Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

ELA
http:/repository.uwyo.edu/ela

EXTREMAL GRAPHS FOR NORMALIZED LAPLACIAN SPECTRAL RADIUS AND ENERGY*

KINKAR CH. DAS† AND SHAOWEI SUN†

Abstract. Let $G = (V, E)$ be a simple graph of order n and the normalized Laplacian eigenvalues $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{n-1} \geq \rho_n = 0$. The normalized Laplacian energy (or Randić energy) of G without any isolated vertex is defined as

$$RE(G) = \sum_{i=1}^{n} |\rho_i - 1|.$$

In this paper, a lower bound on ρ_1 of connected graph G (G is not isomorphic to complete graph) is given and the extremal graphs (that is, the second minimal normalized Laplacian spectral radius of connected graphs) are characterized. Moreover, Nordhaus-Gaddum type results for ρ_1 are obtained. Recently, Gutman et al. gave a conjecture on Randić energy of connected graph [I. Gutman, B. Furtula, Ş. B. Bozkurt, On Randić energy, Linear Algebra Appl. 442 (2014) 50–57]. Here this conjecture for starlike trees is proven.

Key words. Normalized Laplaican spectral radius, Randić energy, Normalized Laplacian spread, Nordhaus-Gaddum type results, Vertex cover number

AMS subject classifications. 05C50, 15A18

This paper is dedicated to Professor Ravindra B. Bapat on the occasion of his 60th birthday.

1. Introduction. Let G be a simple graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. Then its order is $|V(G)|$, denoted by n, and its size is $|E(G)|$, denoted by m. The degree d_i of a vertex v_i of a graph G is the cardinality of the neighborhood N_i of the vertex v_i. The maximum and second maximum degrees are denoted by Δ_1 and Δ_2, respectively. If vertices v_i and v_j are adjacent, we denote that

*Received by the editors on March 28, 2016. Accepted for publication on December 21, 2016. Handling Editor: Manjunatha Prasad Karantha

†Department of Mathematics, Sungkyunkwan University, Suwon 440-746, Republic of Korea (kinkardas2003@googlemail.com, sunshaowei2009@126.com). The first author was supported by the National Research Foundation funded by the Korean government with the grant no. 2013R1A1A2009341.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

ELA
http:/repository.uwyo.edu/ela

by $v_i v_j \in E(G)$. Let $D(G)$ be the diagonal matrix of vertex degrees of a graph G. The Laplacian matrix $L(G)$ is defined as $D(G)-A(G)$, where $A(G)$ is the adjacency matrix of G. Then the normalized Laplacian matrix of graph G without any isolated vertex is defined as $\mathcal{L}(G) = D^{-1/2}(G)L(G)D^{-1/2}(G)$. Let $\rho_1(G) \geq \rho_2(G) \geq \cdots \geq \rho_{n-1}(G) \geq \rho_n(G) = 0$ be its eigenvalues. When only one graph is under consideration, then we write ρ_i instead of $\rho_i(G)$. The normalized Laplacian energy (or Randić energy) of G is defined as

$$RE(G) = \sum_{i=1}^{n} |\rho_i - 1|.$$

Several bounds on Randić energy can be found in the literature, see [9, 10, 14]. Within classes of molecular graphs, a relatively good (increasing) linear correlation between Randić energy and energy of graphs has been investigated in the recent article by Furtula and Gutman [13]. For more information on Randić energy, we refer to the reader [11, 15].

The general Randić index $R_\alpha(G)$ of graph G is defined as the sum of $(d_i\, d_j)^\alpha$ over all edges $v_i\, v_j$ of G, where α is an arbitrary real number, that is,

$$R_\alpha(G) = \sum_{v_i v_j \in E(G)} (d_i\, d_j)^\alpha\,.$$

The general Randić index when $\alpha = -1$ is

$$R_{-1}(G) = \sum_{v_i v_j \in E(G)} \frac{1}{d_i d_j}\,.$$

Some basic mathematical properties of $R_{-1}(G)$ can be found in [5, 18] and the references therein. Throughout this paper we use P_n, C_n, S_n, K_n and $K_{p,\,q}$ $(p+q=n)$ to denote the path graph, the cycle graph, the star graph, the complete graph and the complete bipartite graph on n vertices, respectively. For other undefined notations and terminologies from graph theory, the readers are referred to [2].

The tree of odd order $n\,(\geq 1)$, containing with $\frac{n-1}{2}$ pendant vertices, each attached to a vertex of degree 2, and a vertex of degree $\frac{n-1}{2}$, will be called the $\left(\frac{n-1}{2}\right)$-sun. The tree of even order $n\,(\geq 2)$, obtained from a $\lceil\frac{n-2}{4}\rceil$-sun and a $\lfloor\frac{n-2}{4}\rfloor$-sun, by connecting their central vertices, will be called a $\left(\lceil\frac{n-2}{4}\rceil, \lfloor\frac{n-2}{4}\rfloor\right)$-double sun.

CONJECTURE 1.1. [14] *If $n \geq 1$ is odd, then the connected graph of order n, having greatest Randić energy is the sun. If $n \geq 2$ is even, then the connected graph of order n, having greatest Randić energy is the balanced double sun. Thus, if n is*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

odd, then the tree with maximal RE is the $\left(\frac{n-1}{2}\right)$*-sun. If* n *is even, then this tree is the* $\left(\lceil\frac{n-2}{4}\rceil, \lfloor\frac{n-2}{4}\rfloor\right)$*-double sun.*

The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we obtain a lower bound on $\rho_1(G)$ for connected graphs and characterize the extremal graphs. Using this result we present some bounds on the normalized Laplaican spread $(\rho_1 - \rho_{n-1})$ and Nordhaus-Gaddum type results for ρ_1. In Section 4, we prove Conjecture 1.1 for starlike trees.

2. Preliminaries. Here we recall some basic properties of the normalized Laplacian eigenvalues which will be used in next two sections.

