m_2) = \tilde{L}(u_k^* \cdots u_2^* \iota_1 v_2 \cdots v_\ell) + \tilde{L}(u_k^* \cdots u_2^* \omega_1 v_2 \cdots v_\ell).$$

We have $\tilde{L}(u_k^* \cdots u_2^* \iota_1 v_2 \cdots v_\ell) = 0$ since $\tilde{L}(I) = \{0\}$. Next, the degree $2d$ terms of $u_k^* \cdots u_2^* \omega_1 v_2 \cdots v_\ell$ cannot be squares since the middle two letters of each degree $2d$ word of come from pieces of terms of ω_1, which is either analytic or antianalytic—the middle piece of a square word is always of the form yy^*, where y is a letter. Therefore $\tilde{L}(u_k^* \cdots u_2^* \omega_1 v_2 \cdots v_\ell)$ does not depend on c_d since $u_k^* \cdots u_2^* \omega_1 v_2 \cdots v_\ell$ has no squares of degree $2d$ in it. This completes the proof of the claim.

Let M_d be a vector whose entries are all monomials of degree d in W. Consider $A = L(M_d^* M_d)$, which is defined by evaluating the entries of $M_d^* M_d$ with the functional L. First, since L is hermitian, clearly A is as well. Each monomial $m_1^* m_2$, with $\deg m_1 = \deg m_2 = d$, is distinct. If $m_1 \neq m_2$, then, by the Claim, $L(m_1^* m_2)$ does not depend on c_d. Finally, if $m_1 = m_2$, then $L(m_1^* m_1) = c_d$. Therefore the matrix A is of the form

$$A = c_d \mathrm{Id} + F_d$$

Id is an identity matrix of appropriate size, and where F_d is hermitian and depends only on $c_0, \ldots, c_{d-1}$. Further, by the Claim, the value of L on $\mathbb{F}\langle x, x^* \rangle_{2d-1}$ depends only on $c_0, \ldots, c_{d-1}$. Therefore Lemma 4.5, below, gives the result. $\square$

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The following technical lemma was used at the end of the proof of Proposition 4.3. It will be also used in the proof of Proposition 4.6. If $A = (a_{ij})_{1\le i,j\le n}$ is a matrix of polynomials, and L is a linear functional on $\mathbb{F}\langle x, x^*\rangle$, let $L(A)$ denote the matrix $(L(a_{ij}))_{1\le i,j\le n}$.

LEMMA 4.5. *Let $I \subseteq \mathbb{F}\langle x, x^*\rangle$ be a left ideal. Fix a graded order $\prec$, and let W be the space spanned by all monomials which are not the leading monomial of an element of I so that $\mathbb{F}\langle x, x^*\rangle = I \oplus W$. For each degree d, let M_d be the row vector whose entries are all monomials of degree d in W.*

Suppose there exist positive definite matrices $A_0, \dots, A_d, \dots$ such that for any positive constants $c_0, \dots, c_d, \dots$, a well-defined linear functional L on $\mathbb{F}\langle x, x^*\rangle$ can be defined inductively with the following properties:

1. *$L(I + I^*) = \{0\}$.*
2. *If $\deg(p) < 2d$ for some d, then the definition of $L(p)$ depends only on the choice of $c_0, \dots, c_{d-1}$.*
3. *$L(M_d^* M_d) = c_d A_d + F_d$, where the hermitian matrix F_d depends only on the choice of $c_0, \dots, c_{d-1}$.*

Then there exist values of $c_0, \dots, c_d, \dots$ such that the linear functional so defined satisfies $L(a^*a) \ge 0$ for each $a \in \mathbb{F}\langle x, x^*\rangle$ and equals 0 if and only if $a \in I$.

Proof. If $1 \in I$, then the problem is trivial. Otherwise, for $d = 0$, we must have $L(1) = c_0 A_0 + F_0$, where A_0 is a positive scalar. Therefore, for a sufficiently large value of c_0 we get $L(1) > 0$.

Next assume inductively assume inductively that $c_0, \dots, c_{d-1}$ are defined so that $L(b^*b) > 0$ for each $b \in \mathbb{F}\langle x, x^*\rangle_{d-1} \setminus I_{d-1}$. Let $a \in \mathbb{F}\langle x, x^*\rangle_d$ so that a can be decomposed as $a = \iota + \omega$, where $\iota \in I$ and $\omega \in W$. By Lemma 4.2, $\deg(\iota), \deg(\omega) \le d$. Since $L(I + I^*) = \{0\}$,

$$L(a^*a) = L(\iota^*\iota) + L(\iota^*a) + L(\omega^*\iota) + L(\omega^*\omega) = L(\omega^*\omega).$$

Further, suppose $a \notin I$, which implies that $\omega \ne 0$.

Let N_{d-1} be the row vector whose entries are all words in W of length less than d. Then $\omega = M_d\alpha_d + N_{d-1}\alpha_{d-1}$ for some constant column vectors α_d, α_{d-1}. We see that

$$L(\omega^*\omega) = \begin{bmatrix} \alpha_d \\ \alpha_{d-1} \end{bmatrix}^* \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \begin{bmatrix} \alpha_d \\ \alpha_{d-1} \end{bmatrix}, \tag{4.4}$$

where

$$A = L(M_d^* M_d), \quad B = L(M_d^* N_{d-1}) \quad \text{and} \quad C = L(N_{d-1}^* N_{d-1}).$$

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If $\alpha_d = 0$, then $\deg(\omega) < d$, so $L(\omega^*\omega) = \alpha_{d-1}^* C\alpha_{d-1} > 0$ since $\omega \notin I$. Since L is hermitian, clearly C is also hermitian. Since α_{d-1} is arbitrary, this implies that C is positive definite. Next, B depends on polynomials of degree less than $2d$, so by assumption B depends only on $c_0, \ldots, c_{d-1}$, which are already determined. Next consider $A = c_d A_d + F_d$. We see that $c_0, \ldots, c_{d-1}$ are already determined, and since $A_d \succ 0$, we can choose c_d sufficiently large so that the matrix

$$\begin{bmatrix} c_d A_d + F_d & B \\ B^* & C \end{bmatrix}$$

is positive definite. Given (4.4), this implies that $L(\omega^*\omega) > 0$. The result therefore follows by induction. ∎

4.3. Homogeneous analytic ideals. An (two-sided, left, right) ideal $I \subseteq \mathbb{F}\langle x \rangle$ is called **homogeneous** if it is generated by homogeneous polynomials, not necessarily of the same degree.

