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FAMILIES OF GRAPHS HAVING FEW DISTINCT DISTANCE EIGENVALUES WITH ARBITRARY DIAMETER*

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Abstract. The distance matrix of a simple connected graph G is $D(G) = (d_{ij})$, where d_{ij} is the distance between ith and jth vertices of G. The multiset of all eigenvalues of $D(G)$ is known as the distance spectrum of G. Lin et al.(On the distance spectrum of graphs. *Linear Algebra Appl.*, 439:1662-1669, 2013) asked for existence of graphs other than strongly regular graphs and some complete k-partite graphs having exactly three distinct distance eigenvalues. In this paper some classes of graphs with arbitrary diameter and satisfying this property is constructed. For each $k \in \{4, 5, \dots, 11\}$ families of graphs that contain graphs of each diameter grater than $k-1$ is constructed with the property that the distance matrix of each graph in the families has exactly k distinct eigenvalues. While making these constructions we have found the full distance spectrum of square of even cycles, square of hypercubes, corona of a transmission regular graph with K_2, and strong product of an arbitrary graph with K_n.

Key words. Distance matrix, Distance spectrum, Power of graph, Hypercube.

AMS subject classifications. 05C50.

1. Introduction. All graphs considered in this paper are simple graphs, that is, undirected, loop free and having no multiple edges. Consider an n-vertex connected graph $G = (V, E)$, where $V = V(G)$ is the vertex set and E is the edge set of G. The *distance matrix* $D(G)$ of G is an $n \times n$ matrix (d_{ij}), where d_{ij} is the distance (length of a shortest path) between the ith and jth vertices in G. The eigenvalues, eigenvectors, and spectrum of $D(G)$ are said to be the *distance eigenvalues* (D-*eigenvalues*), *distance eigenvectors* (D-*eigenvectors*), and *distance spectrum* (D-*spectrum*) of G respectively. The matrix $D(G)$ is symmetric, so that all of its eigenvalues are real, say $\mu_i, i = 1, 2, \dots, n$, and can be ordered as $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$. If $\mu_{i_1} > \mu_{i_2} > \cdots > \mu_{i_p}$ are the distinct D-eigenvalues and $m_1, m_2, \dots, m_p$ are the algebraic multiplicities of them respectively, then the D-*spectrum* can be represented as

$$\mathrm{spec}_D(G) = \begin{pmatrix} \mu_{i_1} & \mu_{i_2} & \cdots & \mu_{i_p} \\ m_1 & m_2 & \cdots & m_p \end{pmatrix}.$$

The *transmission* $\mathrm{Tr}(v)$ of a vertex v is defined to be the sum of the distances from v to all other vertices in G, i.e., $\mathrm{Tr}(v) = \sum_{u \in V} d(u, v)$. A connected graph G is said to

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be *s-transmission* regular if $\mathrm{Tr}(v) = s$ for every vertex $v \in V$. A connected graph G is called *distance regular* if it is regular of valency k, and if for any two vertices $x, y \in G$ at distance $i = d(x, y)$, there are precisely c_i neighbors of y in $G_{i-1}(x)$ and b_i neighbors of y in $G_{i+1}(x)$, where $G_i(x)$ is the set of all vertices with distance i from x. The *kth power* G^k of a graph G is a graph with same set of vertices $V(G)$ and two vertices are adjacent when their distance in G is at most k. The *corona* of two graphs G_1 and G_2, denoted by $G_1 \circ G_2$, is the graph which is the disjoint union of one copy of G_1 and $|V(G_1)|$ copies of G_2 in which each vertex of the copy of G_1 is adjacent to all vertices of the corresponding copy of G_2. The *Cartesian Product* of G_1 and G_2 is the graph $G_1 \times G_2$ with vertex set $\{(x_1, x_2)|x_1 \in V(G_1),\ x_2 \in V(G_2)\}$ and two vertices (x_1, x_2) and (y_1, y_2) are adjacent if and only if (i) x_1 is adjacent to y_1 in G_1 and $x_2 = y_2$ in G_2 or (ii) x_2 is adjacent to y_2 in G_2 and $x_1 = y_1$ in G_1. The *Strong Product* of G_1 and G_2 is the graph $G_1 \boxtimes G_2$ with vertex set $\{(x_1, x_2)|x_1 \in V(G_1),\ x_2 \in V(G_2)\}$ and two vertices (x_1, x_2) and (y_1, y_2) are adjacent if and only if one of the following hold (i) x_1 is adjacent to y_1 in G_1 and $x_2 = y_2$ in G_2 (ii) x_2 is adjacent to y_2 in G_2 and $x_1 = y_1$ in G_1 (iii) x_1 is adjacent to y_1 in G_1 and x_2 is adjacent to y_2 in G_2. The *Johnson graph* $J(n, m)$ is the graph whose vertex set is the set of all m-subsets of an n-element set and two m-subsets are adjacent if they have $m-1$ elements in common. The *Hamming graph* $H(n, d)$ has vertex set X^n, where X is a finite set of cardinality $d \geq 2$, and two vertices of $H(n, d)$ are adjacent whenever they differ in precisely one coordinate. In particular the n-dimensional hypercube Q_n is $H(n, 2)$. The theorem below gives the diameter of these graphs defined here.

