Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 29, pp. 120-143, October 2015

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THE P_0^+-MATRIX COMPLETION PROBLEM*

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Abstract. A real $n \times n$ matrix B is a P_0^+-matrix if for each $k \in \{1, 2, \ldots, n\}$ every $k \times k$ principal minor of B is nonnegative, and at least one $k \times k$ principal minor is positive. A digraph D is said to have P_0^+-completion if every partial P_0^+-matrix specifying D can be completed to a P_0^+-matrix. In this paper, some necessary and sufficient conditions for a digraph to have P_0^+-completion are discussed and those digraphs of order at most four that have P_0^+-completion are singled out.

Key words. Partial matrix, Matrix completion, P_0^+-matrix, P_0^+-completion, Digraph.

AMS subject classifications. 15A48.

1. Introduction. A real $n \times n$ matrix B is a

(i) *P-matrix* (*P_0-matrix*) if every principal minor of B is positive (nonnegative).
(ii) P_0^+-matrix, if for each $k \in \{1, 2, \ldots, n\}$, every $k \times k$ principal minor of B is nonnegative, and at least one $k \times k$ principal minor is positive.
(iii) *Q-matrix* if for every $k \in \{1, 2, \ldots, n\}$, the sum $S_k(B)$ of all the $k \times k$ principal minors of B is positive.

Clearly, a P-matrix is a P_0^+-matrix, and a P_0^+-matrix is both a P_0-matrix and a Q-matrix.

A *partial matrix* M is a rectangular array of numbers in which some entries are specified while others are free to be chosen. A *completion* of M is a matrix obtained by assigning numbers to the unspecified entries in M.

For a class Π of matrices (e.g., P-, P_0- or Q-matrices) a *partial* Π-*matrix* is one whose specified entries satisfy the required properties of a Π-matrix. For example, a *partial P-matrix* (*P_0-matrix*) has all fully specified minors positive (nonnegative), and for a *partial Q-matrix* M, $S_k(M) > 0$ for each k for which all $k \times k$ principal submatrices are fully specified. A *partial* P_0^+-*matrix* M is a partial matrix in which all fully specified principal minors are nonnegative and $S_k(M) > 0$ for every $k = 1, 2, \ldots, n$, whenever all $k \times k$ principal submatrices are fully specified. A Π-*completion* of a partial Π-matrix is a Π-matrix obtained by some choices of the unspecified entries.

*Received by the editors on May 1, 2015. Accepted for publication on September 12, 2015. Handling Editor: Steve Kirkland.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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Graphs and digraphs have played an important role in the study of matrix completion problems. In many cases, the positions of the specified entries determine the existence of completions of partial matrices of a given class.

A *pattern* for $n \times n$ partial matrices is a subset of $N \times N$, where $N = \{1, 2, \ldots, n\}$. A partial matrix *specifies a pattern* if its specified entries lie exactly in those positions listed in the pattern. Patterns are usually specified by digraphs. An $n \times n$ partial matrix M is said to *specify* a digraph D on vertices $\{v_1, \ldots, v_n\}$ if (v_i, v_j) is an arc in D if and only if the entry m_{ij} of M is specified.

For a class Π of matrices, the (*combinatorial*) Π-*matrix completion problem* attempts to study the digraphs D having the property that any partial Π-matrix specifying D has a Π-completion. Such a digraph D is then said to have Π-*completion*. For an exposition in matrix completion problems, see the survey articles [7] and [8]. The completion problems for the classes of P- and P_0-matrices have been studied by a number of researchers (see [2, 3, 5, 8, 10], for example). DeAlba *et al.* have discussed the Q-matrix completion problem in 2009 in their paper [4].

1.1. Digraphs. In this paper, we will use commonly used graph theoretic terms which can be found in [1], [6] or any other standard book. However, in our discussion, a *directed graph* or *digraph* D is a pair (V, A), where V is a finite nonempty set of objects, called *vertices*, and A a set of ordered pairs of vertices, called *arcs* or *directed edges*. The vertex set and the arc set of D are denoted by $V(D)$ and $A(D)$, respectively.

Note that the above definition allows an arc $x = (u, u)$ in the arc set of a digraph D, which is called a *loop* at the vertex u. Several works on matrix completion problems used *marked digraphs*, i.e., digraphs with some vertices marked instead of considering loops at those vertices (see [2, 8]). The current definition is in use in some recent works on matrix completion problems, including [4].

Sometimes, we simply write $v \in D$ (resp. $(u, v) \in D$) to mean $v \in V(D)$ (resp. $(u, v) \in A(D)$). The *order* of D, denoted by $|D|$, is the number of vertices of D. If $u \neq v$ and $x = (u, v)$ is an arc in D, we say that x is *incident* with u and v; u is *adjacent to* v; and v is *adjacent from* u. The *outdegree* (resp. *indegree*) of a vertex v in D is the number of vertices of D adjacent from (resp. to) v.