LEMMA 2.1. [7] *Let* G *be a connected graph of order* $n \geq 2$*. Then* $\rho_{n-1} \leq \frac{n}{n-1}$ *with equality holding if and only if* $G \cong K_n$*. If* G *is not the complete graph* K_n*, then* $\rho_{n-1} \leq 1$*.*

LEMMA 2.2. [7] *Let* G *be a connected graph of order* $n \geq 2$*. Then* $\frac{n}{n-1} \leq \rho_1 \leq 2$ *with left equality holding if and only if* $G \cong K_n$*, and right equality holding if and only if* $G \cong K_{p,\,q}$ *with* $n = p + q$*.*

LEMMA 2.3. [7] *Let* G *be a bipartite graph of order* n*. For each* ρ_i*, the value* $2 - \rho_i$ *is also an eigenvalue of* G*.*

The join $G_1 \bigvee G_2$ of graphs G_1 and G_2 is the graph obtained from the disjoint union of G_1 and G_2 by adding all edges between $V(G_1)$ and $V(G_2)$. Let $K_{s,t}^r = K_r \bigvee (K_s \bigcup K_t)$ with $r,\, s,\, t \geq 1$.

LEMMA 2.4. [17] *Let* G *be a connected graph with order* $n \geq 3$*. Then* $G \cong K_{s,t}^r$ *if and only if* G *is* $\{S_4,\, P_4,\, C_4\}$*-free.*

In [4], Cavers found the normalized Laplacian spectrum of $G \cong K_1 \bigvee q\, K_s$ with $n = 1 + q\, s$:

$$NLS(G) = \left\{ \underbrace{\frac{s+1}{s}, \cdots, \frac{s+1}{s}}_{s\,q-q+1}, \underbrace{\frac{1}{s}, \cdots, \frac{1}{s}}_{q-1}, 0 \right\}. \tag{2.1}$$

Note that $G \cong K_{s,\,s}^1$ when $q = 2$.

LEMMA 2.5. [12] *Let* $G = (V,\, E)$ *be a graph of order* n*. If* $N_i = N_j$*,* $i\,(\neq)\,j = 1,\, 2, \ldots,\, k$*, then* G *has at least* $k-1$ *equal normalized Laplacian eigenvalues 1.*

Let ρ be an eigenvalue with a corresponding eigenvector $\mathbf{x} = (x_1,\, x_2, \ldots,\, x_n)^T$ of the matrix $Q\,(= D^{-1}L)$ with non-isolated vertices of graph G. Since the normalized Laplacian matrix $\mathcal{L}$ and Q are similar, then they have the same spectrum. We have

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

$Q\mathbf{x} = \rho\,\mathbf{x}$, that is,

$$\rho\,x_i = x_i - \frac{1}{d_i} \sum_{v_j:\, v_i v_j \in E(G)} x_j, \quad i = 1,\, 2, \ldots,\, n. \tag{2.2}$$

By multiplying $d_i\,x_i$ to each side of the above equation, we get

$$(1-\rho)\,d_i\,x_i^2 = \sum_{v_j:\, v_i v_j \in E(G)} x_i\,x_j, \quad i = 1,\, 2, \ldots,\, n.$$

Taking summation on the above from $i = 1$ to n, we get

$$\rho = 1 - \frac{2 \sum\limits_{v_i v_j \in E(G)} x_i\,x_j}{\sum\limits_{i=1}^{n} d_i\,x_i^2}.$$

From this we get the following result:

Lemma 2.6. [7] *Let G be a graph of order n with non-isolated vertices. Then*

$$\rho_1 \geq 1 - \frac{2 \sum\limits_{v_i v_j \in E(G)} x_i\,x_j}{\sum\limits_{i=1}^{n} d_i\,x_i^2}, \tag{2.3}$$

for any non-zero vector $\mathbf{x} = (x_1,\, x_2, \ldots,\, x_n)^T$.

Lemma 2.7. *Let K_t be a complete graph on t vertices with $V(K_t) = \{v_1,\, v_2, \ldots,\, v_t\}$. Also let H be a graph of order $n-t$ with $V(H) = \{v_{t+1},\, v_{t+2}, \ldots,\, v_n\}$. Fix q $(1 \leq q \leq n-t)$, then the resultant graph $G = (V,\, E)$ is as follows:*

$$V(G) = V(K_t) \cup V(H) = \{v_1,\, v_2, \ldots,\, v_n\},$$
$$E(G) = E(K_t) \cup E(H) \cup \{v_i\,v_j \,:\, i = 1,\, 2, \ldots,\, t;\ j = t+1,\, t+2, \ldots,\, t+q\}.$$

Then G has at least $t-1$ equal normalized Laplacian eigenvalues $\frac{t+q}{t+q-1}$.

Proof. From (2.2), we get

$$\rho\,x_i = x_i - \frac{1}{q+t-1} \sum_{v_j: v_i v_j \in E(G)} x_j, \quad i = 1,\, 2, \ldots,\, t.$$

One can easily see that $\rho = \frac{t+q}{t+q-1}$ with corresponding eigenvectors

$$(\underbrace{1,\, -1}_{2},\, 0, \ldots,\, 0)^T,\ (\underbrace{1,\, 0,\, -1}_{3},\, 0, \ldots,\, 0)^T, \ldots,\ (\underbrace{1,\, 0, \ldots,\, -1}_{t},\, 0, \ldots,\, 0)^T$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

satisfy the above relation. Since these eigenvectors are linearly independent, we get the required result. ◻

LEMMA 2.8. *Let $G \cong \overline{K}_a \bigvee K_{n-a}$ $(1 \le a \le n-1)$. Then the normalized Laplacian spectrum of graph G is*

$$NLS(G) = \left\{1 + \frac{a}{n-1},\, \underbrace{\frac{n}{n-1},\, \cdots,\, \frac{n}{n-1}}_{n-a-1},\, \underbrace{1,\, \cdots,\, 1}_{a-1},\, 0\right\}.$$

Proof. By Lemmas 2.5 and 2.7, we have two normalized Laplacian eigenvalues of G are $\frac{n}{n-1}$ with multiplicity $n-a-1$ and 1 with multiplicity $a-1$. Since $\rho_n = 0$ and $\sum\limits_{i=1}^{n} \rho_i = n$, we get the required result. ◻

For a graph $G = (V,\, E)$, a vertex cover of G is a set of vertices $C \subset V$ such that every edge of G is incident to at least one vertex in C. A minimum vertex cover is a vertex cover C such that no other vertex cover is smaller in size than C. The vertex cover number c is the size of the minimum vertex cover.