THEOREM 1.1. *Let G, G_1, and G_2 be graphs having diameters d, d_1, and d_2 respectively. Then*

Graph	*Diameter*
G^k	$\lceil \frac{d}{2} \rceil$
$G_1 \circ G_2$	$d_1 + 2$
$G_1 \times G_2$	$d_1 + d_2$
$G_1 \otimes G_2$	$max\{d_1, d_2\}$
$J(n, m)$	$d = min(m, n - m)$
$H(n, d)$	n

Recall that, the Kronecker product of matrices $A = (a_{ij})$ of size $m \times n$ and B of size $p \times q$, denoted by $A \otimes B$, is defined to be the $mp \times nq$ partition matrix $(a_{ij}B)$. It is known [13] that for matrices M, N, P and Q of suitable sizes, $MN \otimes PQ = (M \otimes P)(N \otimes Q)$. Suppose a real symmetric matrix A can be partitioned as

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix},$$

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where each A_{ij} is a submatrix (block) of A. If q_{ij} denotes the average row sum of A_{ij} then the matrix $Q = (q_{ij})$ is called a *quotient matrix* of A. If the row sum of each block A_{ij} is a constant then the partition is called *equitable*. The following is an well known result on equitable partition of matrices.

THEOREM 1.2. [5] *Let Q be a quotient matrix of a square matrix A corresponding to an equitable partition. Then the spectrum of A contains the spectrum of Q.*

The distance eigenvalues of graphs have been studied by researchers for many years. For early work, see Graham and Lovász [10], where they have discussed about the characteristic polynomial of distance matrix of a tree. Ruzieh and Powers [23] have found all the eigenvalues and eigenvectors of the distance matrix of the path P_n on n vertices. In [9] Fowler et al. gave all the D-eigenvalues of the cycle C_n with n vertices. Ramane et al. [22] obtained the D-eigenvalues of the join of two graphs whose diameter is less than or equal to 2. In [16] Indulal and Gutman have found the distance spectrum of graphs obtained by some operations. The D-spectrum of the cartesian product of two transmission regular graphs and that of the lexicographic product of two graphs G and H when H is regular are obtained by Indulal [19]. Stevanović and Indulal [24] described the D-spectrum of the join-based compositions of regular graphs in terms of their adjacency spectrum. Ilić [14] characterized the D-spectrum of integral circulant graphs and calculated the D-spectrum of unitary Cayley graphs. Lin et al. [20] characterized all connected graphs with least D-eigenvalue -2 and all connected graphs of diameter 2 with exactly three D-eigenvalues when largest D-eigenvalue is not an integer. For more results related to D-spectrum see [8, 11, 12, 17, 18, 14].

In this paper we find the full distance spectrum of the square of even cycles, the square of hypercubes, the corona of a transmission regular graph with K_2, and the strong product of an arbitrary graph with K_n. Using these and some of the earlier results we have constructed infinite classes of graphs with any diameter but having fixed number, say k, of distinct D-eigenvalues where $k = 4, 5, \ldots, 11$. Here we have proved square of hypercubes Q_n^2 has exactly three distinct D-eigenvalues. Lin et al. [20] asked “Are there any graphs other than strongly regular graphs and some complete k-partite graphs which have three distinct D-eigenvalues?”. So the graph Q_n^2 is a partial answer to this. The authors of [1] asked “Are there connected graphs other than distance regular graphs with diameter d and having less than $d+1$ distinct D-eigenvalues?” We have also partially answered this question.