It is customary to represent a digraph by a diagram with nodes representing the vertices and directed line segments (or arcs) representing the arcs of the digraph. A digraph H is a *subdigraph* of a digraph D if $V(H) \subseteq V(D)$ and $A(H) \subseteq A(D)$. Further, H is an *induced subdigraph* (*induced by* $V(H)$) if $A(H) = \big(V(H) \times V(H)\big) \cap A(D)$, and is a *spanning subdigraph* if $V(H) = V(D)$. The *complement* of D is the digraph $\overline{D}$, where $V(\overline{D}) = V(D)$ and $(v, w) \in A(\overline{D})$ if and only if $(v, w) \notin A(D)$. For

Electronic Journal of Linear Algebra ISSN 1081-3810
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$w \in V(D)$ the subdigraph of D induced by $V(D) \setminus \{w\}$ is denoted by $D - w$.

Two digraphs $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ are *isomorphic* if there is a bijection $\phi : V_1 \to V_2$ such that $A_2 = \{(\phi(u), \phi(v)) : (u, v) \in A_1\}$. An *unlabelled* digraph is an equivalence class of isomorphic digraphs. Any particular member of an unlabelled digraph is referred as the digraph obtained by a *labelling* of the unlabelled digraph.

Let $\widehat{D}$ and $\widehat{H}$ be unlabelled digraphs. We say $\widehat{H}$ is an (unlabelled) *subdigraph* of $\widehat{D}$, if some member of $\widehat{H}$ is a subdigraph of a member of $\widehat{D}$, i.e., if the digraph obtained by a labelling of $\widehat{H}$ is a subdigraph of the digraph obtained by some labelling of $\widehat{D}$.

The digraph D is *symmetric* if $(v, w) \in A(D)$ implies $(w, v) \in A(D)$. For a (simple) graph G with vertex set V and edge set E, we define the *digraph associated to* G to be the symmetric digraph with vertex set V and the arc set $A = \{(u, v) :$ either $u = v$ or u is adjacent to v in $G\}$. Note that the digraph associated to a graph includes all loops. We call the digraph associated to the complete graph K_n on n vertices the *complete symmetric digraph* (or simply the *complete digraph*) of order n, and denote it by K_n^*. A digraph D is said to be *asymmetric* if whenever $(v, w) \in A(D)$ and $v \neq w$ we have $(w, v) \notin A(D)$. Note that an asymmetric digraph may have loops.

A (*directed*) *cycle* C of *length* k (or a k-*cycle*) in a digraph D is a subdigraph with (distinct) vertices $v_1, v_2, \ldots, v_k$ and with arcs $(v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k), (v_k, v_1)$. Note that a 1-cycle is a loop, and a 2-cycle consists of a pair of opposite arcs.

Let π be a permutation of V. A *permutation digraph* is a digraph of the form $D_\pi = (V, A_\pi)$ where $A_\pi = \{\big(v, \pi(v)\big) : v \in V\}$. Clearly, each component of a permutation digraph is a cycle. A *permutation subdigraph of order* k of a digraph D is a permutation digraph that is a subdigraph of D of order k. A digraph D of order n is *stratified* if D has a permutation subdigraph of order k for every $k = 2, 3, \ldots, n$. A digraph D is *weakly stratified* if for each $k = 2, 3, \ldots, n$, either

- (i) D has a permutation subdigraph of order k, or
- (ii) for each $v \in V(D)$ the digraph $D - v$ has a permutation subdigraph of order $k - 1$.

In this paper, we study the (combinatorial) P_0^+-matrix completion problem. The property of being a P_0^+-matrix (or a Q-matrix) is not inherited by principal submatrices, though it is preserved under similarity and transposition. This fact distinguishes the P_0^+- and Q-matrix completion problems from the completion problems of several other classes of P_0-matrices.

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A publication of the International Linear Algebra Society
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2. Partial P_0^+-matrices and the P_0^+-completion problem. Recall that a *partial P_0^+-matrix* is a partial matrix M in which all fully specified principal minors are nonnegative and $S_k(M) > 0$ for every $k \in \{1, 2, \ldots, n\}$, whenever all $k \times k$ principal submatrices are fully specified.

Let M be a partial P_0^+-matrix. If all 1×1 principal submatrices (i.e., all diagonal entries) in M are specified, then $trace(M) > 0$. If for some $k \geq 2$ all $k \times k$ principal submatrices are fully specified, then M is fully specified (and therefore M is a P_0^+-matrix). Thus, the following proposition which provides a more useful characterization of a partial P_0^+-matrix is immediate.

