# RANK AND DIMENSION FUNCTIONS* 

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#### Abstract

In this paper, we invoke the theory of generalized inverses and the minus partial order on the study of regular matrices over a commutative ring to define rank-function for regular matrices and dimension-function for finitely generated projective modules which are direct summands of a free module. Some properties held by the rank of a matrix and the dimension of a vector space over a field are generalized. Also, a generalization of rank-nullity theorem has been established when the matrix given is regular.


Key words. generalized inverse, minus partial order, matrices over commutative ring, projective module, rankfunction, dimension-function.

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## Dedicated to Prof. Ravindra B. Bapat, on the occasion of his 60th birthday.

1. Introduction and Preliminaries. Given a matrix $A$ and a vector space $V$ over a field $\mathbb{F}$, the notion of rank of matrix $A(\operatorname{rank}(A))$ and dimension of vector space $V(\operatorname{dim}(V))$ are well defined and folklore in the literature. Similarly, the rank condition for solvability of a linear system and addition theorem for the dimension are well-known. Whenever we consider the matrices over a commutative ring, not necessarily a field, the known definition of rank and dimension do not hold good. The determinantal rank of a matrix (the size of largest submatrix with nonzero determinant), denoted by $\rho(\cdot)$, is widely used as an alternative notion for the rank of a matrix. In [3], [11] and [17], some attempts have been made to provide some necessary conditions and some sufficient conditions for the solvability of linear system $A x=b$ over a commutative ring with identity. For the purpose, McCoy rank of a matrix has been introduced.

Throughout this paper, $\mathscr{A}$ denotes a commutative ring with identity, $E$ is the set of all nonzero idempotents from $\mathscr{A}$, and the matrices are with entries from $\mathscr{A}$, unless indicated otherwise. Let $A$ be an $m \times n$ matrix over $\mathscr{A}$.

DEFINITION 1.1. McCoy rank of an $m \times n$ matrix $A$, denoted by $\rho_{M}(A)$, is the largest integer ' $t$ ' such that $\operatorname{Ann}\left(D_{t}(A)\right)=0$, where $D_{t}(A)$ is the ideal in $\mathscr{A}$ generated by $t \times t$ minors

[^0]of $A$, and $\operatorname{Ann}\left(D_{t}(A)\right)=\left\{a \in \mathscr{A} \mid a d=0 \forall d \in D_{t}(A)\right\}$.
Note that $\rho_{M}(A) \leq \rho(A)$, and equality holds if $\mathscr{A}$ is a field.
In an attempt to find a necessary and sufficient condition for the solvability of linear system $A x=b$ over a commutative ring with identity, the author in [7] confined himself to the case when the matrix $A$ is regular and proved that the linear system is consistent if and only if $\rho(e A)=\rho(e(A: b))$ for every nonzero idempotent in the ring. Here $(A: b)$ is the matrix obtained by augmenting an additional column $b$ to $A$. This study leads to introducing an integer valued rank-function of a matrix, defined on the set of nonzero idempotents of the ring. In this article, we further study the rank-function and then define the dimensionfunction which inherits some of the known properties of rank and dimension.

In the following, we recall some basics related to the theory of generalized inverses and minus partial order on the class of regular matrices.

Given an $m \times n$ matrix $A$ over $\mathscr{A}$, the notation $\mathscr{C}(A)$ represents the submodule generated by the columns of $A$ in $\mathscr{A}^{m}$. Matrix $A$ is said to be a regular matrix if there exists a matrix $G$ such that $A G A=A$, in which case $G$ is said to be a generalized inverse ( $g$-inverse) of $A$. An arbitrary generalized inverse of $A$ is denoted by $A^{-}$. For $r \leq \min (m, n), C_{r}(A)$ denotes the $r$-th compound matrix of $A$. The notation $\otimes$ represents tensor product and $\wedge^{r}$ represents $r$-th exterior power. Given $a \in \mathscr{A},[a]$ represents the ideal generated by $a$ in $\mathscr{A}$. Readers are referred to [11] for the fundamentals of rings and modules, referred to [14] for the basics of determinants, $[1,8,16]$ for the generalized inverse, and $[4,9,12,13,15]$ for the minus partial order.
2. Rank-Function and Dimension-Function. First, we recall some developments in the theory of generalized inverses of a matrix over a commutative ring $\mathscr{A}$, relevant to the present article, before studying some properties of rank-function and define dimensionfunction.
2.1. Regular Matrices and Minus Partial Order. In characterizing the regular matrices, some properties of compound matrices were found useful (for example, see [2]). It has been noted in [6] that if $C_{r}(A)$ - the $r$-th compound matrix ( $r$-th exterior power) of an $m \times n$ matrix $A$, has determinantal rank one and there exists an idempotent $e$ such that $[e]$ is the same as the ideal generated by entries of $A$ as well as the ideal generated by the entries of $C_{r}(A)$, then the matrix $A$ has a generalized inverse. A regular matrix with such a special property, called a Rao-regular matrix, was introduced in [6]. The definition of the same has been reworded, in terms of ideals, in the following.