Next we state some of the known results which will be used in the sequel.

LEMMA 1.3. [7] *Let G be a graph with adjacency matrix A and* $\mathrm{spec(G)} = \{\lambda_1, \lambda_2, \ldots, \lambda_\mathrm{p}\}$. *Then* $\det\ A = \prod_{i=1}^{p} \lambda_i$. *In addition, for any polynomial $P(x)$, $P(\lambda_i)$ is an eigenvalue of $P(A)$ and hence* $\det\ P(A) = \prod_{i=1}^{p} P(\lambda_i)$.

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LEMMA 1.4. [7] *Let* $C = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$ *be a symmetric* 2×2 *block matrix. Then the spectrum of* C *is the union of the spectra of* $A+B$ *and* $A-B$.

THEOREM 1.5. [21] *Let* S *be a complex square matrix, which is partitioned into blocks, each of size* $n \times n$*:*

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix},$$

where S_{33} *is an invertible matrix. Then the determinant of* S *is given by*

$$\begin{aligned}\det(S) =& \det\left([S_{11} - S_{13}S_{33}^{-1}S_{31}] - [S_{12} - S_{13}S_{33}^{-1}S_{32}][S_{22} - S_{23}S_{33}^{-1}S_{32}]^{-1}[S_{21} - S_{23}S_{33}^{-1}S_{31}]\right) \\ &\times \det([S_{22} - S_{23}S_{33}^{-1}S_{32}]) \times \det(S_{33}).\end{aligned}$$

THEOREM 1.6. [16] *Let* D *be the distance matrix of a connected transmission regular graph* G *of order* p*. Then* D *is irreducible and there exists a polynomial* $P(x)$ *such that* $P(D) = J$*. In this case*

$$P(x) = p \times \frac{(x-\lambda_2)(x-\lambda_3)\cdots(x-\lambda_g)}{(k-\lambda_2)(k-\lambda_3)\cdots(k-\lambda_g)},$$

where k *is the unique sum of each row which is also the greatest simple eigenvalue of* D*, whereas* $\lambda_2, \lambda_3, \ldots, \lambda_g$ *are the other distinct eigenvalues of* D.

THEOREM 1.7. [9] *If* $n = 2p$*, then the characteristic polynomial of* $D(C_n)$*, the distance matrix of an* n*-vertex cycle* C_n*, is given by*

$$p(t) = t^{p-1}\left(t - \frac{n^2}{4}\right)\prod_{j=1}^{p}\left(t + \csc^2\left(\frac{\pi(2j-1)}{n}\right)\right).$$

THEOREM 1.8. [1] *The distance spectrum of the Johnson graph* $J(n,m)$ *is given by*

$$\operatorname{spec}_D(J(n,m)) = \begin{pmatrix} s & 0 & -\frac{s}{n-1} \\ 1 & \binom{n}{m} - n & n-1 \end{pmatrix},$$

where $s = \sum_{j=0}^{m} jk_j$ *and* $k_j = \binom{m}{j}\binom{n-m}{j}$ *for* $j = 0, 1, \ldots, m$.

THEOREM 1.9. [19] *Let* $H(n,d)$ *be the Hamming graph of diameter* n*. Then the distance spectrum of* $H(n,d)$ *is given by*

$$\mathrm{spec_D(D(H(n,d)))} = \begin{pmatrix} nd^{n-1}(d-1) & 0 & -d^{n-1} \\ 1 & d^n - n(d-1) - 1 & n(d-1) \end{pmatrix}.$$

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THEOREM 1.10. [19] *Let G and H be two transmission regular graphs on p and n vertices with transmission regularity k and t respectively. Let* $\mathrm{spec_D(G)} = \{\mathrm{k}, \mu_2, \mu_3, \ldots, \mu_\mathrm{p}\}$ *and* $\mathrm{spec_D(H)} = \{\mathrm{t}, \eta_2, \eta_3, \ldots, \eta_\mathrm{p}\}$. *Then the distance spectrum of cartesian product of G and H is given by*

$$\mathrm{spec_D(G \times H)} = \{\mathrm{nk + pt, n\mu_i, p\eta_j}, 0\},$$

where $i = 2, 3, \ldots, p$, $j = 2, 3, \ldots, n$ and 0 is with multiplicity $(p-1)(n-1)$.