Proposition 4.6. *If $I \subseteq \mathbb{F}\langle x, x^* \rangle$ is a real, homogeneous left ideal (not necessarily finitely generated), then there exists a positive hermitian $\mathbb{F}$-linear functional L on $\mathbb{F}\langle x, x^* \rangle$ such that*

$$I = \{\iota \colon L(\iota^*\iota) = 0\}.$$

Proof. By the proof of [4, Theorem 4.1], for each degree d there exists a positive hermitian $\mathbb{F}$-linear functional L_{2d} on $\mathbb{F}\langle x, x^* \rangle_{2d}^{\text{hom}}$ such that

$$I_d^{\text{hom}} = \{\iota \in \mathbb{F}\langle x, x^* \rangle_d^{\text{hom}} \colon L_{2d}(\iota^*\iota) = 0\},$$

and such that $L_{2d}(\iota) = 0$ for each $\iota \in (I + I^*)_{2d}^{\text{hom}}$. Define the linear functional L on $\mathbb{F}\langle x, x^* \rangle$ to be 0 on odd degree monomials and to be $c_d L_{2d}$ on $\mathbb{F}\langle x, x^* \rangle_{2d}^{\text{hom}}$, where each c_d is a positive constant to be chosen. Note that since I is homogeneous, by construction $L(I + I^*) = \{0\}$ since each homogeneous polynomial in I is mapped to 0. Also, clearly L is hermitian.

Consider $A = L(M_d^* M_d)$. First, since L is hermitian, clearly A is as well. Next, if $M_d\alpha \in \mathbb{F}\langle x, x^* \rangle_d^{\text{hom}}$ for some constant column vector $\alpha \neq 0$, then by linearity

$$L(\alpha^* M_d^* M_d \alpha) = c_d \alpha^* L_d(M_d^* M_d)\alpha$$

which is positive by assumption. Therefore $L_d(M_d^* M_d) \succ 0$ and $A = c_d L_d(M_d^* M_d)$. Further, the definition of L on $\mathbb{F}\langle x, x^* \rangle_{2d-1}$ depends only on $c_0, \ldots, c_{d-1}$. An application of Lemma 4.5 gives the result. ∎

Proposition 4.7. *If $I \subseteq \mathbb{R}\langle x, x^* \rangle$ is a $*$-ideal generated by homogeneous analytic polynomials, then for each degree d, there exists a tuple of matrices X such that $\iota(X) = 0$ for each $\iota \in I$ and $q(X) \neq 0$ for each $q \notin I$ with degree at most d.*

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Proof. Fix $d \in \mathbb{N}$. Let $I^{(d)}$ be the $*$-ideal generated by I as well as by all analytic monomials of degree $d+1$. In this case, $(I^{(d)})_d = I_d$. By Proposition 4.3 there exists a nonnegative hermitian linear functional L_d such that

$$I^{(d)} = \{a \in \mathbb{F}\langle x, x^*\rangle \colon L_d(a^*a) = 0\}.$$

We follow the GNS construction to define $\mathcal{H}$ to be the pre-Hilbert space defined as the vector space $\mathbb{F}\langle x, x^*\rangle / I^{(d)}$ with inner product

$$\langle [a], [b]\rangle := L_d(b^*a),$$

and a tuple of linear operators $\widetilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_g)$ on $\mathcal{H}$ such that $\tilde{X}_i[r] = [x_i r]$ for each $i = 1, \ldots, g$ and each $r \in \mathbb{F}\langle x, x^*\rangle$. Clearly, $\tilde{X}_i^*[r] = [x_i^* r]$ for each $i = 1, \ldots, g$ and for each $r \in \mathbb{F}\langle x, x^*\rangle$, which implies that $q(\widetilde{X})[r] = [qr]$ for each $q, r \in \mathbb{F}\langle x, x^*\rangle$.

Define $\mathcal{W} \subseteq \mathcal{H}$ to be the space

$$\mathcal{W} = \{[ab] \colon a, b \in \mathbb{F}\langle x, x^*\rangle,\ a \text{ analytic},\ \deg(b) \leq d\}.$$

Since every analytic monomial of degree greater than d is in $I^{(d)}$, the space $\mathcal{W}$ is finite dimensional. Let X be the tuple of operators on $\mathcal{W}$ defined by

$$X = (P_{\mathcal{W}} \tilde{X}_1 P_{\mathcal{W}}, \ldots, P_{\mathcal{W}} \tilde{X}_g P_{\mathcal{W}}),$$

where $P_{\mathcal{W}}$ is the self-adjoint projection map onto $\mathcal{W}$. If a is analytic and $\deg(b) \leq d$, then $P_{\mathcal{W}} \tilde{X}_i P_{\mathcal{W}}[ab] = P_{\mathcal{W}} \tilde{X}_i[ab] = P_{\mathcal{W}}[x_i ab] = [x_i ab]$ for each $i = 1, \ldots, g$ and hence,

$$\vartheta(X)[ab] = [\vartheta ab]$$

for each analytic ϑ. If $\iota \in I$ is one of the analytic generators of I, this implies that

$$\iota(X)[ab] = [\iota ab] = 0.$$

Therefore $p(X) = 0$ for each $p \in I$. Also, if $\deg(q) \leq d$, then

$$q(X)[1] = [q].$$

It is clear that $q \in I^{(d)}$ if and only if $q \in I$. Therefore if $q \notin I$, then $q(X) \neq 0$. ∎

THEOREM 4.8. *If $I \subseteq \mathbb{R}\langle x, x^*\rangle$ is a $*$-ideal generated by homogeneous analytic polynomials, then* $\sqrt[\text{hard}]{I} = I$.

Proof. This follows directly from Proposition 4.7. ∎

Proof of Theorem 1.4. One can construct tuples of matrices $X^{(d)}$ on finite-dimensional Hilbert spaces $\mathcal{H}^{(d)}$ by Proposition 4.7 such that $\iota(X^{(d)}) = 0$ for each $\iota \in I$ and $p(X^{(d)}) \neq 0$ if $\deg(p) \leq d$ and $p \notin I$. Since I is homogeneous, one can scale

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each $X^{(d)}$ by a scalar and still preserve $\iota(X^{(d)}) = 0$ for each $\iota \in I$ and $p(X^{(d)}) \neq 0$ if $\deg(p) \leq d$ and $p \notin I$. Therefore choose each $X^{(d)}$ to have norm bounded by 1. Let $X := \bigoplus_{d\in\mathbb{N}} X^{(d)}$ be an operator on $\mathcal{H} := \bigoplus_{d\in\mathbb{N}} \mathcal{H}^{(d)}$. Then clearly $\|X\| \leq 1$ and $p(X) = 0$ if and only if $p \in I$. $\square$

We end this section with two remarks.