Proposition 2.1. *A partial matrix M is a partial P_0^+-matrix if and only if exactly one of the following holds:*

(i) *At least one diagonal entry of M is unspecified, and each fully specified principal minor of M is nonnegative.*
(ii) *All diagonal entries are specified and nonnegative, with at least one of them positive; at least one off-diagonal entry is unspecified and each fully specified principal minor of M is nonnegative.*
(iii) *All entries of M are specified and M is a P_0^+-matrix.*

Definition 2.2. A partial P_0^+-matrix M is said to have a P_0^+-*completion* if there is a completion of M which is a P_0^+-matrix. A digraph D is said to have P_0^+-*completion* if every partial P_0^+-matrix specifying D has a P_0^+-completion. A graph G is said to have P_0^+-*completion*, if the digraph associated to G has P_0^+-completion.

The P_0^+-*matrix completion problem* aims to classify all digraphs which have P_0^+-completion. To distinguish the P_0^+-completion problem from those of P_0- and P-matrix classes, we furnish the following example.

Example 2.3. For $m, n \geq 2$ consider the graph G obtained by identifying a vertex of K_m with a vertex of K_n. Then, G is called a 1-*chordal graph* with two maximal cliques K_m and K_n [5]. It is known that for any m and n, G (i.e., the digraph associated to G) has P-completion and P_0-completion (see [8]). With an appropriate labeling of vertices, a partial matrix specifying G is given by

$$M = \begin{bmatrix} A_{11} & A_{12} & X \\ A_{21} & a_{22} & A_{23} \\ Y & A_{32} & A_{33} \end{bmatrix},$$

where A_{ii} are square, a_{22} is a scalar, and X and Y are fully unspecified. If M is a partial P_0-matrix, then a P_0-completion, called the *zero completion* of M, is obtained by putting $X = Y = 0$ (see [8]). Similarly, if M is a partial P-matrix, then a P-completion, called the *asymmetric completion* of M, is obtained by putting $X = A_{12}a_{22}^{-1}A_{23}$ and $Y = 0$ (see [8]). However, these techniques do not work for the

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P_0^+-completion problem. Note that in case M is a partial P_0^+-matrix, a_{22} may be zero. The zero completion of the partial P_0^+-matrix

$$M = \begin{bmatrix} 1 & 0 & 1 & ? & ? \\ 1 & 0 & 0 & ? & ? \\ -1 & 1 & 0 & 0 & -1 \\ ? & ? & 0 & 1 & 0 \\ ? & ? & 1 & 1 & 0 \end{bmatrix}$$

specifying the 1-chordal graph with two maximal cliques K_3 is not a P_0^+-matrix, even though the submatrices of M corresponding to the maximal cliques are P_0^+-matrices. However, M has P_0^+-completions, e.g.,

$$B_1 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

It will follow from Corollary 4.9 that for any 1-chordal graph G with two maximal cliques there is a partial P_0^+-matrix specifying G which does not have a P_0^+-completion. Nevertheless, in case one of the cliques is K_2, we have the following positive result in a restrictive situation.

THEOREM 2.4. *Let M be a partial matrix specifying a 1-chordal graph with two maximal cliques K_n and K_2. If the principal submatrices of M corresponding to K_n and K_2 are P_0^+-matrices, then M can be completed to a P_0^+-matrix.*

Proof. Without any loss of generality we assume that $V(K_n) = \{1, \ldots, n\}$ and $V(K_2) = \{n, n+1\}$. Thus, we can write

$$M = \begin{bmatrix} A_{11} & A_{12} & X \\ A_{21} & a_{22} & a_{23} \\ Y & a_{32} & a_{33} \end{bmatrix},$$

where X is $(n-1)\times 1$, Y is $1\times(n-1)$, and they are fully unspecified. Moreover, the two principal submatrices specifying K_n and K_2, viz.,

$$M_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad M_2 = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

are P_0^+-matrices. If $a_{22} > 0$, consider the asymmetric completion $\widehat{A}$ of M, i.e., one obtained by setting $X = A_{12}a_{22}^{-1}a_{23}$ and $Y = 0$. Then

$$\det \widehat{A} = \frac{\det M_1 \det M_2}{a_{22}} > 0. \tag{2.1}$$

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Let A be any principal submatrix of $\widehat{A}$. That A has nonnegative determinant follows from an equation similar to (2.1), if a_{22} is included in A, and from the fact that A is block triangular, if a_{22} is excluded. Since M_1 has a positive minor of order k for $1 \leq k \leq n$, $\widehat{A}$ is a P_0^+-matrix.