THEOREM 1.11. [16] *Let G be a k-transmission regular graph of order n and having D-spectrum $\{k, \mu_2, \mu_3, \ldots, \mu_n\}$. Then the D-spectrum of $G \circ K_1$ consists of the numbers*
$n + k - 1 \pm \sqrt{(n+k)^2 + (n-1)^2}$, $\mu_i - 1 \pm \sqrt{\mu_i^2 + 1}$ *for $i = 2, 3, \ldots, n$.*

2. Full D-spectrum of some graphs.

Here we first prove a lemma which will be used in the proof of some of the results of this section.

LEMMA 2.1. *Let $M = (m_{ij})$ be a symmetric matrix of order n with sum of the entries of each row is a constant s and let the spectrum of M be $\{\lambda_1 = \lambda_2 = \ldots = \lambda_k = s, \lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n\}$ for some integer $k \geq 1$. Let J be the square matrix of order n with all entries equal to 1. Then for any real number r, the spectrum of the matrix $M + rJ$ is $\{s + nr, s, \ldots, s, \lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n\}$.*

Proof. Since M is a symmetric square matrix of order n with sum of the entries of each row is s, $M + rJ$ is also a square matrix of order n with sum of the entries of each row is $s + nr$. Therefore $s + nr$ is an eigenvalue of $M + rJ$.

As the symmetric matrices M and rJ commute, they are simultaneously diagonalizable. Then the eigenvalues of $M + rJ$ are the sum of the sum of the eigenvalues of M and rJ for a certain ordering. But the matrix rJ has rank 1, so it has eigenvalues rn and 0 with multiplicity $n - 1$. Hence spectrum of the matrix $M + rJ$ is $\{s + nr, s, \ldots, s, \lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n\}$ ☐

The theorem below gives the full D-spectrum of square of even cycles.

THEOREM 2.2. *Let $\{\frac{n^2}{2}, 0, \lambda_3, \lambda_4, \ldots, \lambda_n\}$ or $\{\frac{n^2}{2}, -1, \lambda_3, \lambda_4, \ldots, \lambda_n\}$ be the D-spectrums of C_n depending on whether $\frac{n}{2}$ is even or odd. Then the D-spectrum of C_n^2 is given by $\{\frac{n^2}{8} + \frac{n}{4}, -\frac{n}{4}, \frac{\lambda_3}{2}, \frac{\lambda_4}{2}, \ldots, \frac{\lambda_n}{2}\}$ if $\frac{n}{2}$ is even and $\{\frac{n^2}{8} + \frac{n}{4}, \frac{-1}{2} - \frac{n}{4}, \frac{\lambda_3}{2}, \frac{\lambda_4}{2}, \ldots, \frac{\lambda_n}{2}\}$ if $\frac{n}{2}$ is odd.*

Proof. Let $\{u_1, u_2, \ldots, u_n\}$ be the vertex set of C_n, such that u_i is adjacent to u_{i+1} (where subscripts are taken mod n). Let us partition the vertex set of C_n as $V_1 \cup V_2$ where V_1 is a set of all even index vertices and V_2 is a set of all odd index vertices. Then every pair of vertices within V_1 or within V_2 are of even distance from each other. Again any vertex of V_1 and any vertex of V_2 are of odd distance from

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each other. Now if we index the rows and columns of the distance matrix by taking the vertices of V_1 followed by the vertices of V_2 and by considering a suitable ordering we get the distance matrix of C_n in the form

$$D(C_n) = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where each entry of the block A is even and sum of the entries of any row in A is equal to the sum of the distances from any vertex in V_1 to all other vertices in V_1, and hence A has constant row sum $\frac{n^2-4}{8}$ or $\frac{n^2}{8}$ depending on whether $\frac{n}{2}$ is odd or even. Again each entry of the block B is odd and sum of the entries of any row in B is equal to the sum of the distances from any vertex in V_1 to all vertices in V_2 and hence B has constant row sum $\frac{n^2+4}{8}$ or $\frac{n^2}{8}$ depending on whether $\frac{n}{2}$ is odd or even.

By Lemma 1.4. the eigenvalues of $D(C_n)$ are the union of the eigenvalues of $A+B$ and $A-B$. Now the matrix $A+B$ has constant row sum $\frac{n^2}{4}$ for all n and the matrix $A-B$ has constant row sum -1 or 0 depending on whether $\frac{n}{2}$ is odd or even. We note that for vertices u and v in C_n if $d(u,v) = a$ in C_n then $d(u,v) = \lceil \frac{a}{2} \rceil$ in C_n^2.