DEFINITION 2.1 (Rao-regular Matrix). A matrix A over $\mathscr{A}$ is a Rao-regular matrix if there exists an idempotent $e$ in $\mathscr{A}$ such that $D_{1}(A)=[e]=D_{t}(A)$, where $t$ is $\rho(A)$ and $[e]$ represents the ideal generated by e in the ring. The idempotent e is called the Rao-idempotent
of Rao-regular matrix A.
In fact, a Rao-regular matrix is regular and $\rho_{M}(A)=\rho(A)$ over the subring $e \otimes \mathscr{A}$. Conveniently, we consider zero matrix is a Rao-regular matrix with its Rao-idempotent equals to 0 .

Lemma 2.2. Let $A$ be an $m \times n$ matrix with determinantal rank $t$. Then the following assertions hold.
(i) If A is a Rao-regular matrix over $\mathscr{A}$, then it is a regular matrix.
(ii) Matrix $A$ with determinantal rank one is Rao-regular if and only if it is regular.
(iii) For any idempotent $f$ in $\mathscr{A}$, if $A$ is Rao-regular then so is $B=f A$.

Proof. The idempotent $e$ in $\mathscr{A}$ satisfying $D_{1}(A)=[e]=D_{t}(A)$, is the Rao-idempotent of the matrix $A$ as defined in [6]. So, there exists a linear combination of elements of $D_{t}(A)$ such that $\sum_{I J} c_{I}^{J}\left|A_{J}^{I}\right|=e$, where $\left|A_{J}^{I}\right| \in D_{t}(A)$ is a $t \times t$ minor obtained by the submatrix with rows determined by $t$-element subset $I$ of $\{1,2, \ldots, m\}$ and columns determined by $t$-element subset $J$ of $\{1,2, \ldots, n\}$. Further, we have $\left(\sum_{I J} c_{I}^{J}\left|A_{J}^{I}\right|\right) A=A$ and the matrix $G=\left(g_{j i}\right)$ defined by $g_{j i}=\sum_{I J} c_{I}^{J} \frac{\partial}{\partial a_{i j}}\left|A_{J}^{I}\right|$ is a generalized inverse of $A$, as proved in the Theorem 1 of [6].

The part (ii) follows easily from the definition of Rao-regular matrix and the construction of $G$ given by $g_{j i}=c_{i j}$.

For $B=f A$, we have $D_{1}(A)=[e]=D_{t}(A)$ implies $D_{1}(B)=[f e]=D_{t}(B)$. So, (iii) follows.

With the clear understanding that an idempotent $e$ in $\mathscr{A}$ such that $D_{1}(A)=[e]=D_{t}(A)$ is the Rao-idempotent of the matrix $A$, as defined in [6], we have the following decomposition theorem. For the proof, we refer to [6].

THEOREM 2.3 (Decomposition Theorem). An $m \times n$ matrix A with determinantal rank $r$ over $\mathscr{A}$ is regular if and only if there exist idempotents $e_{0}, e_{1}, e_{2}, \ldots, e_{r}$ in $\mathscr{A}$ such that
(i) $e_{0}+e_{1}+e_{2}+\cdots+e_{r}=1$ and $e_{i} e_{j}=0$ for $0 \leq i, j \leq r$ and $i \neq j$,
(ii) each $A_{i}=e_{i} A, 0 \leq i \leq r$ is either a zero matrix or else a Rao-regular matrix with determinantal rank $i$ satisfying $D_{1}\left(A_{i}\right)=\left[e_{i}\right]=D_{i}\left(A_{i}\right)$.

Further, the above decomposition is unique.
REMARK 2.4. From the properties of $e_{i}$ given in the Theorem 2.3, it is clear that $e_{0} A=0$ and for any idempotent $e \in \mathscr{A}$, we get $\rho(e A)=\max _{i} \rho\left(e e_{i} A\right)$. Also, eA is Rao-regular implies that $e=e e_{i}$ for some $e_{i}$ given in (i) of the Theorem 2.3. For any $g$-inverse $A^{-}$of $A$, the decomposition for $A A^{-}$(similarly, for $A^{-} A$ ) is obtained by replacing $A_{i}$ with $A_{i} A^{-}$(similarly, with $A^{-} A_{i}$ ) in the decomposition given for $A$ in the theorem.

REMARK 2.5. Let $E$ be the set of all nonzero idempotents from $\mathscr{A}$. An atom in $E$ is an
element e in that set such that ef $=f$ for some $f \in E$ implies $f=e$. In other words, the atoms are the smallest elements in $E$ with reference to the partial order defined by $g<h$ if $g h=g$. If $E$ has only finitely many idempotents (like in the case of a ring satisfying descending chain condition), we find finitely many atoms $e_{1}, e_{2}, \ldots, e_{k}$ in $E$ which are orthogonal to each other (i.e., $e_{i} e_{j}=0$ for $i \neq j$ ), such that $e_{1}+e_{2}+\cdots+e_{k}=1$. In such a case, we can rephrase the Theorem 2.3 as
$A$ is regular if and only if $A_{i}=e_{i} A$ is Rao-regular, for all atoms $e_{i}$ from $E$.
If $\mathscr{A}$ has no nontrivial idempotents (i.e., the only idempotents in $\mathscr{A}$ are 0 and 1), then a nonzero matrix $A$ is regular (Rao-regular) if and only if $D_{1}(A)=\mathscr{A}=D_{t}(A)$, where $t$ is $\rho(A)$ - the determinantal rank of $A$.