REMARK 4.9. Returning to the example at the outset of this section of the $*$-ideal I_p of $\mathbb{F}\langle x, x^*\rangle$ generated by $p = 1 + x_1x_2 - x_2x_1$, note that it does not satisfy the condition

$$\sqrt[\text{hard}]{I(P)} \cap \mathbb{F}\langle x\rangle = I(P) \cap \mathbb{F}\langle x\rangle$$

which is, at least formally, weaker than the hard Nullstellensatz property.

REMARK 4.10. The real radical of the two-sided ideal in $\mathbb{F}\langle x, x^*\rangle$ generated by a collection of analytic polynomials P is the the $*$-ideal generated by P.

5. Left zeroes. There is a theory of Nullstellensatz for a left ideal I in a $*$-algebra $\mathcal{A}$ (see [3, 4] for the most recent results and for historical references) and this section briefly explores the connections between Nullstellensatz for left ideals and the Nullstellensatz in this article for two-sided ideals.

The main result of this section is that, for the important representation classes, the left radical of a two-sided ideal coincides with its hard radical. The machinery developed in this section leads to an alternate proof of Proposition 1.1 (ii). This machinery will also be used in Section 6 where the relations between hard and soft zeros (defined later) are established.

Given a representation class $\mathcal{C}$ of the $*$-algebra $\mathcal{A}$, let

$$\mathcal{C}_{\text{left}} = \{(\pi, v) : \pi \in \mathcal{C}, v \in V_\pi\}.$$

The elements of $\mathcal{C}_{\text{left}}$ will be considered as "left real points" of $\mathcal{A}$. We say that an element (π, v) of $\mathcal{C}_{\text{left}}$ is a **left zero** of an element $a \in \mathcal{A}$ if $\pi(a)v = 0$. If $T \subseteq \mathcal{C}_{\text{left}}$, then

$$\mathcal{I}^{\mathcal{C}}_{\text{left}}(T) = \{a \in \mathcal{A} : \pi(a)v = 0 \text{ for all } (\pi, v) \in T\}$$

is a left ideal in $\mathcal{A}$ — the **left vanishing ideal** of T. In the case that T is a singleton $\{(\pi, v)\}$, it is convenient to abbreviate $\mathcal{I}^{\mathcal{C}}_{\text{left}}(\{(\pi, v)\})$ to $\mathcal{I}^{\mathcal{C}}_{\text{left}}(\pi, v)$. Given $\mathcal{S} \subseteq \mathcal{A}$, let

$$V^{\mathcal{C}}_{\text{left}}(\mathcal{S}) = \{(\pi, v) \in \mathcal{C}_{\text{left}} : \pi(s)v = 0 \text{ for all } s \in S\}$$

be its **left variety** and let

$$\sqrt[\mathcal{C}-\text{left}]{\mathcal{S}} := \mathcal{I}^{\mathcal{C}}_{\text{left}}(V^{\mathcal{C}}_{\text{left}}(\mathcal{S}))$$

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be its **left radical**. If $J(\mathcal{S})$ is the left ideal generated by $\mathcal{S}$, then clearly

$$V^{\mathcal{C}}_{\text{left}}(\mathcal{S}) = V^{\mathcal{C}}_{\text{left}}(J(\mathcal{S})) \quad \text{and} \quad \sqrt[\mathcal{C}-\text{left}]{\mathcal{S}} = \sqrt[\mathcal{C}-\text{left}]{J(\mathcal{S})}.$$

When $T \subseteq \Pi_{\text{left}}$ and $\mathcal{S} \subseteq \mathcal{A}$ and $\mathcal{C} = \Pi$ (recall that Π is the class of all finite-dimensional $*$-representations), we will use the abbreviations

$$\mathcal{I}_{\text{left}}(T) = \mathcal{I}^{\Pi}_{\text{left}}(T), \quad V_{\text{left}}(\mathcal{S}) = V^{\Pi}_{\text{left}}(\mathcal{S}) \quad \text{and} \quad \sqrt[\text{left}]{\mathcal{S}} = \sqrt[\Pi-\text{left}]{\mathcal{S}}.$$

5.1. Real left ideals of finite codimension. By Corollary 2.9, a two-sided ideal I of $\mathcal{A}$ has the form $\mathcal{I}_{\text{hard}}(\pi)$ for some $\pi \in \Pi$ (where $\mathcal{I}_{\text{hard}}(\pi) := \mathcal{I}_{\text{hard}}(\{\pi\}) = \ker \pi$) if and only if I is real and $\dim \mathcal{A}/I < \infty$. Lemma 5.1 is the one-sided version of this fact.

LEMMA 5.1. *If I is a left ideal of $\mathcal{A}$, then the following are equivalent.*

(1) I is real and $\dim \mathcal{A}/I < \infty$;
(2) There exist $\pi \in \Pi$ and $v \in V_\pi$ such that $I = \mathcal{I}_{\text{left}}(\pi, v)$ and $\pi(\mathcal{A})v = V_\pi$.

Moreover, for every representation class $\mathcal{C}$ and every element $(\pi, v) \in \mathcal{C}_{\text{left}}$ such that $\pi(\mathcal{A})v = V_\pi$ the following is true. If $I \subseteq \mathcal{I}^{\mathcal{C}}_{\text{left}}(\pi, v)$ is a two-sided ideal, then $I \subseteq \mathcal{I}^{\mathcal{C}}_{\text{hard}}(\pi)$. In particular, if $\mathcal{I}^{\mathcal{C}}_{\text{left}}(\pi, v)$ is a two-sided ideal, then $\mathcal{I}^{\mathcal{C}}_{\text{hard}}(\pi) = \mathcal{I}^{\mathcal{C}}_{\text{left}}(\pi, v)$.

Proof. Clearly, (2) implies (1). To prove the converse, suppose that I is real and $\dim \mathcal{A}/I < \infty$. Let π be the left regular representation of $\mathcal{A}$ on $\mathcal{A}/I$ and $v = 1 + I$. Let $V_\pi = \mathcal{A}/I$ and note that $\pi(\mathcal{A})v = V_\pi$. It remains to show that there exists an inner product on $\mathcal{A}/I$ such that π is a $*$-representation.