Therefore the distance matrix of C_n^2 is given by

$$D(C_n^2) = \frac{1}{2}\begin{pmatrix} A & B+J \\ B+J & A \end{pmatrix},$$

where J is a square matrix of order $\frac{n}{2}$ with all entry 1. Again using Lemma 1.4. the eigenvalues of $D(C_n^2)$ are the union of the eigenvalues of $\frac{A+B+J}{2}$ and $\frac{A-B-J}{2}$. Using Lemma 2.1. we get that eigenvalues of $A+B+J$ are same as the eigenvalues of $A+B$ except the eigenvalue $\frac{n^2}{4}$ which is replaced by the eigenvalue $\frac{n^2}{4}+\frac{n}{2}$ for all n. Similarly the eigenvalues of $A-B-J$ are same as the eigenvalues of $A-B$ except the eigenvalues -1 and 0 which are replaced by the eigenvalues $-1-\frac{n}{2}$ and $-\frac{n}{2}$ according as $\frac{n}{2}$ is odd or even respectively. Hence we get the desired result. $\square$

In [20] Lin at al. have asked "Are there any graphs other than strongly regular graphs and some complete k-partite graphs which have three distinct D-eigenvalues?" Our next theorem gives a partial answer to this question.

Theorem 2.3. *Let Q_n be the hypercube graph of dimension n. Then the distance spectrum of Q_n^2 is given by*

$$\mathrm{spec_D}(\mathrm{D}(\mathrm{Q_n^2})) = \begin{pmatrix} \frac{1}{2}\sum\limits_{r=1}^{n} r\,\binom{n}{r} + 2^{n-2} & 0 & -2^{n-2} \\ 1 & 2^n-(n+2) & n+1 \end{pmatrix}.$$

Proof.

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We recall that an n-dimensional hypercube Q_n is a graph with vertex set $V(Q_n) = \{(a_1, a_2, \ldots, a_n) : a_i = 0 \text{ } or \text{ } 1\}$ and two vertices of Q_n are adjacent if and only if they differ at exactly one coordinate. For $u, v \in V(Q_n)$, it is clear that $d(u, v) = r$ if and only if coordinates of u and v differ in exactly r places.

Consider the vertex $u = (0, 0, ..., 0)$. Let V_1 be the set of vertices of Q_n which are of even (may be zero) distance from u and let V_2 be the set of vertices of Q_n which are of odd distance from u. All vertices within V_1 and those within V_2 are of even distance from each other. Again any vertex of V_1 and any vertex of V_2 are of odd distance from each other. Clearly $V_1 \cup V_2$ partitions of $V(Q_n)$. Also in Q_n, from any vertex v, the number of vertices in Q_n with distance i is $\binom{n}{i}$.

We have in the structure of Q_n it has two copies of Q_{n-1}. Consider a suitable ordering of the vertices of V_1 and V_2 and by taking the vertices of V_1 followed by the vertices of V_2 the distance matrix of Q_n is of the form

$$D(Q_n) = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where A and B have same properties as in the Theorem 2.2. Now we find the constant row sum of the matrices A and B according as n is odd or even. The sum of the distances from any vertex in V_1 to all other vertices in V_1 is given by

$$k_1 = \begin{cases} 2\binom{n}{2} + 4\binom{n}{4} + \cdots + (n-1)\binom{n}{n-1}, & \text{if } n \text{ is odd,} \\ \\ 2\binom{n}{2} + 4\binom{n}{4} + \cdots + n\binom{n}{n}, & \text{if } n \text{ is even.} \end{cases}$$

Again the sum of the distances from any vertex in V_1 to all vertices in V_2 is given by

$$k_2 = \begin{cases} 1\binom{n}{1} + 3\binom{n}{3} + \cdots + n\binom{n}{n}, & \text{if } n \text{ is odd,} \\ \\ 1\binom{n}{1} + 3\binom{n}{3} + \cdots + (n-1)\binom{n}{n-1}, & \text{if } n \text{ is even.} \end{cases}$$

By using Lemma 1.4. the eigenvalues of $D(Q_n)$ are the union of the eigenvalues of $A + B$ and $A - B$. The matrix $A + B$ has constant row sum $k_1 + k_2$ and the matrix $A - B$ has constant row sum $k_1 - k_2$.