THEOREM 2.6 (Theorem 2.2, [7]). Let A be an $m \times n$ regular matrix with determinantal rank $r$ and let $B, C$ and $D$ be matrices of size $m \times p, q \times n$, and $q \times p$, respectively. For $T=\left(\begin{array}{cc}A & B \\ C & D\end{array}\right)$, the following statements are equivalent.
(i) $\rho(e A)=\rho(e T)$ for every idempotent $e \in \mathscr{A}$.
(ii) Matrix equations $A X=B, Y A=C$ are solvable and $D=C A^{-} B$ is invariant under the choices of $A^{-}$.

In [7], the above theorem lead the author to obtain a rank condition for the solvability of linear system $A x=b$ and further to define rank-function of a matrix. For $E=\left\{e \in \mathscr{A} \mid e^{2}=\right.$ $e \neq 0\}$, the rank-function of a matrix $A$ is defined in the following.

DEFINITION 2.7 (Rank-Function [7]). The rank-function of an $m \times n$ matrix $A$, denoted by $\Re_{A}$, is an integer valued function $\Re_{A}: E \rightarrow \mathbb{Z}$ such that $\Re_{A}(e)=\rho(e A)$ for all $e \in E$.

If the ring $\mathscr{A}$ has no nontrivial idempotents, then $E=\{1\}$ and $\mathfrak{R}_{A}$ coincides with wellknown determinantal rank $\rho(A)$. The following property of rank-function was noted in [7].
$\left(R_{1}\right)$ Given an $m \times n$ regular matrix $A$ over $\mathscr{A}$, a linear system $A x=b$ is solvable if and only if $\Re_{A}=\Re_{T}$, where $T=(A: b)$ is a augmented matrix $A$ with column vector $b$.

So, we have
$\left(R_{2}\right)$ Given an $m \times n$ regular matrix $A$ and $b \in \mathscr{A}^{m}$, we have $b \notin \mathscr{C}(A)$ if and only if $\mathfrak{R}_{A}(e)<\mathfrak{R}_{T}(e)$ for some $e \in E$, where $T=(A: b)$.

Note that for every projective submodules $P, Q$ such that $P \oplus Q=\mathscr{A}^{m}$, there exists an $m \times m$ projection (idempotent) matrix $A$ such that $\mathscr{C}(A)=P$ and $\mathscr{C}(I-A)=Q$. Similarly, for a regular matrix $A$ of size $m \times n$ with a generalized inverse $G$ we have that $\mathscr{C}(A)=\mathscr{C}(A G)$ is a projective submodule and $\mathscr{C}(A G) \oplus \mathscr{C}(I-A G)=\mathscr{A}^{m}$. So, the properties $\left(R_{1}\right)$ and $\left(R_{2}\right)$ of rank-function invites us to define a dimension-function for a projective submodule of $\mathscr{A}^{m}$.

From the equivalence of (i) and (iv) given in the Theorem 2.9, we are convinced that the theory of 'minus partial order' has an important role in studying the properties of dimensionfunction to be defined. In the following, we have a definition of minus partial order on the class of regular matrices, adopted from [5].

DEFINITION 2.8 (Minus Partial Order). The minus partial order denoted by $\leq^{-}$is $a$ relation on the class of regular matrices defined by $B \leq^{-} A$ if there exists a $B^{-}$such that

$$
\begin{align*}
& B^{-} A=B^{-} B  \tag{2.1}\\
& A B^{-}=B B^{-} . \tag{2.2}
\end{align*}
$$

The following theorem is well-known for the matrices over a field and readers are referred to [12] for the proof. In general, when the matrices are regular over a commutative ring with identity, the techniques used in [12] fails to prove the theorem, but it has been proved with different techniques in [10].

Theorem 2.9. Let $A, B, C \in \mathscr{A}^{m \times n}$ such that $A$ is regular and $A=B+C$. Then the following statements are equivalent.
(i) $B$ is regular and $B \leq^{-} A$.
(ii) $B$ is regular and $\left\{A^{-}\right\} \subseteq\left\{B^{-}\right\}$.
(iii) $B, C$ are regular and both $B, C \leq^{-} A$.
(iv) $\mathscr{C}(A)=\mathscr{C}(B) \oplus \mathscr{C}(C)$ ('Range Summability' condition).

Corollary 2.10. Let $A$ be an $m \times n$ regular matrix over $\mathscr{A}$. Then we have the following.
(i) $B \leq^{-} A$ if and only if $e B \leq^{-}$eA for all idempotents $e$ of $\mathscr{A}$.
(ii) If $A_{i}$ is any component of decomposition as given in the Theorem 2.3, then $A_{i} \leq^{-} A$.

Proof. Since $(e A) A^{-}(e A)=e A$ for every idempotent $e \in \mathscr{A}$ and every generalized inverse $A^{-}$of $A$, the proof follows immediately from Theorem 2.9.

The following theorem is an interesting case of Theorem 2.6 with $A=B+C$ and $D=0$. Also, it helps in verifying the rank additive property for the Rao-regular matrices $A, B, C$ satisfying $A=B+C$ and $B, C \leq^{-} A$.