LEMMA 2.9. [6] *For a tree with vertex cover number c, the multiplicity of the normalized Laplacian eigenvalue 1 is exactly $n - 2\,c$.*

A matching M in G is a set of pairwise non-adjacent edges, that is, no two edges share a common vertex. A maximum matching is a matching that contains the largest possible number of edges. The matching number ν of a graph G is the size of a maximum matching.

LEMMA 2.10. (König's matching theorem) [2] *For any bipartite graph G, $\nu = c$.*

Denoted by $T(n,\, k,\, n_1,\, n_2, \ldots,\, n_k)$, is a tree of order n formed by joining the center v_i of star S_{n_i} to a new vertex v for $i = 1,\, 2, \ldots,\, k$ with $n_1 \ge n_2 \ge \cdots \ge n_k \ge 1$, that is,

$$T(n,\, k,\, n_1,\, n_2, \ldots,\, n_k) - \{v\} = S_{n_1} \cup S_{n_2} \cup \cdots \cup S_{n_k}\,.$$

In particular, we have $T(n,\, k,\, 2,\, 2, \ldots,\, 2) \cong \left(\frac{n-1}{2}\right)$-sun, where $k = \frac{n-1}{2}$ and n is odd.

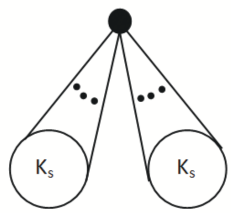
LEMMA 2.11. [12] *Let $T = (V,\, E)$ be a tree of order $n \ge 3$. Also let v_1 and v_2 be the maximum and the second maximum degree vertices of degrees Δ_1 and Δ_2, respectively. If $v_1 v_2 \notin E(T)$, then*

$$\rho_{n-1}(T) \le 1 - \sqrt{1 - \frac{1}{\Delta_2}}$$

with equality holding if and only if $T \cong T(n,\, k,\, n_1,\, n_2, \ldots, n_k)$, $n_1 = n_2$.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

3. Normalized Laplacian spectral radius of graphs.



$K^1_{s,\,s}$ $(2\,s+1=n)$

Fig. 1: Graph $K^1_{s,\,s}$ with $2\,s+1=n$.

Several lower bounds on ρ_1 in terms of graph parameters (n, m, Δ_1 etc.) are known, see [16] and the references therein. Moreover, it is already known that the complete graph gives the minimal normalized Laplacian spectral radius of a connected graph G of order n. In this section our aim is to find the second minimal normalized Laplacian spectral radius of a connected graph G of order n. For this we give a lower bound on ρ_1 of a connected graph G and characterize the extremal graphs.

THEOREM 3.1. *Let $G\,(\neq K_n)$ be a connected graph of order n. Then*

$$\rho_1 \geq \frac{n+1}{n-1} \tag{3.1}$$

with equality holding if and only if $G \cong K_n - e$ or $K^1_{s,\,s}$ with $2\,s+1=n$ (see, Fig. 1).

Proof. First we prove this result when P_4 or C_4 or S_4 is an induced subgraph of G. Suppose that the path $P_4 : v_iv_jv_kv_\ell$ is an induced subgraph of the graph G. If $d_i+d_j+d_k+d_\ell < 3\,n-3$, then we set $x_i=1$, $x_j=-1$, $x_k=1$, $x_\ell=-1$ and all the other components are 0 in (2.3). We have

$$\rho_1 \geq 1+\frac{6}{d_i+d_j+d_k+d_\ell} > \frac{n+1}{n-1}.$$

Otherwise, $d_i+d_j+d_k+d_\ell \geq 3\,n-3$. Let $CN = N_i\cap N_j\cap N_k\cap N_\ell$ with $|CN|=a$, where $0\leq a\leq n-4$. Since each vertex in CN is adjacent to $\{v_i,\,v_j,\,v_k,\,v_\ell\}$ and $|CN|=a$, we must have

$$d_i+d_j+d_k+d_\ell \leq 3(n-4-a)+4a+6 = 3n+a-6$$

and hence $3\leq a\leq n-4$. Setting $x_i=x_j=x_k=x_\ell=1$, $x_r=-\frac{4}{a+1}$ for $v_r\in CN$,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

ELA
http:/repository.uwyo.edu/ela

and all the other components are 0 in (2.3), we have

$$\rho_1 \geq 1+2\,\frac{\dfrac{16\,a}{a+1}-3-\displaystyle\sum_{\substack{v_r v_s \in E(G),\\ v_r, v_s \in CN}} \frac{16}{(a+1)^2}}{d_i+d_j+d_k+d_\ell+\dfrac{16}{(a+1)^2}\displaystyle\sum_{v_r \in CN} d_r}\,.$$

Since

$$\sum_{\substack{v_r v_s \in E(G),\\ v_r, v_s \in CN}} \frac{16}{(a+1)^2} \leq \frac{8\,a\,(a-1)}{(a+1)^2} \quad \text{and} \quad \sum_{v_r \in CN} d_r \leq a(n-1),$$

from the above result, we have

$$\rho_1 \geq 1+\frac{2}{n-1}\cdot\frac{5\,a^2+18\,a-3}{3\,a^2+22\,a+3+(a-3)(a+1)^2/(n-1)}\,. \tag{3.2}$$

For $4 \leq a \leq n-4$, we have $a^2-2\,a-3=(a-3)(a+1)>0$ and $a+1<n-1$ and hence

$$5\,a^2+18\,a-3>3\,a^2+22\,a+3+(a-3)(a+1)>3\,a^2+22\,a+3+(a-3)(a+1)^2/(n-1).$$

Then by (3.2), again we get $\rho_1 > \frac{n+1}{n-1}$. For $a=3$, by (3.2), we have $\rho_1 \geq 1+\frac{2}{n-1}$ with equality holding if and only if $x_i=x_j=x_k=x_\ell=1$, $x_r=-1$ for $v_r \in CN$, and all the other components are 0. By (2.2), we have

$$1+\frac{2}{n-1}=1+\frac{2}{d_i}$$

and hence $d_i=n-1$, which is a contradiction with $d_i \leq n-3$. Therefore we get $\rho_1 > \frac{n+1}{n-1}$.