Here $k_1 + k_2 = \sum_{r=1}^{n} r \binom{n}{r}$ and $k_1 - k_2 = 0$ for each n.

Then the distance matrix of Q_n^2 is given by

$$D(Q_n^2) = \frac{1}{2}\begin{pmatrix} A & B+J \\ B+J & A \end{pmatrix},$$

where J is a square matrix of order 2^{n+1} with all entries equal to 1. The matrix $\frac{A+B+J}{2}$ has constant row sum $\frac{1}{2}\sum_{r=1}^{n} r \binom{n}{r} + 2^{n-2}$ and the matrix $\frac{A-B-J}{2}$ has constant

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row sum -2^{n-2}. Using Lemma 1.6, Theorem 1.9, and Lemma 2.1. we get the desired result. □

Next we determine the distance spectrum of the corona of a k-transmission regular graph G with K_2, in terms of the distance spectrum of G.

THEOREM 2.4. *Let G be a k-transmission regular graph of order n with D-spectrum $\{k, \lambda_2, \lambda_3, \ldots, \lambda_n\}$. Then the D-spectrum of $G \circ K_2$ consists of*
$\frac{1}{2}\left(4n+3k-3 \pm \sqrt{(4n+3k-3)^2+4(2n^2+3k)}\right)$,
$\frac{1}{2}\left(3\lambda_i-3 \pm \sqrt{(3\lambda_i-3)^2+12\lambda_i}\right)$ *for $i = 2, 3, \ldots, n$, and -1 with multiplicity n.*

Proof. Let D be the distance matrix of the graph G. Then by definition of corona the distance matrix of $G \circ K_2$ can be written as

$$D(G \circ K_2) = \begin{pmatrix} D & D+J & D+J \\ D+J & D+2(J-I) & D+2J-I \\ D+J & D+2J-I & D+2(J-I) \end{pmatrix},$$

where J is a square matrix of order n with all entries 1 and I is the identity matrix of order n. Now characteristic equation of $D(G \circ K_2)$ is given by

$$\det \begin{pmatrix} xI-D & -(D+J) & -(D+J) \\ -(D+J) & xI-(D+2(J-I)) & -(D+2J-I) \\ -(D+J) & -(D+2J-I) & xI-(D+2(J-I)) \end{pmatrix} = 0$$

Since any two blocks of the above determinant commute, by applying Theorem 1.5. we get the characteristic equation is

$$\begin{aligned} &\det((xI-D)[\{xI-(D+2(J-I))\}^2-\{D+2J-I\}^2] \\ &\quad +(D+J)[-(D+J)\{xI-(D+2(J-I))\}-(D+J)(D+2J-I)] \\ &\quad -(D+J)[(D+J)(D+2J-I)+(D+J)(xI-(D+2(J-I)))]) = 0 \end{aligned}$$

Since $J^2 = nJ$ and G is a k-transmission regular graph, each row sum and column sum of D is the constant k. So $DJ = JD = kJ$ and the above equation becomes

$$\det[(1+x)I\{3(1+x)D+(2n+4x)J-(x^2+3x)I\}] = 0 \tag{2.1}$$

Then in equation (2.1) we put the value of J in terms of D which is given in Theorem 1.6. Finally the D-spectrum of $G \circ K_2$ is obtained by applying Lemma 1.3. □

REMARK 2.5. Note that if the graph G has r distinct D-eigenvalues then $G \circ K_2$ has $2r+1$ distinct D-eigenvalues provided that G is a transmission regular graph and the functional values of the expressions in the Theorem 2.4 of two distinct D-eigenvalues of G are not equal.

THEOREM 2.6. *Let G be a graph of order m with D-eigenvalues $\lambda_i, i = 1, 2, \ldots, m$ and let K_n be the complete graph of order n. Then the D-spectrum of $G \boxtimes K_n, n \geq 2$ is given by $n\lambda_i + (n-1)$, $i = 1, 2, \ldots, m$ and -1 with multiplicity $m(n-1)$.*