On the other hand, if $a_{22} = 0$, consider the zero completion $\widehat{A}$ of M, i.e., one obtained by setting $X = 0 = Y$. Since M_2 is a P_0^+-matrix, a_{33} must be positive. Then, we have $\det \widehat{A} = a_{33} \det M_1 + \det A_{11} \det M_2 > 0$. That a principal minor given by a principal submatrix A of $\widehat{A}$ is nonnegative follows from the determinant equality when n-th row is present and because the principal submatrix is block diagonal when n-th row is absent. Since M_1 has a positive minor of order k for $1 \leq k \leq n$, $\widehat{A}$ is a P_0^+-matrix. $\square$

Unlike several other matrix completion problems, a P_0^+-completion of the principal submatrix corresponding to the specified diagonal positions of a partial P_0^+-matrix does not provide a P_0^+-completion of the partial matrix. For example, consider the partial P_0^+-matrix

$$M = \begin{bmatrix} 2 & 1 & u & 1 \\ 1 & \frac{1}{2} & 0 & 1 \\ -1 & v & 0 & x \\ 1 & 1 & y & z \end{bmatrix},$$

with u, v, x, y, z as unspecified entries. The principal submatrix $M[\{1,2,3\}]$ of M induced by the specified diagonal entries has a P_0^+-completion, e.g., one obtained by putting $u = 2, v = 0$. However, for any completion of M we have $\det M[\{1,2,4\}] = -1/2$ and hence M cannot be completed to a P_0^+-matrix.

3. Digraphs having P_0^+-completion. We observe that if a digraph D omits all loops, then D has P_0^+-completion. Indeed, a completion of a partial P_0^+-matrix M specifying D can be obtained by assigning a sufficiently large value to each of the diagonal entries.

A digraph D may not have P_0^+-completion even if the subdigraph of D induced by the vertices at which D includes loops has P_0^+-completion. For example, the digraph D_1 in Figure 3.1 includes a loop at the vertex 1. The subdigraph induced by the vertex 1 has P_0^+-completion, being a complete digraph. However, the partial P_0^+-matrix

$$M = \begin{bmatrix} 0 & 0 \\ 0 & z \end{bmatrix}$$

specifying D_1 does not have a P_0^+-completion.

The following example shows that a digraph may not have P_0^+-completion even

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FIG. 3.1. *A digraph D_1 that does not have P_0^+-completion*

if each of its components has P_0^+-completion.

EXAMPLE 3.1. Consider the digraph $D_2 = K_2^* \cup K_3^*$. The components K_2^* and K_3^* of D_2 have P_0^+-completion. However, the partial P_0^+-matrix

$$M = \begin{bmatrix} 1 & 0 & ? & ? & ? \\ 0 & 1 & ? & ? & ? \\ ? & ? & 0 & 0 & 0 \\ ? & ? & 0 & 0 & 0 \\ ? & ? & 0 & 0 & 0 \end{bmatrix}$$

specifying D_2 does not have a P_0^+-completion, because for any completion of M the last three rows are linearly dependent.

The following example shows that the property of having P_0^+-completion is not inherited by the induced subdigraphs.

EXAMPLE 3.2. The digraph D_3 in Figure 3.2 has P_0^+-completion whereas its subdigraph D_0 induced by the vertices 1 and 2 does not have. To see this consider a partial P_0^+-matrix

$$M = \begin{bmatrix} d_1 & a_{12} & u \\ v & d_2 & -s \\ a_{31} & s & d_3 \end{bmatrix}$$

specifying D_3, where $u, v, \pm s$ are the unspecified entries. Now, for $t > 0$ choose a

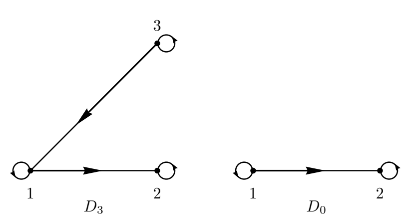

FIG. 3.2. *D_3 has P_0^+-completion, but its induced subdigraph D_0 does not have.*

completion $B(t)$ of M with $|u| = |v| = |s| = t$ and with appropriate signs for u, v and s so that $a_{12}v \leq 0$, $a_{31}u \leq 0$ and $uvs > 0$. Then, all 2×2 principal minors of B are

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nonnegative and $B(2,3) > 0$. Further,

$$\det B = t^3 + p(t),$$

where $p(t)$ is a polynomial in t of degree at most 2. Thus, $B(t)$ is a P_0^+-matrix for large values of t. On the other hand, the partial P_0^+-matrix

$$M_0 = \begin{bmatrix} 0 & 0 \\ ? & 1 \end{bmatrix}$$

specifying D_0 does not have a P_0^+-completion.