The kernel $Z(I)$ of the left regular representation π is the largest two-sided ideal contained in I by Lemma 2.1. Moreover, $\mathcal{A}/Z(I)$ is isomorphic to a subspace of the linear maps on the finite dimensional space $\mathcal{A}/I$. Hence $\mathcal{A}/Z(I)$ is finite dimensional. By Lemma 2.1 and Lemma 2.2(i), $Z(I)$ is a real $*$-ideal. By Lemma 2.4 and Proposition 2.7, $\mathcal{A}/Z(I)$ is $*$-isomorphic to a finite direct sum $\oplus_i M_{n_i}(F_i)$ where $n_i \in \mathbb{N}$, $F_i \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and the involution is conjugate transpose.

Let $\rho\colon \mathcal{A} \to \oplus_i M_{n_i}(F_i)$ be the composition of the canonical mapping $\mathcal{A} \to \mathcal{A}/Z(I)$ and the isomorphism $\mathcal{A}/Z(I) \to \oplus_i M_{n_i}(F_i)$. Since $\mathcal{A}/I$ is isomorphic to $\rho(\mathcal{A})/\rho(I)$ as a vector space, it suffices to construct a positive linear functional L on $\rho(\mathcal{A})$ such that $\rho(I) = \{b \in \rho(\mathcal{A})\colon L(b^*b) = 0\}$. It is well-known that every left ideal in a semisimple algebra is generated by an idempotent; see e.g. [7, Corollary 2.1A]. Therefore, $\rho(I) = \rho(\mathcal{A})e$ for some idempotent $e \in \rho(\mathcal{A})$ and we can take

$$L(b) := \operatorname{tr}\big((1-e)^* b (1-e)\big)$$

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where 1 is the identity matrix and b runs through $\rho(\mathcal{A}) = \oplus_i M_{n_i}(F_i)$.

To prove the moreover statement, observe since $\pi(\mathcal{I}^{\mathcal{C}}_{\text{left}}(\pi, v))v = 0$ and also I is a two-sided ideal, that $\pi(I)\pi(\mathcal{A})v = 0$. Since $\pi(\mathcal{A})v = V_\pi$, it follows that $\pi(I) = 0$. Hence $I \subseteq \mathcal{I}^{\mathcal{C}}_{\text{hard}}(\pi)$. In the case that $\mathcal{I}^{\mathcal{C}}_{\text{left}}(\pi, v)$ is a two-sided ideal, $\mathcal{I}^{\mathcal{C}}_{\text{left}}(\pi, v) \subseteq \mathcal{I}^{\mathcal{C}}_{\text{hard}}(\pi)$. The reverse inclusion is evident and hence $\mathcal{I}^{\mathcal{C}}_{\text{hard}}(\pi) = \mathcal{I}^{\mathcal{C}}_{\text{left}}(\pi, v)$. ☐

Remark 5.2. For $\mathcal{A} = \mathbb{F}\langle x, x^*\rangle$, Lemma 5.1 can also be deduced from [4, Theorem 4.1] and [11, Lemma 3]. ☐

5.2. Left radical of a two-sided ideal is two-sided. Note, if I is a two-sided ideal in $\mathcal{A}$, then

$$\sqrt[\mathcal{C}-\text{left}]{I} \subseteq \sqrt[\mathcal{C}-\text{hard}]{I}.$$

We would like to know when the opposite inclusion holds.

We say that a representation class $\mathcal{C}$ is **regular** if for every $(\pi, v) \in \mathcal{C}_{\text{left}}$ there exists $(\tilde{\pi}, \tilde{v}) \in \mathcal{C}_{\text{left}}$ such that $\tilde{\pi}(\mathcal{A})\tilde{v} = V_{\tilde{\pi}}$ and $\|\pi(a)v\| = \|\tilde{\pi}(a)\tilde{v}\|$ for every $a \in \mathcal{A}$. Recall that Π is the class of all finite-dimensional $*$-representations, $\mathcal{B}$ is the class of all bounded $*$-representations and $\mathcal{R}$ is the class of all $*$-representations.

Proposition 5.3. *The representation classes Π, $\mathcal{B}$ and $\mathcal{R}$ are regular.*

Proof. Let $\mathcal{C}$ be a representation class of $\mathcal{A}$ and $(\pi, v) \in \mathcal{C}_{\text{left}}$. Then $I := \{a \in \mathcal{A} \colon \pi(a)v = 0\}$ is a left ideal of $\mathcal{A}$. Let $\tilde{\pi}$ be the left regular representation of $\mathcal{A}$ on $V_{\tilde{\pi}} := \mathcal{A}/I$. We endow $\mathcal{A}/I$ with the inner product $\langle a + I, b + I\rangle := \langle \pi(a)v, \pi(b)v\rangle_{V_\pi}$ so that $\tilde{\pi}$ becomes a $*$-representation. Let us define $\tilde{v} := 1 + I$. Clearly, $\tilde{\pi}(\mathcal{A}) = V_{\tilde{\pi}}$ and $\|\tilde{\pi}(a)\tilde{v}\| = \|a + I\| = \|\pi(a)v\|$ for every $a \in \mathcal{A}$.

It remains to show that $\tilde{\pi} \in \mathcal{C}$ if $\mathcal{C}$ is one of the classes Π, $\mathcal{B}$. If π is finite-dimensional, then $\tilde{\pi}$ is also finite-dimensional because $\dim \mathcal{A}/\operatorname{Ker}\pi < \infty$ and $\operatorname{Ker}\pi \subseteq I$ implies $\dim \mathcal{A}/I < \infty$. If π is bounded, then $\tilde{\pi}$ is also bounded because $\|\tilde{\pi}(a)(b + I)\| = \|ab + I\| = \|\pi(ab)v\| = \|\pi(a)\pi(b)v\| \le \|\|\pi(a)\|\| \, \|\pi(b)v\| = \|\|\pi(a)\|\| \, \|b + I\|$. ☐

Proposition 5.4. *If $\mathcal{C}$ is a regular representation class of $\mathcal{A}$ and I is a two-sided ideal in $\mathcal{A}$, then*

$$\sqrt[\mathcal{C}-\text{hard}]{I} \subseteq \sqrt[\mathcal{C}-\text{left}]{I}.$$