THEOREM 2.11. Let $A$ be an $m \times n$ regular matrix and $A=B+C$. For $T=\left(\begin{array}{ll}A & B \\ C & 0\end{array}\right)$, the following statements are equivalent.
(i) $B$ and $C$ are regular and $B \leq^{-} A, C \leq^{-} A$.
(ii) $\mathfrak{R}_{A}=\Re_{T}$.
(iii) $A X=B$ and $Y A=C$ are solvable and $C A^{-} B=0$ for every choice of $A^{-}$.
(iv) $B$ and $C$ are regular and $\left\{A^{-}\right\} \subset\left\{B^{-}\right\} \cap\left\{C^{-}\right\}$.

Proof. (i) $\Rightarrow$ (ii). Let $B^{-}$and $C^{-}$be any g-inverses of $B$ and $C$, respectively, such that

$$
\begin{equation*}
B^{-} A=B^{-} B, \quad A B^{-}=B B^{-} \tag{2.3}
\end{equation*}
$$

$$
\begin{equation*}
C^{-} A=C^{-} C, \quad A C^{-}=C C^{-} . \tag{2.4}
\end{equation*}
$$

Note that $P=\left(\begin{array}{cc}I & 0 \\ -C C^{-} & I\end{array}\right)$ and $Q=\left(\begin{array}{cc}I & -B^{-} B \\ 0 & I\end{array}\right)$ are invertible matrices. Now using (2.3) and (2.4), we get $P T Q=\left(\begin{array}{cc}A & 0 \\ 0 & 0\end{array}\right)$. Therefore $\mathfrak{R}_{A}=\mathfrak{R}_{T}$.
(ii) $\Rightarrow$ (iii) follows immediately from (i) $\Rightarrow$ (ii) of Theorem 2.6.
(iii) $\Rightarrow$ (iv). Since $A X=B$ and $Y A=C$ are solvable, we have

$$
\begin{equation*}
A A^{-} B=B \tag{2.5}
\end{equation*}
$$

$$
\begin{equation*}
C A^{-} A=C \tag{2.6}
\end{equation*}
$$

for every choice of $A^{-}$. Now substitute $C A^{-} B=0$ in (2.5) to get $B A^{-} B=B$ for all $A^{-}$and similarly, substitute the same in (2.6) to obtain $C A^{-} C=C$ for all $A^{-}$. Thus, (iii) $\Rightarrow$ (iv).
(iv) $\Rightarrow$ (i) follows from (ii) $\Rightarrow$ (i) of Theorem 2.9. $\square$

For a regular matrix $A$ such that $A=B+C, T=\left(\begin{array}{cc}A & B \\ C & 0\end{array}\right)$ and $S=\left(\begin{array}{cc}0 & B \\ C & 0\end{array}\right)$, we have
$\left(R_{3}\right) B, C \leq^{-} A$ if and only if $\mathscr{R}_{A}=\mathscr{R}_{T}=\mathscr{R}_{S}$.
REMARK 2.12. Note that $\mathscr{R}_{A}=\mathscr{R}_{S}$ need not imply the rank-additive property, as known in the literature i.e., $\mathscr{R}_{A}=\mathscr{R}_{B}+\mathscr{R}_{C}$. For example, choose $A=\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right), B=\left(\begin{array}{ll}4 & 0 \\ 0 & 4\end{array}\right)$ and $C=\left(\begin{array}{ll}3 & 0 \\ 0 & 3\end{array}\right)$ over $\mathbb{Z}_{6}$, the commutative ring of integers modulo 6. Clearly, the property $\left(R_{3}\right)$ holds but $\mathscr{R}_{A}(1)=2 \neq 4=\mathscr{R}_{B}(1)+\mathscr{R}_{C}(1)$.

The rank-additive property has been addressed in [10], when the matrices $A, B$ are Raoregular. However, we will establish rank additive property for Rao-regular matrices, independently, as a consequence of Theorem 2.11 and proceed to establish rank-additive property
in the class of regular matrices in terms of rank-function.
Lemma 2.13. Let $B$ and $C$ be Rao-regular matrices of size $m \times n$ and with same Raoidempotent. For $T=\left(\begin{array}{ll}B & 0 \\ 0 & C\end{array}\right)$, we have $\rho(T)=\rho(B)+\rho(C)$, in which case $T$ is also a Rao-regular matrix.

Proof. Clearly, $\rho(T) \leq \rho(B)+\rho(C)$. We will prove the equality, whenever the matrices $B$ and $C$ are Rao-regular with same Rao-idempotents. Let $\rho(B)=p$ and $\rho(C)=q$. Since $B$ and $C$ are Rao-regular, $D_{1}(B)=D_{p}(B)=[e]=D_{1}(C)=D_{q}(C)$ for some idempotent $e \in$ $\mathscr{A}$. Clearly, $e b_{\alpha}=b_{\alpha}$ and $e c_{\beta}=c_{\beta}$, for all $b_{\alpha} \in D_{p}(B)$ and $c_{\beta} \in D_{q}(C)$. Now for $r=$ $p+q$, consider any $r \times r$ submatrix $T_{1}$ of $T$ in the form $\left(\begin{array}{cc}B_{1} & 0 \\ 0 & C_{1}\end{array}\right)$, where $B_{1}$ and $C_{1}$ are appropriate submatrices of $B$ and $C$, respectively. Since $\rho(B)=p$ and $\rho(C)=q$, the determinant $\left|T_{1}\right|$ is zero unless $B_{1}$ is of size $p \times p$ and $C_{1}$ is of size $q \times q$. So, let $B_{1}$ and $C_{1}$ be of size $p \times p$ and $q \times q$, respectively, and $\left|T_{1}\right|=b_{\alpha} c_{\beta}$ for $b_{\alpha}=\left|B_{1}\right| \in D_{p}(B)$ and $c_{\beta}=\left|C_{1}\right| \in D_{1}(C)$.