Next suppose that the cycle $C_4=v_iv_jv_kv_\ell v_i$ is an induced subgraph of the graph G. Then $d_t \leq n-2$, $t=i,\,j,\,k,\,\ell$. Setting $x_i=1$, $x_j=-1$, $x_k=1$, $x_\ell=-1$ and all the other components are 0 in (2.3), we have

$$\rho_1 \geq 1+\frac{8}{d_i+d_j+d_k+d_l} \geq 1+\frac{2}{n-2} > \frac{n+1}{n-1}.$$

Next suppose that the star S_4 is an induced subgraph of the graph G with vertex set $V(S_4)=\{v_1,\,v_2,\,v_3,\,v_4\}$, where $v_1\,v_i \in E(G)$, $i=2,\,3,\,4$. First we assume that the vertices v_2, v_3 and v_4 have at least two common neighbors $\{v_1,\,v_5\}$, (say). Then

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

$d_i \le n-3$ for $i = 2, 3, 4$. Setting $x_1 = 1$, $x_2 = -1$, $x_3 = -1$, $x_4 = -1$, $x_5 = 1$ and all the other components are 0 in (2.3), we have

$$\rho_1 \ge \begin{cases} 1 + \dfrac{10}{d_1+d_2+d_3+d_4+d_5} & \text{if } v_1v_5 \in E(G) \text{ with } d_1,\, d_5 \le n-1 \\[2ex] 1 + \dfrac{12}{d_1+d_2+d_3+d_4+d_5} & \text{if } v_1v_5 \notin E(G) \text{ with } d_1,\, d_5 \le n-2 \end{cases}$$

$$\ge 1 + \frac{10}{2(n-1)+3(n-3)} > \frac{n+1}{n-1}\,.$$

Next we assume that v_2, v_3 and v_4 have exactly one common neighbor. Hence $d_2 + d_3 + d_4 \le 2(n-4) + 3 = 2n-5$. Setting $x_1 = 1$, $x_2 = -1$, $x_3 = -1$, $x_4 = -1$ and all the other components are 0 in (2.3), we have

$$\rho_1 \ge 1 + \frac{6}{d_1+d_2+d_3+d_4} \ge 1 + \frac{6}{n-1+2n-5} > \frac{n+1}{n-1}.$$

We now consider the graph G with $\{S_4, P_4, C_4\}$-free. By Lemma 2.4, we have $G \cong K^r_{s,\,t}$ $(r+s+t=n)$ as G is connected. Then by Lemma 2.7, one can easily get that the normalized Laplacian eigenvalues of G are

$$\left\{ \underbrace{\frac{s+r}{s+r-1}, \dots, \frac{s+r}{s+r-1}}_{s-1}, \underbrace{\frac{t+r}{t+r-1}, \dots, \frac{t+r}{t+r-1}}_{t-1}, \underbrace{\frac{n}{n-1}, \dots, \frac{n}{n-1}}_{r-1} \right\}$$

and the remaining three eigenvalues satisfy the following equations:

$$\rho x_1 = x_1 - \frac{1}{n-1}\,\left((r-1)x_1 + sx_2 + tx_3\right)$$

$$\rho x_2 = x_2 - \frac{1}{r+s-1}\,\left((s-1)x_2 + rx_1\right)$$

$$\rho x_3 = x_3 - \frac{1}{r+t-1}\,\left((t-1)x_3 + rx_1\right).$$

Therefore the remaining three normalized Laplacian eigenvalues of G satisfy the following equation

$$x\, f(x) = 0,$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

where

$$f(x) = x^2 - \left(\frac{r}{s+r-1} + \frac{s+t}{n-1} + \frac{r}{r+t-1}\right)x + \frac{sr}{(r+t-1)(n-1)} + \frac{r^2}{(s+r-1)(r+t-1)} + \frac{rt}{(s+r-1)(n-1)}.$$

Now,

$$\begin{aligned} f\left(\frac{n+1}{n-1}\right) &= \left(\frac{n+1}{n-1} - \frac{r}{n-t-1} - \frac{n-r}{n-1} - \frac{r}{n-s-1}\right)\left(\frac{n+1}{n-1}\right) \\ &\quad + \frac{sr}{(n-s-1)(n-1)} + \frac{r^2}{(n-t-1)(n-s-1)} + \frac{rt}{(n-t-1)(n-1)} \\ &= \left(\frac{r+1}{n-1} - \frac{r\,(2n-s-t-2)}{(n-s-1)\,(n-t-1)}\right)\left(\frac{n+1}{n-1}\right) \\ &\quad + \frac{n\,(n-1)\,r - 2rst}{(n-1)\,(n-s-1)\,(n-t-1)} \\ &= \frac{2nr - n^2 - 2r + 1 + (n+1)st - (n-3)rst}{(n-1)^2\,(n-s-1)\,(n-t-1)}. \qquad (3.3) \end{aligned}$$

Without loss of generality, we can assume that $t \geq s$. We now consider two cases (i) $r = 1$, (ii) $r \geq 2$.

Case (i) : $r = 1$. Since $s \leq \frac{n-1}{2} \leq t$, from (3.3), we get

$$f\left(\frac{n+1}{n-1}\right) = \frac{4st - (n-1)^2}{st(n-1)^2} \leq 0\,.$$

Thus we have $\rho_1 \geq \frac{n+1}{n-1}$ with equality holding if and only if $s = t = \frac{n-1}{2}$, that is, if and only if $G \cong K^1_{s,\,s}$ with $2\,s+1 = n$.