THEOREM 3.3. *Suppose $D \neq K_n^*$ is a digraph having P_0^+-completion and $\widehat{D}$ is a spanning subdigraph of D. If $\widehat{D}$ has P_0-completion, then $\widehat{D}$ has P_0^+-completion.*

Proof. Let $\widehat{M} = [\widehat{m}_{ij}]$ be a partial P_0^+-matrix specifying $\widehat{D}$. Then, being a partial P_0-matrix, $\widehat{M}$ has a P_0-completion $\widehat{B} = [\widehat{a}_{ij}]$. Let $M = [m_{ij}]$ be the partial matrix specifying D defined by

$$m_{ij} = \begin{cases} \widehat{m}_{ij}, & \text{if } (i,j) \in A(\widehat{D}), \\ \widehat{a}_{ij}, & \text{if } (i,j) \in A(D) \setminus A(\widehat{D}), i \neq j, \\ \max\{1, \widehat{a}_{ij}\}, & \text{if } (i,j) \in A(D) \setminus A(\widehat{D}), i = j. \end{cases}$$

Then, M is a partial P_0^+-matrix specifying D, and therefore has a P_0^+-completion B. Clearly, B is a P_0^+-completion of $\widehat{M}$. ☐

For all digraphs D that we have examined, including the digraphs of order at most 4, the following has been observed: if D has P_0-completion and its complement $\overline{D}$ is stratified, then D has P_0^+-completion. However, we do not know whether the result is true in general.

4. Necessary conditions for P_0^+–completion. In this section, we present some necessary conditions for a digraph to have P_0^+-completion.

THEOREM 4.1. *Let $D \neq K_n^*$ be a digraph having P_0^+-completion. Then, every proper induced subdigraph of D has P_0-completion.*

Proof. Let D be of order n and α be a proper subset of $\{1, 2, \dots, n\}$. Consider the subdigraph D_α of D induced by α and let M_α be a partial P_0-matrix specifying D_α. We extend M_α to a partial matrix M specifying D by setting all remaining specified off-diagonal entries as 0 and the remaining specified diagonal entries, if any, as 1. Then, all fully specified principal minors of M are nonnegative, and M is a partial P_0^+-matrix. Now, since D has P_0^+-completion, M can be completed to a P_0^+-matrix $\widehat{B}$. Clearly, the principal submatrix of $\widehat{B}$ induced by α is a P_0-completion of M_α. ☐

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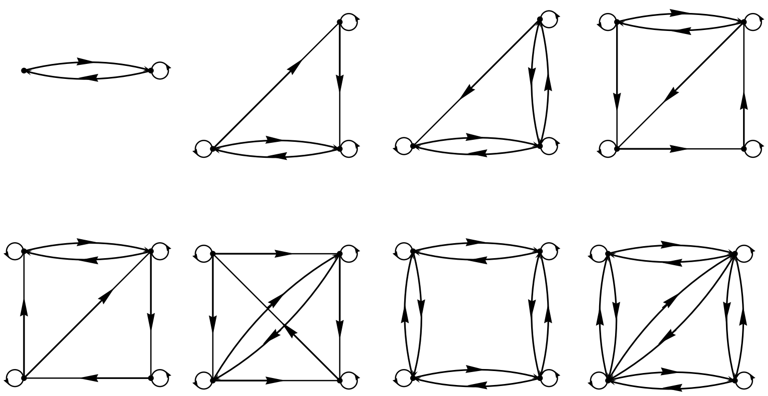

FIG. 4.1. *Digraphs not having P_0-completion*

In [2], Choi *et al.* showed that any digraph which contains one of the (unlabelled) digraphs in Figure 4.1 as an induced subdigraph does not have P_0-completion. In particular, these digraphs do not have P_0-completion. Therefore, an immediate implication of Theorem 4.1 is the following.

COROLLARY 4.2. *Any digraph which contains one of the (unlabelled) digraphs in Figure 4.1 as a proper induced subdigraph does not have P_0^+-completion.*

That the converse of the Theorem 4.1 is not true can be seen from the following example.

EXAMPLE 4.3. Consider the digraph D_4 in Figure 4.2. Each of the strongly connected components of D_4, being complete, has P_0-completion. It follows from [8, Theorem 5.8] that D_4 has P_0-completion. Consequently, each of the induced subdigraphs of D_4 has P_0-completion. However, D_4 does not have P_0^+-completion (see Remark 4.6).