Proof. Pick any $b \in \sqrt[\mathcal{C}-\text{hard}]{I}$ and any $(\pi, v) \in \mathcal{C}_{\text{left}}$ such that $\pi(I)v = 0$. Let $(\tilde{\pi}, \tilde{v}) \in \mathcal{C}_{\text{left}}$ be such that $\tilde{\pi}(\mathcal{A})\tilde{v} = V_{\tilde{\pi}}$ and $\|\pi(a)v\| = \|\tilde{\pi}(a)\tilde{v}\|$ for every $a \in \mathcal{A}$. In particular, $\tilde{\pi}(I)\tilde{v} = 0$. Since $I \subseteq \mathcal{I}^{\mathcal{C}}_{\text{left}}(\tilde{\pi}, \tilde{v})$ is a two-sided ideal, it follows from Lemma 5.1 that $I \subseteq \mathcal{I}^{\mathcal{C}}_{\text{hard}}(\tilde{\pi})$; i.e., $\tilde{\pi}(I) = 0$. Now $b \in \sqrt[\mathcal{C}-\text{hard}]{I}$ implies that $\tilde{\pi}(b) = 0$. Hence, $\tilde{\pi}(b)\tilde{v} = 0$ which implies that $\pi(b)v = 0$. This proves that $b \in \sqrt[\mathcal{C}-\text{left}]{I}$. ☐

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As an illustration of Propositions 5.4 and 5.3 we give an alternative proof of Proposition 1.1 (ii).

Proof. Let I be a two-sided ideal in a $\mathcal{A} = \mathbb{F}\langle x, x^*\rangle$ such that $\dim \mathcal{A}/I < \infty$. By [11, Lemma 3], I is finitely generated as a left ideal. Therefore, $\sqrt[\text{real}]{I} = \sqrt[\text{left}]{I} = \sqrt[\text{hard}]{I}$ where the first equality comes from [4, Theorem 1.6] and the second one from Propositions 5.4 and 5.3. By [4, Theorem 4.1] and the GNS construction, there exists $(\pi, v) \in \Pi_{\text{left}}$ such that $\sqrt[\text{real}]{I} = \mathcal{I}^{\mathcal{C}}_{\text{left}}(\pi, v)$ and $\pi(\mathcal{A})v = V_\pi$. By the moreover portion of Lemma 5.1, $\mathcal{I}^{\mathcal{C}}_{\text{left}}(\pi, v) = \mathcal{I}^{\mathcal{C}}_{\text{hard}}(\pi)$. □

6. Soft zeros. A tuple X in $M(\mathbb{F})^g$ is a **soft zero** of a polynomial p from $\mathbb{F}\langle x, x^*\rangle$ if $\det p(X) = 0$. Replacing hard with soft zeros in the definitions in Subsection 1.1 produces the notions of the soft vanishing set, soft variety and soft radical. This also works for a general $*$-algebra $\mathcal{A}$ and a general representation class $\mathcal{C}$ of $\mathcal{A}$. We say that a "real point" $\pi \in \mathcal{C}$ is a **soft zero** of a "polynomial" $a \in \mathcal{A}$ if $\pi(a)$ is not invertible. Again, we can define the soft vanishing set, soft variety and soft radical. We choose to work with general $\mathcal{A}$ but only the with the simplest $\mathcal{C}$, i.e., $\mathcal{C} = \Pi$, the finite-dimensional $*$-representations.

Given a subset T of Π, the **soft vanishing set** of T is

$$\mathcal{I}_{\text{soft}}(T) = \{a \in \mathcal{A} \colon \det \pi(a) = 0, \text{ for all } \pi \in T\}.$$

The set $\mathcal{I}_{\text{soft}}(T)$ satisfies $\mathcal{A}\mathcal{I}_{\text{soft}}(T)\mathcal{A} \subseteq \mathcal{I}_{\text{soft}}(T)$ but in general it is not closed under sums. (Thus it is not an ideal.) Likewise, given a subset $\mathcal{S}$ of $\mathcal{A}$, the **soft variety** of $\mathcal{S}$ is

$$V_{\text{soft}}(\mathcal{S}) = \{\pi \in \Pi \colon \det \pi(a) = 0, \text{ for all } a \in \mathcal{S}\}$$

and the **soft radical** of $\mathcal{S}$ is

$$\sqrt[\text{soft}]{\mathcal{S}} = \mathcal{I}_{\text{soft}}(V_{\text{soft}}(\mathcal{S})).$$

For $\pi \in \Pi$, it is convenient to abbreviate $\mathcal{I}_{\text{soft}}(\{\pi\})$ by $\mathcal{I}_{\text{soft}}(\pi)$. In subsection 6.1, we describe the structure of $\mathcal{I}_{\text{soft}}(\pi)$ and in subsection 6.2 we describe exactly when $\mathcal{I}_{\text{soft}}(\pi) = \mathcal{I}_{\text{hard}}(\pi)$. This is used in subsection 6.3 to characterize when $\mathcal{I}_{\text{soft}}(\pi)$ is a two-sided ideal.

6.1. The structure of $\mathcal{I}_{\text{soft}}(\pi)$. For a left ideal I of $\mathcal{A}$, let

$$\widehat{I} := \{p \in \mathcal{A} \colon \text{ there exists } q \in \mathcal{A} \setminus I \text{ such that } pq \in I\}.$$

PROPOSITION 6.1. *For a subset $\mathcal{S}$ of a $*$-algebra $\mathcal{A}$ the following are equivalent:*

1. *$\mathcal{S} = \mathcal{I}_{\text{soft}}(\pi)$ for some finite-dimensional $*$-representation π of $\mathcal{A}$.*

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2. *$\mathcal{S} = \bigcup_{i=1}^{k} \widehat{I_i}$ for some $k \in \mathbb{N}$ and some real left ideals $I_1, \ldots, I_k$ of $\mathcal{A}$ with $\dim \mathcal{A}/I_i < \infty$.*

Proof. To prove that (1) implies (2), recall that every finite-dimensional $*$-representation is a finite direct sum of irreducible $*$-representations, see e.g., [16, Proposition 9.2.4]. Furthermore, if $\pi = \oplus_i \pi$, then clearly $\mathcal{I}_{\text{soft}}(\pi) = \bigcup_i \mathcal{I}_{\text{soft}}(\pi_i)$. Therefore, in view of Lemma 5.1, it suffices to show that for every irreducible $*$-representation π of $\mathcal{A}$ and every nonzero $w \in V_\pi$ we have that

$$\mathcal{I}_{\text{soft}}(\pi) = \widehat{\mathcal{I}_{\text{left}}(\pi, w)}.$$

Clearly, for each $p \in \mathcal{I}_{\text{soft}}(\pi)$ there exists a nonzero $v \in V_\pi$ such that $\pi(p)v = 0$. We claim that for each nonzero $w \in V_\pi$ there exists $q \in \mathcal{A}$ such that $\pi(q)w = v$. This claim implies that $\pi(p)\pi(q)w = 0$ and $\pi(q)w \neq 0$, so that $p \in \widehat{\mathcal{I}_{\text{left}}(\pi, w)}$. We will prove the claim by contradiction. If $v \notin \pi(\mathcal{A})w$, then $\pi(\mathcal{A})w$ is a proper nontrivial invariant subspace for $\pi(\mathcal{A})$. Now, [16, Proposition 9.2.4] implies that π is reducible.