Case (ii) : $r \geq 2$. If $s = t = 1$, we have $G \cong K_n - e$. By Lemma 2.8, $\rho_1(G) = \frac{n+1}{n-1}$ and hence the equality holds in (3.1). Otherwise, we consider the following two subcases:

Subcase (a) : $s = 1$ and $t \geq 2$. In this subcase, from (3.3), we get

$$f\left(\frac{n+1}{n-1}\right) = \frac{2\,nr - n^2 - 2r + 1 + (n+1)t - (n-3)rt}{(n-1)^2\,(n-2)\,(n-t-1)}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

For $n = 5$ or 6, one can check directly that

$$f\left(\frac{n+1}{n-1}\right) < 0. \tag{3.4}$$

Otherwise, $n \geq 7$. Since $t \geq 2$ and $n \geq r+3$, we have

$$\begin{aligned} t\left[(n-3)r-(n+1)\right]-2nr+n^2+2r-1 &\geq 2\left[(n-3)r-(n+1)\right]-2nr+n^2+2r-1 \\ &= (n-1)^2-4(r+1) > 0. \end{aligned}$$

Again (3.4) holds and hence the inequality in (3.1) is strict.

Subcase (b) : $t \geq s \geq 2$. If $n = 6$, then $G \cong K^2_{2,\,2}$ and hence the inequality in (3.1) is strict. Otherwise, $n \geq 7$. Now,

$$\begin{aligned} st\left[(n-3)r-(n+1)\right]-2nr+n^2+2r-1 &\geq 4\left[(n-3)r-(n+1)\right]-2nr+n^2+2r-1 \\ &= 2r\,(n-5)+(n-2)^2-9 > 0. \end{aligned}$$

Again (3.4) holds and hence the inequality in (3.1) is strict. □

Using the above theorem, we can obtain a lower bound for the normalized Laplacian spread $(\rho_1 - \rho_{n-1})$ of all graphs G $(\neq K_n,\, \overline{K_n})$ with fixed order n and characterize the extremal graphs.

THEOREM 3.2. *Let* G $(\neq K_n,\, \overline{K}_n)$ *be a graph with order* n. *Then* $\rho_1 - \rho_{n-1} \geq \frac{2}{n-1}$ *with equality holding if and only if* $G \cong K_n - e$.

Proof. First we assume that G is a disconnected graph. Since $G \neq \overline{K}_n$, then G has at least one edge. Thus we have $\rho_1 > 1$ and $\rho_{n-1} = 0$. Hence $\rho_1 - \rho_{n-1} > \frac{2}{n-1}$.

Next we assume that G is a connected graph. Since $G \neq K_n$, by Lemma 2.1 and Theorem 3.1, we have

$$\rho_1 \geq \frac{n+1}{n-1} \ \text{ and } \ \rho_{n-1} \leq 1.$$

Thus $\rho_1 - \rho_{n-1} \geq \frac{2}{n-1}$. Moreover, $\rho_1 - \rho_{n-1} = \frac{2}{n-1}$ if and only if $\rho_{n-1} = 1$ and $\rho_1 = \frac{n+1}{n-1}$. Again by Theorem 3.1, we have $\rho_1 = \frac{n+1}{n-1}$ if and only if $G \cong K_n - e$ or $G \cong K^1_{s,\,s}$ with $2\,s+1 = n$. By (2.1), we get $\rho_{n-1}(K^1_{s,\,s}) = \frac{2}{n-1}$ $(2\,s+1 = n)$. By Lemma 2.8, we have $\rho_{n-1}(K_n - e) = 1$. This completes the proof of the theorem. □

We now give the first $\lfloor \frac{n+2}{2} \rfloor$ minimal normalized Laplacian spectral radius of graphs with order n.

THEOREM 3.3. (i) *Let* $G \notin \{K_n,\, K_{n-1} \cup \overline{K}_1,\, K_{n-2} \cup \overline{K}_2, \ldots,\, K_{p+1} \cup \overline{K}_p,\, K_p \cup \overline{K}_{p+1},\, K_n - e,\, K^1_{p,\,p}\}$ *be a graph of odd order* n $(= 2p+1)$. *Then*

$$\rho_1(K_n) < \rho_1(K_{n-1} \cup \overline{K}_1) < \rho_1(K_{n-2} \cup \overline{K}_2) < \cdots < \rho_1(K_{p+2} \cup \overline{K}_{p-1})$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

$$< \rho_1(K_{p+1} \cup \overline{K}_p) = \rho_1(K_n - e) = \frac{n+1}{n-1} = \rho_1(K^1_{p,\,p}) < \rho_1(G). \tag{3.5}$$

(ii) Let $G \notin \{K_n,\, K_{n-1} \cup \overline{K}_1,\, K_{n-2} \cup \overline{K}_2, \ldots,\, K_{p+2} \cup \overline{K}_{p-2},\, K_{p+1} \cup \overline{K}_{p-1},\, K_n - e\}$ *be a graph of even order* n $(= 2p)$*. Then*

$$\begin{aligned} &\rho_1(K_n) < \rho_1(K_{n-1} \cup \overline{K}_1) < \rho_1(K_{n-2} \cup \overline{K}_2) < \cdots < \rho_1(K_{p+2} \cup \overline{K}_{p-2}) \\ &\qquad < \rho_1(K_{p+1} \cup \overline{K}_{p-1}) < \rho_1(K_n - e) = \frac{n+1}{n-1} < \rho_1(G). \end{aligned} \tag{3.6}$$

Proof. We have

$$\rho_1(K_{n-i} \cup \overline{K}_i) = \frac{n-i}{n-i-1}\,,\ i = 1,\,2, \ldots,\, n-2.$$

For $n = 2p+1$, we have

$$\begin{aligned} &\rho_1(K_n) < \rho_1(K_{n-1} \cup \overline{K}_1) < \rho_1(K_{n-2} \cup \overline{K}_2) < \cdots < \rho_1(K_{p+2} \cup \overline{K}_{p-1}) \\ &\qquad < \rho_1(K_{p+1} \cup \overline{K}_p) = \frac{n+1}{n-1}\,. \end{aligned} \tag{3.7}$$

For $n = 2p$, we have

$$\begin{aligned} &\rho_1(K_n) < \rho_1(K_{n-1} \cup \overline{K}_1) < \rho_1(K_{n-2} \cup \overline{K}_2) < \cdots < \rho_1(K_{p+2} \cup \overline{K}_{p-2}) \\ &\qquad < \rho_1(K_{p+1} \cup \overline{K}_{p-1}) = \frac{n+2}{n} < \frac{n+1}{n-1}\,. \end{aligned} \tag{3.8}$$

First we assume that G is a connected graph. For $G \neq K_n$, by Theorem 3.1, we have

$$\rho_1(G) \geq \frac{n+1}{n-1}$$

with equality holding if and only if $G \cong K_n - e$ or $K^1_{s,\,s}$ with $2s+1 = n$. From the above results, we get the required results in (3.5) and (3.6).