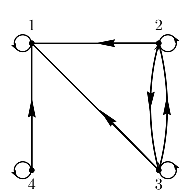

FIG. 4.2. *The digraph D_4 does not have P_0^+-completion*

THEOREM 4.4. *Let D be a digraph of order n that omits at least one loop. If D has P_0^+-completion, then $\overline{D}$ is stratified.*

Proof. Suppose D has P_0^+-completion. Let $k \geq 2$, and assume $\overline{D}$ has no per-

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mutation subdigraph of order k. If M is the partial matrix that specifies D with all specified entries zero, and B is a completion of M, then all $k \times k$ principal minors of B are zero, so B is not a P_0^+-matrix. This implies that $\overline{D}$ must be stratified. ▯

To see that the converse of Theorem 4.4 is not true, consider the digraph D_5 in Figure 4.3, which omits loop at the vertex 4. Though $\overline{D_5}$ is stratified, D_5 does not

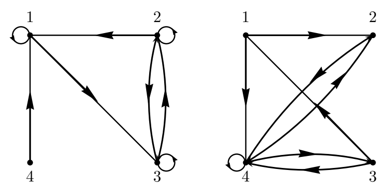

FIG. 4.3. *The digraph D_5 and its complement $\overline{D_5}$*

have P_0^+-completion, in view of Corollary 4.2, since the subdigraph induced by the vertices $\{1, 2, 3\}$ does not have P_0-completion (see Figure 4.1).

THEOREM 4.5. *Let $D \neq K_n^*$ be a digraph of order $n \geq 2$ that includes all loops and has P_0^+-completion. Then $\overline{D}$ is weakly stratified.*

Proof. Suppose for some k $(2 \leq k \leq n)$ $\overline{D}$ has no permutation subdigraph of order k and there is a vertex v in D such that $\overline{D} - v$ does not have a permutation subdigraph of order $k-1$. Let $M = [m_{ij}]$ be the partial matrix specifying D with $m_{vv} = 1$ and all other specified entries zero. Then for any completion B of M all $k \times k$ principal minors of B are zero, and therefore, B is not a P_0^+-matrix. ▯

REMARK 4.6. The digraph D_4 in Figure 4.2 includes all loops, and $\overline{D_4}$ is not weakly stratified. Thus, D_4 does not have P_0^+-completion.

The converse of Theorem 4.5 is not true which can be seen from the following example.

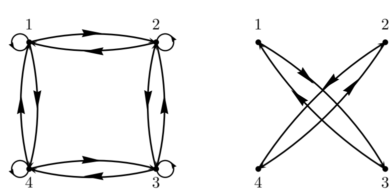

FIG. 4.4. *The symmetric 4-cycle C_4 and its complement*

EXAMPLE 4.7. Consider the symmetric 4-cycle C_4 (Figure 4.4). It is easy to see that $\overline{C_4}$ is weakly stratified. To see that C_4 does not have P_0^+-completion, consider

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the partial P_0^+-matrix

$$M = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 1 & 0 & 0 & x_{24} \\ x_{31} & -1 & 1 & 1 \\ 1 & x_{42} & 0 & 0 \end{bmatrix}$$

specifying C_4. For a completion B of M the 3×3 principal minor $B(1,2,3)$ of B indexed by $\{1,2,3\}$ is $-x_{13}$. Similarly, $B(1,3,4) = x_{13}$. Therefore, for B to be a P_0^+-matrix, $x_{13} = 0$. However, this yields $\det B = 0$. Thus, B cannot be a P_0^+-matrix.

Corollary 4.8. *If a digraph D of order $n \geq 2$ contains a vertex v with indegree or outdegree n, then D does not have P_0^+-completion.*

Proof. Clearly, v has either indegree or outdegree zero in $\overline{D}$, and therefore v does not lie on any cycle in $\overline{D}$. Consequently, $\overline{D}$ does not have a spanning permutation subdigraph, i.e., $\overline{D}$ is not stratified. Further, for any vertex $u \neq v$ in D the digraph $\overline{D} - u$ does not have a spanning permutation subdigraph. Hence, $\overline{D}$ is not weakly stratified. □

Corollary 4.9. *A 1-chordal graph G with two maximal cliques and with all loops does not have P_0^+-completion.*

Proof. Let G be of order n and v be the vertex in G common to the maximal cliques. Then, v has indegree as well as outdegree n in the digraph associated to G, and the result follows from Corollary 4.8. □

Corollary 4.10. *Let D be a digraph of order n that includes all loops and has P_0^+-completion. Then, $\overline{D}$ has a 2-cycle.*

Proof. Since D has P_0^+-completion and D includes all loops, $\overline{D}$ must be weakly stratified. Because $\overline{D}$ does not contain any loop, $\overline{D}$ has a permutation subdigraph of order 2 only if it has a 2-cycle. □

Remark 4.11. It is known that any asymmetric digraph has P_0-completion; see for example [2, Theorem 2.2]. In contrast, a maximal asymmetric digraph with all loops does not have P_0^+-completion, since the complement of such a digraph does not have a 2-cycle.