Suppose $\left|T_{1}\right|=0$ for every choice of $B_{1}$ and $C_{1}$. Since $D_{p}(B)=[e]$, there exists a linear combination of elements of $D_{p}(B)$ which equals to $e$, viz. $\sum a_{\alpha} b_{\alpha}=e$. Now $\left|T_{1}\right|=0$ for every choices of $B_{1}$ and $C_{1}$ implies that $b_{\alpha} c_{\beta}=0$ for all $b_{\alpha} \in D_{p}(B)$ and $c_{\beta} \in D_{1}(C)$. So, $\left(\sum a_{\alpha} b_{\alpha}\right) c_{\beta}=0$. In other words, $e c_{\beta}=0$ for all $c_{\beta} \in D_{q}(C)$, a contradiction. Thus, $\rho(T)=$ $\rho(B)+\rho(C)$.

Note that the product of any two linear combinations of the elements, one from $D_{p}(B)$ and the other from $D_{q}(C)$, produces a linear combination of $D_{r}(T)$. Now choose the linear combinations that equal to $e$, we get a linear combination of $D_{r}(T)$ which equals to $e$. Therefore $T$ is Rao-regular. $\quad$ I

Now, we have the following
$\left(R_{4}\right)$ If $A, B, C$ are Rao-regular matrices with same Rao-idempotents, and $A=B+C$, then $B, C \leq^{-} A$ if and only if $\rho(A)=\rho(B)+\rho(C)$. In fact, we have

$$
\mathfrak{R}_{A}=\mathfrak{R}_{B}+\mathfrak{R}_{C} .
$$

Now with reference to given matrices $M_{1}, M_{2}, \ldots M_{k}$, we define a typical characteristic function on the set of idempotents $E=\left\{e \in \mathscr{A}: e^{2}=e\right\}$ which is useful in establishing rank-additive property in the case of regular matrices.

DEFInItion 2.14. Given an $m \times n$ matrix $M$, we define

$$
\chi_{M}: E \rightarrow\{0,1\}
$$

such that $\chi_{M}(e)=1$, if eM is either a zero matrix or else it is a Rao-regular matrix with Rao-idempotent $e$, and $\chi_{M}(e)=0$, otherwise. Further, $\chi_{\left(M_{1}, \ldots, M_{k}\right)}(e)=\chi_{M_{1}}(e) \cdots \chi_{M_{k}}(e)$.

Let $A, B, C$ be regular matrices and $A=B+C$. From (iii) of Lemma 2.2 and Theorem 2.3, it is clear that there exists $f_{1}, f_{2}, \ldots f_{k} \in E$ such that (i) $f_{i} f_{j}=0$ for $i \neq j$, (ii) $\sum_{i} f_{i}=1$, and (iii) $f_{i} A, f_{i} B$ and $f_{i} C$ are Rao-regular. Note that $B \leq^{-} A$ if and only if $f_{i} B \leq^{-} f_{i} A \leq^{-} A$ for all $i$. So from $\left(R_{4}\right)$, we have
$\left(R_{5}\right)$ Let $A, B, C$ be regular matrices such that $A=B+C$. Then $B \leq^{-} A$ if and only if $\chi_{(A, B, C)}(e) \Re_{A}(e)=\chi_{(A, B, C)}(e) \Re_{B}(e)+\chi_{(A, B, C)}(e) \Re_{C}(e)$ for all $e \in E$. In other words,

$$
B \leq^{-} A \Leftrightarrow \chi_{(A, B, C)} \Re_{A}=\chi_{(A, B, C)} \Re_{B}+\chi_{(A, B, C)} \Re_{C} .
$$

LEMMA 2.15. Let $A$ and $B$ be any regular matrices of same size. Then we have the following.
( $R_{6}$ ) $\mathfrak{R}_{A}=\Re_{B}$ if and only if $\chi_{(A, B)} \Re_{A}=\chi_{(A, B)} \Re_{B}$.
Proof. Since $A, B$ are regular, from (iii) of Lemma 2.2 and Theorem 2.3, we find idempotents $f_{1}, f_{2}, \ldots f_{k} \in E$ such that (i) $f_{i} f_{j}=0$ for $i \neq j$, (ii) $\sum_{i} f_{i}=1$, and (iii) $f_{i} A$ and $f_{i} B$ are Rao-regular. Note that $|D|=\left|f_{1} D\right|+\left|f_{2} D\right|+\cdots+\left|f_{k} D\right|$ for every square matrix $D$ and therefore $\rho(C)=\max _{i} \rho\left(f_{i} C\right)$ for any $C$. So, we have $\rho(e C)=\max _{i} \rho\left(f_{i} e C\right)=$ $\max _{i}\left(\chi_{(A, B)}\left(f_{i}\right) \rho\left(f_{i} e C\right)\right)=\max _{i}\left(\chi_{(A, B)}\left(f_{i} e\right) \rho\left(f_{i} e C\right)\right)$ for $C=A, B$ and 'only if' part follows. As 'if part' is trivial, the lemma is proved.