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Proof. Let $V(G) = \{u_1, u_2, \ldots, u_m\}$ and $V(K_n) = \{v_1, v_2, \ldots, v_n\}$, then $V(G \boxtimes K_n) = \{(u_i, v_j) : i = 1, 2, \ldots, m \text{ and } j = 1, 2, \ldots, n\}$. Now we consider $V(G \boxtimes K_n) = \bigcup\limits_{i=1}^{m} S_i$ where $S_i = \{(u_i, v_j),\ j = 1, 2, \ldots, n\}$. Then $S = \{S_i : i = 1, 2, \ldots, m\}$ is a partition of $V(G \boxtimes K_n)$. Using this partition we consider the blocks of the distance matrix $D(G \boxtimes K_n)$. For $i, j \in \{1, 2, \ldots, m\}$, let S_{ij} be the $(i,j)^{th}$ block of $D(G \boxtimes K_n)$. Now if we maintain the ordering of vertices as in the above partition then the distance matrix of $G \boxtimes K_n$ can be written as

$$D(G \boxtimes K_n) = D(G) \otimes J + I_m \otimes D(K_n),$$

where J is a square matrix of order n with all entries 1, I_m is a unit matrix of order m and $D(G),\ D(K_n)$ are the distance matrices of G and K_n respectively. Now each vertex set $S_i,\ i = 1, 2, \ldots, m$ induces a copy of K_n. So each block $S_{ii} = D(K_n)$ for $i = 1, 2, ..., m$. Consider the vertex $(u_i, v_j) \in S_i$ and the vertex $(u_k, v_l) \in S_k$ for $i \neq k$. Then

$$\begin{aligned} d_{G\boxtimes K_n}((u_i, v_j), (u_k, v_l)) &= max\{d_G(u_i, u_k), d_{K_n}(v_j, v_l)\} \\ &= d_G(u_i, u_k) \quad \text{for all } j, l \in \{1, 2, \ldots, n\}. \end{aligned}$$

Thus we get all the entries of the block S_{ik} is $d_G(u_i, u_k)$ for $i \neq k$ and $i, k \in \{1, 2, \ldots, m\}$. So row sum of each block of the matrix $D(G \boxtimes K_n)$ is a constant. Then corresponding to the equitable partition $\bigcup\limits_{i=1}^{m} S_i$ the quotient matrix of $D(G \boxtimes K_n)$ is given by

$$Q = nD(G) + (n-1)I_m.$$

By Lemma 1.2. the eigenvalues of Q are eigenvalues of $D(G \boxtimes K_n)$. So we get $n\lambda_i + (n-1),\ i = 1, 2, \ldots, m$, are eigenvalues of $D(G \boxtimes K_n)$. For the remaining eigenvalues let X_i be the eigenvector of $D(G)$ corresponding to the eigenvalue λ_i for $i = 1, 2, ..., m$, and since $D(K_n)$ has -1 as an eigenvalue with multiplicity $(n-1)$, let Y_j be the eigenvector of $D(K_n)$ corresponding to the eigenvalue -1 for $j = 2, 3, \ldots, n$. Then

$$D(G)X_i = \lambda_i X_i \text{ for } i = 1, 2, \ldots, m \text{ and } D(K_n)Y_j = -Y_j \text{ for } j = 2, 3, \ldots, n.$$

We have that the $m(n-1)$ vectors $(X_i \otimes Y_j)$ for $i = 1, 2, \ldots, m$ and $j = 2, 3, \ldots, n$ are linearly independent and also

$$\begin{aligned} D(G \boxtimes K_n)(X_i \otimes Y_j) &= (D(G) \otimes J + I_m \otimes D(K_n))(X_i \otimes Y_j) \\ &= (D(G) \otimes J)(X_i \otimes Y_j) + (I_m \otimes D(K_n))(X_i \otimes Y_j) \\ &= (D(G)X_i) \otimes (JY_j) + (I_m X_i) \otimes (D(K_n)Y_j) \\ &= (D(G)X_i) \otimes 0 + X_i \otimes -Y_j \quad \text{[as sum of all entries of } Y_j \text{ is zero]} \\ &= -(X_i \otimes Y_j) \end{aligned}$$

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Thus we get that $X_i \otimes Y_j$ is an eigenvector of $D(G \boxtimes K_n)$ corresponding to the eigenvalue -1 for $i = 1, 2, \ldots, m,\ j = 2, 3, \ldots, n$. Hence -1 is an eigenvalue of $D(G \boxtimes K_n)$ with multiplicity $m(n-1)$. $\square$

Remark 2.7. Note that the graph $G \boxtimes K_n, n \geq 2$ has k or $k+1$ distinct D-eigenvalues depending on whether or not -1 is a D-eigenvalue of G.