5. Classification of small digraphs as to P_0^+-completion.

In this section, we apply the results in the previous sections to classify the digraphs of order at most four that include all loops as to P_0^+-completion.

Any matrix which is permutation similar to a P_0^+-matrix is a P_0^+-matrix. Therefore, if a digraph D has P_0^+-completion, then any digraph which is isomorphic to D

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A publication of the International Linear Algebra Society
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has P_0^+-completion, that is, any digraph obtained by a labelling of the unlabelled digraph associated to D has P_0^+-completion.

The nomenclature of the digraphs considered in the sequel is as per their order in the atlas in [6, Appendix, pp. 233]. Here, $D_p(q,n)$ is the one obtained by attaching a loop at each of the vertices to the n-th member in the list of digraphs with p vertices and q (non-loop) arcs in the atlas. The classification is broken up into a series of lemmas.

LEMMA 5.1. *The digraphs $D_p(q,n)$ which are listed below do not have P_0^+-completion.*

$$\begin{array}{lll}
p=2; & q=1 & \\
p=3; & q=3; & n=2,3 \\
 & q=4; & n=2,3,4 \\
 & q=5 & \\
p=4; & q=6; & n=45\text{–}48 \\
 & q=7; & n=29\text{–}38 \\
 & q=8; & n=16\text{–}27 \\
 & q=9; & n=4\text{–}13 \\
 & q=10; & n=2\text{–}5 \\
 & q=11. &
\end{array}$$

Proof. Each of the digraphs listed contains all loops but its complement does not contain a 2-cycle. Hence, by Corollary 4.10, the digraph does not have P_0^+-completion. $\square$

LEMMA 5.2. *The digraphs $D_p(q,n)$ which are listed below do not have P_0^+-completion.*

$$\begin{array}{lll}
p=3; & q=2; & n=1,3,4 \\
 & q=3; & n=1,4 \\
 & q=4; & n=1 \\
p=4; & q=3; & n=8,11 \\
 & q=4; & n=10,12,14,15,21,27 \\
 & q=5; & n=4\text{–}6,11,14\text{–}17,19,21\text{–}24,26,28,29,31,34,36,37 \\
 & q=6; & n=1,2,9\text{–}23,26,27,29,30,32\text{–}41,43,44 \\
 & q=7; & n=1,3\text{–}28 \\
 & q=8; & n=1,3\text{–}15 \\
 & q=9; & n=1\text{–}3 \\
 & q=10; & n=1.
\end{array}$$

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A publication of the International Linear Algebra Society
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Proof. Each of the digraphs listed contains all loops but its complement is not weakly stratified. Thus, by Theorem 4.5, the digraphs do not have P_0^+-completion. □

LEMMA 5.3. *The digraphs $D_4(q,n)$ which are listed below do not have P_0^+-completion.*

$$\begin{aligned} q &= 4; \quad n = 13 \\ q &= 5; \quad n = 12, 13, 18, 20 \\ q &= 6; \quad n = 24, 25, 28, 31, 42. \end{aligned}$$

Proof. Each of the digraphs listed has an induced subdigraph isomorphic to one of the digraphs in Figure 4.1. Hence, by Corollary 4.2, the digraphs do not have P_0^+-completion. □

LEMMA 5.4. *The digraphs $D_4(q,n)$ which are listed below do not have P_0^+-completion.*

$$\begin{aligned} q &= 6; \quad n = 6, 8 \\ q &= 7; \quad n = 2 \\ q &= 8; \quad n = 2. \end{aligned}$$

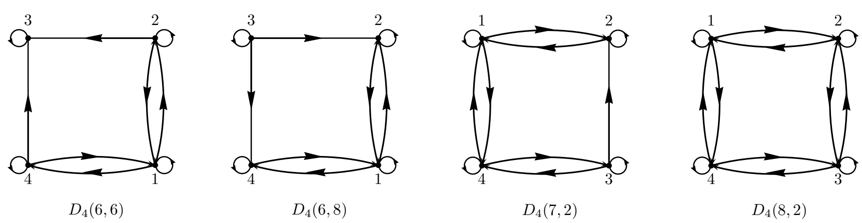

FIG. 5.1. *Digraphs $D_4(q,n)$ do not have P_0^+-completion.*

Proof. The digraph $D_4(8,2)$ is the symmetric 4-cycle C_4, and that it does not have P_0^+-completion has been seen in Example 4.7. Next, for any completion B_1 of the partial P_0^+-matrix