The rank-function defined above inherits the following property from well-known determinantal rank.
$\left(R_{7}\right) \mathfrak{R}_{A B} \leq\left\{\mathfrak{R}_{A}, \mathfrak{R}_{B}\right\}$. Further, $\mathfrak{R}_{A A^{-}}=\mathfrak{R}_{A^{-} A}=\mathfrak{R}_{A}$, where $A^{-}$is any $g$-inverse of $A$.
2.2. Dimension-Function and its properties. Let $P$ be a finitely generated projective module and be a direct summand of $\mathscr{A}^{m}$ for some $m$. An idempotent $e \in \mathscr{A}$ is said to be the supporting idempotent of $P$, if the ideal generated by $1-e$ is the annihilator of $P$ i.e.,

$$
[1-e]=\{x \in \mathscr{A}: x p=0 \quad \forall p \in P\} .
$$

It is well-known that a matrix $A \in \mathscr{A}^{m \times n}$ is regular if and only if $\mathscr{C}(A)$ is a direct summand of $\mathscr{A}^{m}$. In this case, $\mathscr{C}(A)=\mathscr{C}\left(A A^{-}\right)$is a projective module. So, all the projective modules we consider in this paper are finitely generated and direct summand of $\mathscr{A}^{m}$, unless indicated otherwise.

LEMMA 2.16. Given a finitely generated projective module $P$ in $\mathscr{A}^{m}$, supporting idempotent of $P$ exists and it is unique.

Proof. Given a projective module $P$, consider a projection matrix $E$ (an idempotent matrix) such that $\mathscr{C}(E)=P$. It is clear that the sum of Rao-idempotents of the nonzero summands in the unique decomposition, as given in Theorem 2.3, is the supporting idempotent of $P$. $\square$

Now, we define dimension-function for a submodule of $\mathscr{A}^{m}$ as similar to the case of rank-function for the matrices. The notation $\rho$, which denotes the determinantal rank of a matrix, is used in the following definition for the reason that the determinant function and the exterior power are related.

DEFINITION 2.17 (Dimension-Function). For a finitely generated submodule $M$ of $\mathscr{A}^{m}$, let $\rho(M)$ be the largest positive integer $r$ such that $\wedge^{r}(M) \neq(0)$. The dimension-function $\mathfrak{D}_{M}$ for a module $M$ is defined by

$$
\begin{equation*}
\mathfrak{D}_{M}: E \rightarrow \mathbb{Z} \tag{2.7}
\end{equation*}
$$

such that $\mathfrak{D}_{M}(e)=\rho(e M)$ for all $e \in E$, where $e P=<e>\otimes P$.
From the Definition 2.17, it is clear that $\mathfrak{D}_{M}=\mathscr{R}_{A}$ for any choice of matrix $A$ such that $\mathscr{C}(A)=M$. Now from $\left(R_{1}\right)$ and $\left(R_{2}\right)$, we have the following theorem.

THEOREM 2.18. Let $P, Q$ be such that $P \oplus Q=\mathscr{A}^{m}$ and $b \in \mathscr{A}^{m}$. Then the following assertions hold.
( $\left.D_{1}\right) \mathfrak{D}_{P}(e) \leq \mathfrak{D}_{M}(e)$ for all $e \in E$, where $M$ is a module generated by $P$ and $b$.
$\left(D_{2}\right) b \in P$ if and only if $\mathfrak{D}_{P}=\mathfrak{D}_{M}$.
Analogous to a Rao-regular matrix, we introduce 'strong projective module'.
Definition 2.19 (Strong Projective Module). A nontrivial projective module $P$ is said to be a strong projective module if the supporting idempotent of $\wedge^{r} P$ is the same as that of $P$, where $r$ is the largest integer for which $\wedge^{r} P \neq(0)$. We consider $P=(0)$ as a trivial case of strong projective module with 0 as its supporting idempotent.

Note that $\wedge^{r} \mathscr{C}(A)=\mathscr{C}\left(C_{r}(A)\right)$ (where $C_{r}(A)$ is the $r$-th compound matrix or $r$-th exterior power of $A$ ) and therefore, $\mathscr{C}(A)$ is strong projective module and direct summand of $\mathscr{A}^{m}$ if and only if $A$ is Rao-regular. In this case, the supporting idempotent of $P$ is the same as the Rao-idempotent of $A$. Now, we have the following theorem.

ThEOREM 2.20. Let $P \oplus Q=\mathscr{A}^{m}$ and $\rho(P)=k$. Then there exist strong projective modules $P_{1}, P_{2}, \ldots, P_{k}$ with mutually orthogonal supporting idempotents such that $\rho\left(P_{i}\right)=i$, unless $P_{i}=(0)$, and $P=P_{1} \oplus P_{2} \oplus \cdots \oplus P_{k}$.