3. Graphs with few distinct D-eigenvalues. There are several graphs with diameter d and having at least $d+1$ distinct D-eigenvalues. For example, integral circulant graphs [14], complete k-partite graphs $K_{n_1,\ldots,n_k}$ with $n_i \neq n_j,\ i, k \in \{1, 2, \ldots, k\}$ [20], odd cycles [9], square of even cycles (Section 2, Theorem 2.2.) etc. Moreover there are several graphs for which the number of distinct D-eigenvalues depends on the diameter d. For examples, the graphs odd cycles, even cycles, and square of even cycles have number of distinct D-eigenvalues $d+1$, $\lceil \frac{d}{2} \rceil + 2$, and $d+2$ respectively.

Here we construct families of graphs for which number of distinct D-eigenvalues is independent of the diameter. Moreover these families of graphs have arbitrary diameter and few distinct D-eigenvalues. These graphs are listed in the table below. In this table Tr-regular stands for the transmission regular.

Graphs	Tr-regular	D-eigenvalues	Diameter
$J(n,m) \boxtimes K_p$	Yes	4	$d \geq 2$
$H(n,d) \boxtimes K_p$	Yes	4	$d \geq 2$
$Q_n^2 \boxtimes K_p$	Yes	4	$d \geq 2$
$J(n,m) \times H(n,d)$	Yes	4	$d \geq 4$
$(J(n,m) \times H(n,d)) \boxtimes K_p$	Yes	5	$d \geq 4$
$J(n,m) \circ K_1$	No	6	$d \geq 4$
$H(n,d) \circ K_1$	No	6	$d \geq 4$
$Q_n^2 \circ K_1$	No	6	$d \geq 4$
$J(n,m) \circ K_2$	No	7	$d \geq 4$
$H(n,d) \circ K_2$	No	7	$d \geq 4$
$Q_n^2 \circ K_2$	No	7	$d \geq 4$
$(J(n,m) \boxtimes K_p) \times (J(n,m) + H(n,d))$	Yes	7	$d \geq 6$
$(J(n,m) \boxtimes K_p) \times (H(n,d) \boxtimes K_n)$	Yes	8	$d \geq 4$
$(J(n,m) \times H(n,d)) \circ K_2$	No	9	$d \geq 6$
$((J(n,m) \times H(n,d)) \boxtimes K_p) \circ K_1$	No	10	$d \geq 6$
$((J(n,m) \times H(n,d)) \boxtimes K_p) \circ K_2$	No	11	$d \geq 6$

Analysis of the above table. (i) By Theorem 1.8, Theorem 1.9, Theorem 2.2, and Remark 2.7. the graphs $J(n,m) \boxtimes K_p$, $H(n,d) \boxtimes K_p$, and $Q_n^2 \boxtimes K_p$ have exactly four distinct D-eigenvalues. Again by Theorem 1.8, Theorem 1.9, and Theorem 1.10. the graph $J(n,m) \times H(n,d)$ has exactly four distinct D-eigenvalues as both the graphs $J(n,m)$ and

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$H(n,d)$ has zero as a D-eigenvalue.

(ii) Note that to apply Theorem 1.11. and Theorem 2.4, the graph G has to be transmission regular. From Theorem 1.8, Theorem 1.9, Theorem 2.2, and Theorem 1.11 the graphs $J(n,m) \circ K_1$, $H(n,d) \circ K_1$, $Q_n^2 \circ K_1$ have exactly 6 distinct D-eigenvalues. Also by Remark 2.5. the graphs $J(n,m) \circ K_2$, $H(n,d) \circ K_2$, and $Q_n^2 \circ K_2$ have exactly 7 distinct D-eigenvalues.

(iii) Again note that by Theorem 2.6. and Remark 2.7, the graphs $G \boxtimes K_p$ and $(G \boxtimes K_p) \boxtimes K_p$ have the same number of distinct D-eigenvalues. In this way one can calculate the number of distinct D-eigenvalues of the remaining graphs. Also from Theorem 1.1. one can get the restriction on the diameters of the above graphs.

(iv) In [1] the authors have asked "Are there connected graphs other than distance regular graphs with diameter d and having less than $d+1$ distinct D-eigenvalues ?". For a partial answer to this question we refer to the graphs in the above table. For any distance regular graph G, $G \circ K_1$ and $G \circ K_2$ are not distance regular graph as they loose their regularity. In fact one verifies that none of the graphs given in the table are distance regular graphs.

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