Next we assume that G is a disconnected graph. Let $G_1, G_2, \ldots, G_k$ be the k $(k \geq 2)$ connected components in G. Also let n_i be the number of vertices in G_i, $i = 1,\, 2, \ldots,\, k$ such that $\sum\limits_{i=1}^{k} n_i = n$ with $n_1 \geq n_2 \geq \cdots \geq n_k$. If $n_2 \geq 2$, then by Lemma 2.2,

$$\rho_1(G) = \max\Big\{\rho_1(G_1),\, \rho_1(G_2), \ldots,\, \rho_1(G_k)\Big\} \geq 1 + \frac{1}{n_2 - 1} > \frac{n+1}{n-1}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

ELA
http:/repository.uwyo.edu/ela

and hence the two inequalities hold in (3.5) and (3.6). Otherwise, $n_2 = 1$. Then $G \cong G_1 \cup \overline{K}_{k-1}$. We consider the following two cases:

Case (i) : $n = 2p+1$. If $k \geq p+2$, then

$$\rho_1(G) = \rho_1(G_1) \geq \frac{n_1}{n_1 - 1} = \frac{n-k+1}{n-k} > \frac{n+1}{n-1}$$

and hence the inequality (3.5) holds. Otherwise, $k \leq p+1$. For $G_1 \cong K_{n-k+1}$, then $G \cong K_{n-k+1} \cup \overline{K}_{k-1}$ and hence the inequality holds in (3.7), that is, the inequality holds in (3.5). Otherwise, $G_1 \neq K_{n-k+1}$. Using Theorem 3.1, we get

$$\rho_1(G) = \rho_1(G_1) \geq \frac{n-k+2}{n-k} > \frac{n+1}{n-1}$$

and hence the inequality holds in (3.5).

Case (ii) : $n = 2p$. Similarly, as Case (i), we consider $k \geq p+1$ and $k \leq p$, and we can see that the inequality holds in (3.6). □

We now obtain Nordhaus-Gaddum type results on $\rho_1(G)$.

THEOREM 3.4. *Let* $G\,(\neq K_n, \overline{K}_n)$ *be a graph of order* n. *Then*

$$\rho_1(G) + \rho_1(\overline{G}) \geq 2 + \frac{4}{n-1} \tag{3.9}$$

with equality holding if and only if $G \cong K_2 \bigcup K_1$ *or* $G \cong S_3$.

Proof. Since $G \neq K_n, \overline{K}_n$, we have $n \geq 3$. For $n = 3$, $G \cong K_2 \bigcup K_1$ or $G \cong S_3$. Hence the equality holds in (3.9). Otherwise, $n \geq 4$. One can easily see that both G and $\overline{G}$, or one of them must be connected. Without loss of generality, we can assume that G is connected. If $\overline{G}$ is connected, by Theorem 3.1 we have

$$\rho_1(G) + \rho_1(\overline{G}) > 2 + \frac{4}{n-1}.$$

Otherwise, $\overline{G}$ is disconnected. If $\overline{G} \neq K_i \bigcup \overline{K}_{n-i}$ $(2 \leq i \leq n-1)$, then by Theorem 3.3, we have $\rho_1(\overline{G}) > 1 + \frac{2}{n-1}$ and hence $\rho_1(G) + \rho_1(\overline{G}) > 2 + \frac{4}{n-1}$, by Theorem 3.1. Next we only have to consider the remaining case: $\overline{G} \in \Big\{K_i \cup \overline{K}_{n-i} \,|\, 2 \leq i \leq n-1\Big\}$. Therefore $\rho_1(\overline{G}) = \frac{i}{i-1}$ $(2 \leq i \leq n-1)$. By Lemma 2.8, we have $\rho_1(G) = 1 + \frac{i}{n-1}$ $(2 \leq i \leq n-1)$ and hence

$$\rho_1(G) + \rho_1(\overline{G}) = 2 + \frac{1}{i-1} + \frac{i}{n-1}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

It is easy to see that the function $f(x) = 2 + \frac{1}{x-1} + \frac{x}{n-1}$ is decreasing on $[2, \sqrt{n-1}+1]$ and increasing on $(\sqrt{n-1}+1,\, n-1]$. Since $n \geq 4$, we have

$$\rho_1(G) + \rho_1(\overline{G}) \geq f(\sqrt{n-1}+1) = 2 + \frac{2}{\sqrt{n-1}} + \frac{1}{n-1} > 2 + \frac{4}{n-1}.$$

This completes the proof of the theorem. $\square$

THEOREM 3.5. *Let G be a graph of order $n\,(>2)$. Then*

$$\rho_1(G) + \rho_1(\overline{G}) \leq 4 \tag{3.10}$$

with equality holding if and only if

$$G \in \Big\{S_3,\, P_4,\, K_{2,n-2},\, K_2 \bigcup K_{n-2}\Big\}.$$

Proof. One can easily check that S_3, P_4, $K_{2,n-2}$ and $K_2 \bigcup K_{n-2}$ satisfy the equality in (3.10). Otherwise, $n \geq 5$ and $G \neq K_{2,n-2}$, $G \neq K_2 \bigcup K_{n-2}$. By Lemma 2.2, we can immediately get $\rho_1(G)+\rho_1(\overline{G}) \leq 4$. Suppose that equality holds in (3.10). Then by Lemma 2.2, both G and $\overline{G}$ have at least one connected bipartite component. Without loss of generality, we can assume that G is connected. Since G is a bipartite graph with $n \geq 5$, then there is at least one independent vertex subset S of $V(G)$ with $|S| \geq 3$. Then K_3 is an induced subgraph of $\overline{G}$. If $\overline{G}$ is connected, then $\overline{G}$ can not be a bipartite graph, a contradiction. Otherwise, $\overline{G}$ is a disconnected graph. Then G must be complete bipartite graph $K_{p,\,q}$ and hence $\overline{G} \cong K_p \bigcup K_q$ $(p+q=n)$. Since $n \geq 5$ and $\rho_1(\overline{G}) = 2$, then $p = 2$ or $q = 2$, and hence $G \cong K_{2,\,n-2}$, a contradiction. This completes the proof of the theorem. $\square$