Proof. Let $E$ be an $m \times m$ idempotent matrix such that $\mathscr{C}(E)=P$ and $\mathscr{C}(E) \oplus \mathscr{C}(I-E)=$ $\mathscr{A}^{m}$. Referring to Theorem 2.3, we have Rao-regular matrices $E_{1}, E_{2}, \ldots, E_{k}$ with mutually orthogonal Rao-idempotents $e_{1}, e_{2}, \ldots, e_{r}$ such that $E=E_{1}+E_{2}+\cdots+E_{r}$. From (ii) of Corollary 2.10, we have $E_{i} \leq^{-} E$ and from Theorem 2.9, it is easily proved that $P=\mathscr{C}\left(E_{1}\right) \oplus$ $\mathscr{C}\left(E_{2}\right) \oplus \cdots \oplus \mathscr{C}\left(E_{k}\right)$. The cases in which $E_{i}=0$ and $\mathscr{C}\left(E_{i}\right)=(0)$, for any $i$, are conveniently ignored. Since $E_{i}$ is Rao-regular, we have $\mathscr{C}\left(E_{i}\right)(1 \leq i \leq k)$ are strong projective module with desired properties.

Analogous to the rank-additive property in the case of Rao-regular matrices, we have the following theorem for strong projective modules.

THEOREM 2.21. Let $P, Q, R$ be strong projective modules with the same supporting idempotents and be direct summands of $\mathscr{A}^{m}$. Then
$\left(D_{3}\right)$ If $P=Q \oplus R$, then $\mathfrak{D}_{P}=\mathfrak{D}_{Q}+\mathfrak{D}_{R}$
( $D_{4}$ ) Particularly, if $P=\mathscr{A}^{m}$ then $\mathfrak{D}_{Q}+\mathfrak{D}_{R}=m$, a constant function on $E$.
Proof. For projective modules $P, Q, R$ satisfying $P=Q \oplus R$ and any $x \in \mathscr{A}^{m}$, consider the unique decomposition $x=q+r+z$, where $q \in Q, r \in R, z \in P^{\prime}$ and $P \oplus P^{\prime}=\mathscr{A}^{m}$. Now, identify the idempotent matrices $E, F, G$ satisfying $E(x)=q+r, F(x)=q$ and $H(x)=r$. Clearly, $\mathscr{C}(E)=P, \mathscr{C}(F)=Q, \mathscr{C}(G)=R, E=F+G$ and $F G=G F=0$. So, $F, G \leq^{-} E$. Since, $P, Q, R$ are strong projective modules with the same supporting idempotents, we have that $E, F, G$ are Rao-regular matrices with same Rao-idempotents. Now, $\left(D_{3}\right)$ follows from $\left(R_{4}\right)$.
$\left(D_{4}\right)$ is immediate from $\left(D_{3}\right)$.
For finitely generated projective modules $P, P_{1}, P_{2}, \ldots, P_{k}$, being direct summands of $\mathscr{A}^{m}$, define

$$
\chi_{P}: E \rightarrow\{0,1\}
$$

such that $\chi_{P}(e)=1$ if $e P$ is either a zero module or a strong projective module with its supporting idempotent equals to $e$, and $\chi_{P}(e)=0$, otherwise. Further,

$$
\chi_{\left(P_{1}, \ldots P_{k}\right)}(e)=\chi_{P_{1}}(e) \chi_{P_{2}}(e) \cdots \chi_{P_{k}}(e) \text { for all } e \in E
$$

From Theorem 2.20, it can be seen that every projective module is a direct sum of strong projective modules with mutually orthogonal support. Now, from the associations between regular matrices, Rao-regular matrices, projective modules and strong projective modules, we obtain the following result.

THEOREM 2.22. Let $P, Q, R, S$ be projective modules and be direct summands of $\mathscr{A}^{m}$.
( $D_{5}$ ) If $P=Q \oplus R$, then $\chi_{(P, Q, R)}\left(\mathfrak{D}_{Q}+\mathfrak{D}_{R}\right)=\chi_{(P, Q, R)} \mathfrak{D}_{P}$.
( $D_{6}$ ) If $P=Q \oplus R=Q \oplus S$, then $\mathfrak{D}_{R}=\mathfrak{D}_{S}$
( $D_{7}$ ) If $P, Q, P+Q$ and $P \cap Q$ are direct summands of $\mathscr{A}^{m}$, then

$$
\chi_{(P, Q, P+Q, P \cap Q)} \mathfrak{D}_{(P+Q)}=\chi_{(P, Q, P+Q, P \cap Q)}\left(\mathfrak{D}_{(P)}+\mathfrak{D}_{(Q)}-\mathfrak{D}_{(P \cap Q)}\right)
$$

Proof. There exist idempotent matrices $E, F, G$ such that $E=F+G, F, G \leq^{-} E$ and $\mathscr{C}(E)=P, \mathscr{C}(F)=Q, \mathscr{C}(G)=R$, as discussed in the proof of Theorem 2.21. So, $\left(D_{5}\right)$ follows from $\left(R_{5}\right)$.