Suppose now that (2) is true. By Lemma 5.1, every I_i is of the form $\mathcal{I}_{\text{left}}(\pi_i, v_i)$ for some finite-dimensional $*$-representation π_i and some $v_i \in V_{\pi_i}$ such that $\pi_i(\mathcal{A})v_i = V_{\pi_i}$. We claim that $\widehat{I_i} = \mathcal{I}_{\text{soft}}(\pi_i)$. Namely, take any $p \in \mathcal{A}$ and recall that $p \in \widehat{I_i}$ if and only if $pq \in I_i$ for some $q \in \mathcal{A} \setminus I_i$. The latter is true if and only if there exists $q \in \mathcal{A}$ such that $\pi_i(q)v_i \neq 0$ and $\pi_i(p)\pi_i(q)v_i = 0$ which is true if and only if there exists $w_i \in V_{\pi_i}$ such that $w_i \neq 0$ and $\pi_i(p)w_i = 0$. The latter is equivalent to $p \in \mathcal{I}_{\text{soft}}(\pi_i)$. The claim implies (1) since $\cup_{i=1}^{k} \widehat{I_i} = \cup_{i=1}^{k} \mathcal{I}_{\text{soft}}(\pi_i) = \mathcal{I}_{\text{soft}}(\oplus_{i=1}^{k} \pi_i)$. $\square$

6.2. When $\mathcal{I}_{\text{soft}}(\pi)$ Has the Form $\mathcal{I}_{\text{hard}}(\psi)$.

PROPOSITION 6.2. *For a $*$-algebra $\mathcal{A}$ and representation $\pi \in \Pi$ the following are equivalent:*

1. *$\mathcal{I}_{\text{soft}}(\pi) \subseteq \mathcal{I}_{\text{hard}}(\psi)$ for some $\psi \in \Pi$.*
2. *$\mathcal{I}_{\text{soft}}(\pi) = \mathcal{I}_{\text{hard}}(\pi)$.*

If (2) is true and $\mathbb{F} = \mathbb{C}$, then $\pi(\mathcal{A})$ is $*$-isomorphic to $\mathbb{C}$ endowed with the standard involution. If (2) is true and $\mathbb{F} = \mathbb{R}$, then $\pi(\mathcal{A})$ is $*$-isomorphic to either $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ with standard involutions.

Proof. First, $\mathcal{I}_{\text{hard}}(\pi) \subseteq \mathcal{I}_{\text{soft}}(\pi)$ by definition. Next, suppose $a \in \mathcal{I}_{\text{soft}}(\pi)$, which is equivalent to $\det \pi(a) = 0$. Then $\det(\pi(a)^*\pi(a)) = 0$ as well, and since $\mathcal{I}_{\text{soft}}(\pi) \subseteq \mathcal{I}_{\text{hard}}(\psi)$, we have $a^*a \in \mathcal{I}_{\text{hard}}(\psi)$. Further, $\pi(a)^*\pi(a)$ cannot have any nonzero eigenvalues λ since, if it did, then λ would be real, $a^*a - \lambda \in \mathcal{I}_{\text{soft}}(\pi) \subseteq \mathcal{I}_{\text{hard}}(\psi)$, and so $a^*a - (a^*a - \lambda) = \lambda \in \mathcal{I}_{\text{hard}}(\pi)$. Therefore $\pi(a)^*\pi(a) = 0$, which implies that $\pi(a) = 0$. Therefore $\mathcal{I}_{\text{soft}}(\pi) \subseteq \mathcal{I}_{\text{hard}}(\pi)$, which implies that $\mathcal{I}_{\text{soft}}(\pi) = \mathcal{I}_{\text{hard}}(\pi)$.

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Suppose that $\mathcal{I}_{\text{soft}}(\pi) = \mathcal{I}_{\text{hard}}(\pi)$ for some $\pi \in \Pi$. Since $\pi(\mathcal{A})$ is contained in $M_n(\mathbb{F})$ for some n, it is a finite-dimensional formally real $*$-algebra. We claim that $\pi(\mathcal{A})$ has no zero-divisors. Namely, if $\pi(a)\pi(b) = 0$ for some $a, b \in \mathcal{A}$, then either $\det \pi(a) = 0$ or $\det \pi(b) = 0$ which implies that either $\pi(a) = 0$ or $\pi(b) = 0$. The claims about the structure of $\pi(\mathcal{A})$ now follow from Proposition 2.7. ☐

A representation π is **unitarily equivalent** to representation ψ if there exists a unitary operator $T\colon V_\pi \to V_\psi$ such that $\psi(a) = T\pi(a)T^*$ for every $a \in \mathcal{A}$.

Corollary 6.3. *For $*$-algebra $\mathcal{A}$ and every irreducible $\pi \in \Pi$ the following are equivalent.*

1. $\mathcal{I}_{\text{soft}}(\pi) = \mathcal{I}_{\text{hard}}(\pi)$.
2. $\pi(\mathcal{A})$ *has no zero divisors.*
3. *If $\mathbb{F} = \mathbb{C}$, then π is unitarily equivalent to some $*$-representation $\psi\colon \mathcal{A} \to \mathbb{C}$. If $\mathbb{F} = \mathbb{R}$, then π is unitarily equivalent to some $*$-representation of one of the following types: (i) $\psi\colon \mathcal{A} \to \mathbb{R}$ where $\pi(\mathcal{A}) = \mathbb{R}$, (ii) $\psi\colon \mathcal{A} \to M_2(\mathbb{R})$ where $\pi(\mathcal{A}) = \mathbb{C}$ or (iii) $\psi\colon \mathcal{A} \to M_4(\mathbb{R})$ where $\pi(\mathcal{A}) = \mathbb{H}$.*