4. Normalized Laplacian energy of starlike tree.

A tree is said to be starlike if exactly one of its vertices has degree greater than two. By $S(n_1, n_2, \ldots, n_k)$ we denote the starlike tree which has a vertex v_1 of degree $k \geq 3$ with $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$ and which has the property

$$S(n_1, n_2, \ldots, n_k) - v_1 = P_{n_1} \bigcup P_{n_2} \bigcup \cdots \bigcup P_{n_k}.$$

This tree has $n_1+n_2+\cdots+n_k+1 = n$ vertices. In particular, we have $S(2, 2, \ldots, 2) \cong \left(\frac{n-1}{2}\right)$-sun, where n is odd. Let $s = |S|$ ($|S|$ is the cardinality of set S) be the non-negative integer such that

$$S = \{i : n_i \text{ is odd}, 1 \leq i \leq k\}.$$

Banerjee et al. [1] proved that for starlike tree $T \cong S(n_1, n_2, \ldots, n_k)$, T has the matching number $\frac{n-s+1}{2}$ $(s \neq 0)$. This result with Lemma 2.10, we get the following:

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

LEMMA 4.1. *Let $T \cong S(n_1, n_2, \ldots, n_k)$ be a starlike tree of order n with vertex cover number c. Then*

$$c = \begin{cases} \frac{n-1}{2} & s = 0, \\ \\ \frac{n+1-s}{2} & s \geq 1. \end{cases}$$

Banerjee et al. [1, Theorem 2.2 (ii)] gave a result on $R_{-1}(T)$ which is not true. Here we give the correct statement:

LEMMA 4.2. *Let $T \cong S(n_1, n_2, \ldots, n_k)$ be a starlike tree of order n and let p be the largest positive integer such that $n_p > 1$. Then*

$$R_{-1}(T) = \frac{n+1}{4} - \frac{k-p}{4}\left(1 - \frac{2}{k}\right).$$

Proof. Since T is a starlike tree and $n_1 \geq n_2 \geq \cdots \geq n_p \geq 2$, we have $(d_1, d_j) = (k, 2)$ for p edges $v_1 v_j \in E(T)$, $(d_1, d_j) = (k, 1)$ for $k - p$ edges $v_1 v_j \in E(T)$, $(d_i, d_j) = (2, 1)$ for p edges $v_i v_j \in E(T)$ and $(d_i, d_j) = (2, 2)$ for the remaining edges $v_i v_j \in E(T)$. Thus

$$\begin{aligned} R_{-1}(T) &= \sum_{v_i v_j \in E(T)} \frac{1}{d_i\, d_j} \\ &= \frac{p}{2\,k} + \frac{k-p}{k} + \frac{p}{2} + \frac{n-k-p-1}{4} \\ &= \frac{n+1}{4} - \frac{k-p}{4}\left(1 - \frac{2}{k}\right). \end{aligned}$$

□

COROLLARY 4.3. *Let $T \cong S(n_1, n_2, \ldots, n_k)$ be a starlike tree of order n. Then*

$$R_{-1}(T) \leq \frac{n+1}{4}$$

with equality holding if and only if $n_k \geq 2$.

Proof. The proof follows directly from Lemma 4.2. □

We now obtain the Randić energy for the special kind of starlike trees.

LEMMA 4.4. *Let $T \cong S(2, 2, \ldots, 2)$ be a starlike tree of order n. Then*

$$RE(T) = 2 + \frac{(n-3)\sqrt{2}}{2}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

Proof. The two normalized Laplacian eigenvalues of T are $1 \pm \frac{\sqrt{2}}{2}$ with the same multiplicity $\frac{n-3}{2}$ [3]. Since T is bipartite with $\sum\limits_{i=1}^{n} \rho_i(T) = n$, we get

$$NLS(T) = \Big\{2,\, \underbrace{1 \pm \frac{\sqrt{2}}{2}, \ldots,\, 1 \pm \frac{\sqrt{2}}{2}}_{\frac{n-3}{2}},\, 1,\, 0\Big\}.$$

Hence we get the required result. ◻

We are now ready to prove Conjecture 1.1 for starlike trees of odd order n.

THEOREM 4.5. *Let $T \cong S(n_1, n_2, \ldots, n_k)$ be a starlike tree of order n (n is odd). Then*

$$RE(T) \leq 2 + \frac{(n-3)\sqrt{2}}{2} \tag{4.1}$$

with equality holding if and only if $T \cong S(2, 2, \ldots, 2)$.

Proof. Let c be the vertex cover number of T. From the matrix $\mathcal{L}^2$, we have

$$\sum_{i=1}^{n} \rho_i^2 = n + 2 \sum_{v_i\, v_j \in E(T)} \frac{1}{d_i d_j} = n + 2\, R_{-1}(T)\,.$$

Thus we have

$$\sum_{i=1}^{n} (\rho_i - 1)^2 = \sum_{i=1}^{n} \rho_i^2 - n = 2\, R_{-1}(T).$$

By Lemmas 2.3 and 2.9, we have

$$\sum_{i=1}^{n} (\rho_i - 1)^2 = 2 \sum_{i=1}^{c} (\rho_i - 1)^2 = 2 + 2 \sum_{i=2}^{c} (\rho_i - 1)^2\,.$$

Therefore

$$R_{-1}(T) = 1 + \sum_{i=2}^{c} (\rho_i - 1)^2. \tag{4.2}$$

Again by Lemmas 2.3 and 2.9, we conclude that $\rho_c > 1$. Moreover, we have

$$RE(T) = \sum_{i=1}^{n} |\rho_i - 1| = 2 \sum_{i=1}^{c} (\rho_i - 1) = 2 + 2 \sum_{i=2}^{c} (\rho_i - 1). \tag{4.3}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

By Cauchy-Schwartz inequality and (4.2), we have

$$\sum_{i=2}^{c}(\rho_i - 1) \leq \sqrt{(c-1)\sum_{i=2}^{c}(\rho_i-1)^2} = \sqrt{(c-1)\,(R_{-1}(T)-1)} \tag{4.4}$$

with equality holding if and only if $\rho_2 = \rho_3 = \cdots = \rho_c$.