If $P=Q \oplus R=Q \oplus S$, from $\left(D_{5}\right)$ we have $\chi_{(P, Q, R, S)} \mathfrak{D}_{R}=\chi_{(P, Q, R, S)} \mathfrak{D}_{S}$. Now, $\left(D_{6}\right)$ is proved in the lines similar to that in the proof of $\left(R_{6}\right)$ given in the Lemma 2.16.

Note that the modules $U$ and $V$ are direct summands of $\mathscr{A}^{m}$ and $U \subseteq V$ implies that $U$ is a direct summand of $V$. This can be verified easily from (i) $\Rightarrow$ (iv) of Theorem 2.9, by considering the projections $E$ on $U, F$ on $V$ and then $E F \leq^{-} F$. Now, $\left(D_{7}\right)$ follows from the fact that $P, Q$ are direct summands $P+Q$ and $P \cap Q$ is direct summand of each of $P, Q, P+Q$.

Now, we prove the rank-nullity theorem when the matrix given over a commutative ring is regular.

THEOREM 2.23 (Rank-Nullity Theorem). Let $A$ be an $m \times n$ regular matrix over $\mathscr{A}$ and $\mathscr{K}$ be the null-space (kernel) of $A$. Then

$$
\left(D_{8}\right) \chi_{(A)} \chi_{\mathscr{K}}\left(\mathfrak{R}_{A}+\mathfrak{D}_{\mathscr{K}}\right)=n
$$

Proof. If $A^{-}$is any $g$-inverse of $A$, the kernel $\mathscr{K}$ of $A$ is given by $\mathscr{C}\left(I-A^{-} A\right)$. We know that $A^{-} A,\left(I-A^{-} A\right) \leq^{-} I$ and therefore from $\left(R_{5}\right)$ we get

$$
\chi_{\left(A^{-} A, I-A^{-} A\right)} \Re_{A^{-} A}+\chi_{\left(A^{-} A, I-A^{-} A\right)} \Re_{\left(I-A^{-} A\right)}=n .
$$

$\left(D_{7}\right)$ follows from the fact that $\chi_{A^{-} A}=\chi_{A}, \chi_{\left(I-A^{-} A\right)}=\chi_{\mathscr{K}}, \Re_{A^{-} A}=\Re_{A}$, and $\Re_{\left(I-A^{-} A\right)}=$ $\mathfrak{D}_{\mathscr{K}}$. $\square$
2.3. Examples. Now, we shall conclude this paper with examples demonstrating various results presented in the paper. Consider a $2 \times 2$ matrix $A=\left(\begin{array}{ll}1 & 2 \\ 0 & 3\end{array}\right)$ over $\mathscr{A}=\mathbb{Z}_{6}$. Clearly, $\rho(A)=2$ as $|A|=3$. The ideal generated by $1 \times 1$ minors denoted by $D_{1}(A)$ is $\mathscr{A}$, but $D_{2}(A)$ is the ideal generated by 3 in $\mathscr{A}$. Since $\operatorname{Ann}\left(D_{2}(A)\right)=\operatorname{Ann}([3])=[4] \neq(0)$, the McCoy rank of the matrix $\rho_{M}(A)=1$.

The matrix $A$ is regular, as it is an idempotent matrix.
The Rao-decomposition of the matrix, as given in Theorem 2.3 is

$$
\left(\begin{array}{ll}
1 & 2 \\
0 & 3
\end{array}\right)=\left(\begin{array}{ll}
4 & 2 \\
0 & 0
\end{array}\right)+\left(\begin{array}{ll}
3 & 0 \\
0 & 3
\end{array}\right)
$$

where $e_{0}=0, e_{1}=4, e_{2}=3$.
The supporting idempotent of projective module $P=\mathscr{C}(A)=\operatorname{Span}\left\{\binom{1}{0},\binom{0}{3}\right\}$ is 1 . The decomposition $P=P_{1} \oplus P_{2}$, as desired in the Theorem 2.20 , is given by the strongly projective modules $P_{1}=\operatorname{Span}\left\{\binom{4}{0}\right\}$ and $P_{2}=\operatorname{Span}\left\{\binom{3}{0},\binom{0}{3}\right\}$ with supporting idempotents 4 and 3 , respectively.

Note that $E=\{1,3,4\}$. For the regular matrix $A$, we have $\chi_{A}(1)=0, \chi_{A}(3)=1=\chi_{A}(4)$ and $\chi_{A}=\chi_{P}$. The rank-function of $A$, denoted by $\Re_{A}$, is defined by $\Re_{A}(1)=\Re_{A}(3)=2$ and $\mathfrak{R}_{A}(4)=1$. Further, $\mathfrak{D}_{P}=\mathfrak{R}_{A}$.

Note that $B=I-A=\left(\begin{array}{ll}0 & 4 \\ 0 & 4\end{array}\right)$ is a Rao-regular matrix with determinantal rank 1. Clearly, $I=A+B$ and $A, B \leq^{-} I$. As $B$ satisfying $\chi_{B}(1)=0, \chi_{B}(3)=\chi_{B}(4)=1$ and $\Re_{B}(1)=$ $\Re_{B}(4)=1, \Re_{B}(3)=0$, the rank-nullity theorem is easily verified.

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