Proof. In the proof of Proposition 6.2 we showed that (1) implies (2). To show that (3) implies (1) note that every element $A \in \mathbb{C} \subseteq M_2(\mathbb{R})$ satisfies $\det A = 0$ if and only if $A = 0$ and that every element $B \in \mathbb{H} \subseteq M_4(\mathbb{R})$ satifies $\det B = 0$ iff $B = 0$. Finally, (2) implies (3) by the Burnside's theorem for irreducible subalgebras of $M_n(\mathbb{C})$ and its real version [10, Theorem 6]. (Clearly, if π is similar to ψ, then π is also unitarily equivalent to ψ.) ☐

6.3. When $\sqrt[\text{soft}]{\{p\}}$ and $\sqrt[\text{real}]{\mathcal{A}p}$ are two-sided ideals. Given an element p of a $*$-algebra $\mathcal{A}$ let J_p and I_p denote the left and two-sided ideals generated by p respectively. We will write Π_{irr} for the set of all irreducible finite-dimensional $*$-representations of $\mathcal{A}$. Recall that every $\pi \in \Pi$ can be decomposed as an orthogonal sum of finitely many elements from Π_{irr}. Recall also that $\pi(\mathcal{A})v = V_\pi$ for every $\pi \in \Pi_{\text{irr}}$ and every nonzero $v \in V_\pi$.

Lemma 6.4. *For every element p of a $*$-algebra $\mathcal{A}$ we have that*

$$\mathcal{I}_{\text{hard}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}}) \subseteq \sqrt[\text{left}]{J_p}.$$

If $\sqrt[\text{left}]{J_p}$ is a two-sided ideal, then the opposite inclusion holds too.

Recall that $\sqrt[\text{left}]{J_p} = \sqrt[\text{real}]{J_p}$ if $\mathcal{A} = \mathbb{F}\langle x, x^* \rangle$ by the left Nullstellenatz [4, Theorem 1.6].

Proof. Take any $q \in \mathcal{I}_{\text{hard}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}})$ and any $(\pi, v) \in \Pi_{\text{left}}$ such that $\pi(p)v = 0$. Let us decompose $\pi = \pi_1 \oplus \cdots \oplus \pi_k$ where $\pi_i \in \Pi_{\text{irr}}$, and $v = v_1 \oplus \cdots \oplus v_k$ where $v_i \in V_{\pi_i}$. It follows that $\pi_i(p)v_i = 0$ for every $i = 1, \ldots, k$. Therefore, either

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$\det \pi_i(p) = 0$ or $v_i = 0$, which implies that either $\pi_i(q) = 0$ or $v_i = 0$ for every $i = 1, \dots, k$. Consequently, $\pi_i(q)v_i = 0$ for every $i = 1, \dots, k$, and so, $\pi(q)v = 0$. This proves the first part.

To prove the second part, take any $r \in \sqrt[\text{left}]{J_p}$ any $\pi \in V_{\text{soft}}(p) \cap \Pi_{\text{irr}}$. Since $\det \pi(p) = 0$, there exists a nonzero $v \in V_\pi$ such that $\pi(p)v = 0$. Since $\sqrt[\text{left}]{J_p}$ is a two-sided ideal, we have that $rs \in \sqrt[\text{left}]{J_p}$ for every $s \in \mathcal{A}$. It follows that $\pi(rs)v = 0$ for every $s \in \mathcal{A}$. Since $\pi(\mathcal{A})v = V_\pi$, it follows that $\pi(r) = 0$. Therefore, $r \in \mathcal{I}_{\text{hard}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}})$. □

Proposition 6.5. *For $p \in \mathbb{F}\langle x, x^*\rangle$ the following are equivalent:*

1. *The ideal $\sqrt[\text{real}]{J_p}$ is two-sided (i.e., $\sqrt[\text{real}]{J_p} = \sqrt[\text{real}]{I_p}$).*
2. *For every $\pi \in \Pi_{\text{irr}}$, $\det \pi(p) = 0$ implies $\pi(p) = 0$ (i.e., $V_{\text{soft}}(p) \cap \Pi_{\text{irr}} \subseteq V_{\text{hard}}(p)$).*

Proof. If $V_{\text{soft}}(p) \cap \Pi_{\text{irr}} \subseteq V_{\text{hard}}(p)$, then clearly $\sqrt[\text{real}]{I_p} \subseteq \mathcal{I}_{\text{hard}}(V_{\text{hard}}(p)) \subseteq \mathcal{I}_{\text{hard}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}})$. By Lemma 6.4, we have $\mathcal{I}_{\text{hard}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}}) \subseteq \sqrt[\text{left}]{J_p} = \sqrt[\text{real}]{J_p}$. It follows that $\sqrt[\text{real}]{I_p} \subseteq \sqrt[\text{real}]{J_p}$.

Conversely, if $\sqrt[\text{real}]{I_p} = \sqrt[\text{real}]{J_p}$, then $\sqrt[\text{real}]{J_p}$ is a two-sided ideal by Lemma 2.2 and it is finitely generated as a left ideal by the Real Algorithm [4, Theorem 3,1]. Proposition 1.1 now implies that $\sqrt[\text{real}]{J_p} = \mathcal{I}_{\text{hard}}(V_{\text{hard}}(p))$. (Namely, by the first part of Proposition 1.1, $\sqrt[\text{real}]{J_p}$ has finite codimension. Therefore, the assumptions of the second part of Proposition 1.1 are satisfied.) On the other hand, we have that $\sqrt[\text{real}]{J_p} = \mathcal{I}_{\text{hard}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}})$ by Lemma 6.4. Finally,

$$\begin{aligned}
V_{\text{soft}}(p) \cap \Pi_{\text{irr}} &\subseteq V_{\text{hard}}(\mathcal{I}_{\text{hard}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}})) \\
&= V_{\text{hard}}(\sqrt[\text{real}]{J_p}) \\
&= V_{\text{hard}}(\mathcal{I}_{\text{hard}}(V_{\text{hard}}(p))) \\
&= V_{\text{hard}}(p)
\end{aligned}$$

as claimed. □

Lemma 6.6. *For every element p of a $*$-algebra $\mathcal{A}$ we have that*

$$\sqrt[\text{left}]{J_p} \subseteq \sqrt[\text{soft}]{\{p\}} = \mathcal{I}_{\text{soft}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}}).$$

Proof. Since $V_{\text{soft}}(p) \cap \Pi_{\text{irr}} \subseteq V_{\text{soft}}(p)$, we have that $\sqrt[\text{soft}]{\{p\}} = \mathcal{I}_{\text{soft}}(V_{\text{soft}}(p)) \subseteq \mathcal{I}_{\text{soft}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}})$. Conversely, take any $q \in \mathcal{I}_{\text{soft}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}})$ and any $\pi \in V_{\text{soft}}(p)$. Let us decompose $\pi = \pi_1 \oplus \cdots \oplus \pi_k$ where $\pi_i \in \Pi_{\text{irr}}$ for all i. Since $\det \pi(p) = 0$, it follows that $\det \pi_i(p) = 0$ for some i. Therefore, $\det \pi_i(q) = 0$, which implies that $\det \pi(q) = 0$. This proves the equality.