Since n is odd, we have $s = 0$ or $s \geq 2$. By Lemma 4.1, we get $c \leq \frac{n-1}{2}$ with equality holding if and only if $s = 0$ or $s = 2$. Using this result with Corollary 4.3, from (4.4), we get

$$\sum_{i=2}^{c}(\rho_i - 1) \leq \frac{(n-3)\sqrt{2}}{4}\,. \tag{4.5}$$

From (4.3) and (4.5), we get the required result in (4.1). The first part of the proof is done.

Suppose that equality holds in (4.1). Then all the inequalities in the above proof must be equalities. In particular, from equality in (4.4), we have $\rho_2 = \rho_3 = \cdots = \rho_c$.

From equality in (4.5) with Corollary 4.3, we have $n_k \geq 2$. Moreover, we have $c = \frac{n-1}{2}$. Again from equality in (4.5) with the above result, we get

$$\rho_2 = \rho_3 = \cdots = \rho_c = 1 + \frac{\sqrt{2}}{2}\,.$$

By Lemmas 2.3 and 2.9, we conclude that the normalized Laplacian spectrum is

$$NLS(T) = \left\{2,\,1,\,0,\,\underbrace{1 \pm \frac{\sqrt{2}}{2}, \ldots, 1 \pm \frac{\sqrt{2}}{2}}_{\frac{n-3}{2}}\right\}.$$

Since $n_k \geq 2$, we have $k \leq \frac{n-1}{2}$. If $k = \frac{n-1}{2}$, then $T \cong S(2, 2, \ldots, 2)$. Otherwise, $3 \leq k < \frac{n-1}{2}$. In this case $n_1 \geq 3$. Without loss of generality, we can assume that the second maximum degree vertex v_2 of degree 2 is not adjacent to the maximum degree vertex v_1 of degree k in T. Then by Lemma 2.11, we have $\rho_{n-1}(T) < 1 - \frac{\sqrt{2}}{2}$, which is a contradiction (The inequality is strict because the diameter of T is strictly greater than the diameter of $T(n, n_1, n_1, n_2, \ldots, n_k)$).

Conversely. one can see easily that the equality holds in (4.1) for $S(2, 2, \ldots, 2)$, by

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 31, pp. 237-254, December 2016

ELA
http:/repository.uwyo.edu/ela

Lemma 4.4. ∎

Acknowledgment. The authors would like to thank anonymous referee for valuable comments which have considerably improved the presentation of this paper.

REFERENCES

[1] A. Banerjee and R. Mehatari. Characteristics polynomial of normalized Laplacian for trees. *Applied Math. Comput.*, 271:838–844, 2015.

[2] J. A. Bondy and U. S. R. Murty. *Graph Theory with Applications.* Macmillan Press, New York, 1976.

[3] R. O. Braga, R. R. Del-Vechio, V. M. Rodrigues, and V. Trevisan. Trees with 4 or 5 distinct normalized Laplacian eigenvalues. *Linear Algebra Appl.*, 471:615–635, 2015.

[4] M. Cavers. *The normalized Laplacian matrix and general Randić index of graphs.* PhD dissertation, University of Regina, 2010.

[5] M. Cavers, S. Fallat, and S. Kirkland. On the normalized Laplacian energy and general Randić index R_{-1} of graphs. *Linear Algebra Appl.*, 433:172–190, 2010.

[6] H. Chen and J. Jost. Minimum vertex covers and the spectrum of the normalized Laplacian on trees. *Linear Algebra Appl.*, 437:1089–1101, 2012.

[7] F. K. Chung. *Spectral graph theory.* American Mathematical Soc., 1997.

[8] K. C. Das, A. D. Güngör, and Ş. B. Bozkurt. On the normalized Laplacian eigenvalues of graphs. *Ars. Comb.*, 118:143–154, 2015.

[9] K. C. Das and S. Sorgun. On Randić energy of graphs. *MATCH Commun. Math. Comput. Chem.*, 72:227–238, 2014.

[10] K. C. Das, S. Sorgun, and I. Gutman. On Randić energy. *MATCH Commun. Math. Comput. Chem.*, 73:81–92, 2015.

[11] K. C. Das, S. Sorgun, and K. Xu. On Randić energy of graphs. *I. Gutman, X. Li (Eds.), Energies of graphs-Theory and Applications*, Univ. Kragujevac, Kragujevac, 2016, pp. 111-122.

[12] K. C. Das and S. Sun. Normalized Laplacian eigenvalues and energy of trees. *Taiwanese Journal of Mathematics*, 20 (3):491–507, 2016.

[13] B. Furtula and I. Gutman. Comparing energy and Randić energy. *Maced. J. Chem. Chem. Engin.*, 32:117–123, 2013.

[14] I. Gutman, B. Furtula, and Ş. B. Bozkurt. On Randić energy. *Linear Algebra Appl.*, 442:50–57, 2014.

[15] I. Gutman, M. Robbiano, and B. S. Martin Upper bound on Randić energy of some graphs. *Linear Algebra Appl.*, 478:241–255, 2015.

[16] J. Li, J.-M. Guo, and W. C. Shiu. Bounds on normalized Laplacian eigenvalues of graphs. *Journal Inequalities Appl.*, 316:1–8, 2014.

[17] H. Lin, M. Zhai, and S. Gong. On graphs with at least three distance eigenvalues less than -1. *Linear Algebra Appl.*, 458:548–558, 2014.

[18] D. Rautenbach. A note on trees of maximum weight and restricted degrees. *Discrete Math.*, 271:335–342, 2003.