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To prove the inclusion take any $r \in \sqrt[\text{left}]{J_p}$ and any $\pi \in V_{\text{soft}}(p)$. Since $\det \pi(p) = 0$, there exists a nonzero $v \in V_\pi$ such that $\pi(p)v = 0$. It follows that $\pi(r)v = 0$ which implies that $\det \pi(r) = 0$. Therefore, $r \in \sqrt[\text{soft}]{\{p\}}$. $\square$

PROPOSITION 6.7. *For every element p of a $*$-algebra $\mathcal{A}$, the following are equivalent.*

1. *The set $\sqrt[\text{soft}]{\{p\}}$ is a two-sided ideal.*
2. *For every $\pi \in V_{\text{soft}}(p) \cap \Pi_{\text{irr}}$ we have that $\mathcal{I}_{\text{soft}}(\pi) = \mathcal{I}_{\text{hard}}(\pi)$. (cf. Corollary 6.3.)*
3. *$\sqrt[\text{soft}]{\{p\}} = \mathcal{I}_{\text{hard}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}})$.*
4. *$\sqrt[\text{soft}]{\{p\}} = \sqrt[\text{left}]{J_p}$.*

Proof. If (2) is false, then there exist $\pi \in V_{\text{soft}}(p) \cap \Pi_{\text{irr}}$ and $q \in \mathcal{A}$ such that $\det \pi(q) = 0$ but $\pi(q) \neq 0$. Pick $w \in V_\pi$ such that $\pi(q)w \neq 0$. For every v in the unit sphere $S_\pi \subseteq V_\pi$, pick $r_v \in \mathcal{A}$ such that $\pi(r_v)v = w$. The sets

$$U_v := \{u \in S_\pi \colon \pi(q)\pi(r_v)u \neq 0\}$$

are clearly open and they cover S_π because $v \in U_v$ for every $v \in S_\pi$. Since V_π is finite-dimensional, S_π is compact. Pick $v_1, \ldots, v_k \in S_\pi$ such that $S_\pi = U_{v_1} \cup \cdots \cup U_{v_k}$ and consider the element

$$r := \sum_{i=1}^{k} r_{v_i}^* q^* q r_{v_i}.$$

By construction, $\langle \pi(r)v, v\rangle = \sum_{i=1}^{k} \|\pi(q)\pi(r_{v_k})v\|^2 > 0$ for every $v \in S_\pi$. Therefore $\det \pi(r) \neq 0$, and so $r \notin \mathcal{I}_{\text{soft}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}})$. Since $q \in \sqrt[\text{soft}]{\{p\}}$ and $r \notin \sqrt[\text{soft}]{\{p\}}$, it follows that (1) is false.

If (2) is true, then (3) follows from

$$\begin{aligned}
\sqrt[\text{soft}]{\{p\}} &= \mathcal{I}_{\text{soft}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}}) \\
&= \bigcap_{\pi \in V_{\text{soft}}(p) \cap \Pi_{\text{irr}}} \mathcal{I}_{\text{soft}}(\pi) \\
&= \bigcap_{\pi \in V_{\text{soft}}(p) \cap \Pi_{\text{irr}}} \mathcal{I}_{\text{hard}}(\pi) \\
&= \mathcal{I}_{\text{hard}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}}).
\end{aligned}$$

If (3) is true, then (4) follows from Lemmas 6.4 and 6.6. Namely,

$$\sqrt[\text{soft}]{\{p\}} = \mathcal{I}_{\text{hard}}(V_{\text{soft}}(p) \cap \Pi_{\text{irr}}) \subseteq \sqrt[\text{left}]{J_p} \subseteq \sqrt[\text{soft}]{\{p\}}$$

shows that $\sqrt[\text{left}]{J_p} = \sqrt[\text{soft}]{\{p\}}$.

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Clearly, $\sqrt[\mathrm{soft}]{\{p\}}$ is a two-sided semigroup ideal w.r.t. multiplication in $\mathcal{A}$. If (4) is true, then it is also a subgroup w.r.t. addition in $\mathcal{A}$. Hence, (1) is true. □

REMARK 6.8. If $\mathcal{A} = \mathbb{F}\langle x, x^*\rangle$, then $\sqrt[\mathrm{soft}]{\{p\}}$ is a two-sided ideal if and only if $\sqrt[\mathrm{real}]{J_p} = \sqrt[\mathrm{soft}]{\{p\}}$ (since $\sqrt[\mathrm{left}]{J_p} = \sqrt[\mathrm{real}]{J_p}$ by the left Nullstellenatz [4, Theorem 1.6]) if and only if $\sqrt[\mathrm{soft}]{\{p\}} = \mathcal{I}_{\mathrm{hard}}(\psi)$ for some $\psi \in \Pi$ (by the Real Algorithm [4, Theorem 3,1] and Proposition 1.1). However, $\mathcal{I}_{\mathrm{hard}}(\psi)$ may be different from $\mathcal{I}_{\mathrm{soft}}(\psi)$ in this case because ψ may not be irreducible.

EXAMPLE 6.9. If $\mathcal{A} = \mathbb{C}\langle x, x^*\rangle$, then $\mathcal{I}_{\mathrm{hard}}(X) = \mathcal{I}_{\mathrm{soft}}(X)$ if and only if $\mathcal{I}_{\mathrm{hard}}(X) = \mathcal{I}_{\mathrm{hard}}((\lambda_1, \ldots, \lambda_g))$ for some $\lambda_1, \ldots, \lambda_g \in \mathbb{C}$. The polynomial p defined by

$$p = \sum_{i=1}^{g} (x_i - \lambda_i)^* (x_i - \lambda_i) + \sum_{i=1}^{g} (x_i - \lambda_i)(x_i - \lambda_i)^*$$

satisfies $\sqrt[\mathrm{real}]{J_p} = \mathcal{I}_{\mathrm{hard}}((\lambda_1, \ldots, \lambda_g)) = \mathcal{I}_{\mathrm{soft}}((\lambda_1, \ldots, \lambda_g))$ and so $\sqrt[\mathrm{soft}]{\{p\}}$ is a two-sided ideal.

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