$$M_1 = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 1 & 0 & x_{23} & x_{24} \\ x_{31} & -1 & 1 & 1 \\ 1 & x_{42} & 0 & 0 \end{bmatrix}$$

specifying $D_4(7,2)$ we have $B(1,2,3) = -x_{13}$ and $B(1,3,4) = x_{13}$. Therefore, for B_1 to be a P_0^+-matrix, $x_{13} = 0$. However, this yields $\det B = 0$. Thus, B_1 cannot be a P_0^+-matrix. Similarly, the partial P_0^+-matrices

$$M_2 = \begin{bmatrix} 0 & 1 & x_{13} & 1 \\ 0 & 0 & -1 & x_{24} \\ x_{31} & x_{32} & 1 & x_{34} \\ 0 & x_{42} & 1 & 0 \end{bmatrix} \quad \text{and} \quad M_3 = \begin{bmatrix} 0 & 0 & x_{13} & 0 \\ 1 & 0 & x_{23} & x_{24} \\ x_{31} & -1 & 1 & 1 \\ 1 & x_{42} & x_{43} & 0 \end{bmatrix}$$

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specifying the digraphs $D_4(6,6)$ and $D_4(6,8)$, respectively, do not have P_0^+-completions. Indeed, for any completion B_2 of M_2, we have $B_2(1,2,3) = -x_{31}$ and $B_2(1,3,4) = x_{31}$. Further, M_3 is the transpose of M_2. $\square$

The following result can be easily verified.

LEMMA 5.5. *For real numbers a, b, c, d, the inequalities*

$$\begin{aligned} ax + by &\geq 0 \\ cx + dy &\geq 0 \\ xy &< 0 \end{aligned}$$

do not have a solution for x and y only if one of the following holds:

(i) $a = 0, c = 0,\ bd < 0$ *or* $b = 0, d = 0,\ ac < 0$;
(ii) $b = 0, c = 0,\ ad > 0$ *or* $a = 0, d = 0,\ bc > 0$;
(iii) $c > 0, d > 0,\ \dfrac{a}{c} = \dfrac{b}{d} < 0$.

LEMMA 5.6. *The digraphs $D_4(6,n),\ n = 3,4,5,7$, have P_0^+ completion.*

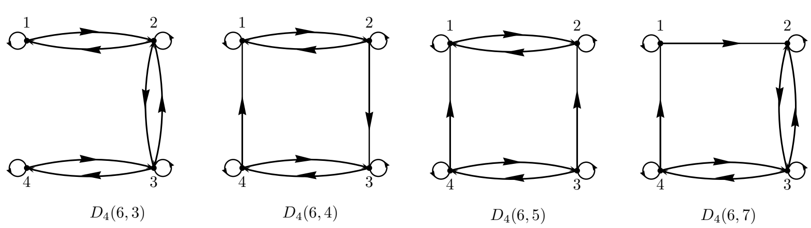

FIG. 5.2. *Digraphs $D_4(q,n)$ have P_0^+-completion.*

Proof. For each partial matrix considered below we denote the specified diagonal entries by d_i and the specified off-diagonal entries by a_{ij}. First, let M be a partial P_0^+-matrix specifying the digraph $D_4(6,3)$. We show that for some choices of s and t the completion

$$B = \begin{bmatrix} d_1 & a_{12} & s & 0 \\ a_{21} & d_2 & a_{23} & t \\ -s & a_{32} & d_3 & a_{34} \\ 0 & -t & a_{43} & d_4 \end{bmatrix}$$

of M is a P_0^+-matrix. Clearly, for any choices of s and t, all 2×2 principal minors of B are nonnegative. Moreover, $B(1,3)$ and $B(2,4)$ are positive for nonzero s and t.

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Now, the 3×3 principal minors of B are

$$B(1,2,3) = d_1 B(2,3) + s(a_{21}a_{32} - a_{12}a_{23}) + (d_2 s^2 - d_3 a_{12}a_{21}) \tag{5.1}$$
$$B(1,2,4) = d_1 t^2 + d_4 B(1,2) \tag{5.2}$$
$$B(1,3,4) = d_1 B(3,4) + d_4 s^2 \tag{5.3}$$
$$B(2,3,4) = d_2 B(3,4) + t(a_{32}a_{43} - a_{23}a_{34}) + (d_3 t^2 - d_4 a_{23}a_{32}). \tag{5.4}$$

Since $B(1,2) \geq 0$, it follows that $a_{12}a_{21} \leq 0$ if $d_2 = 0$. Similarly, $a_{23}a_{32} \leq 0$ if $d_3 = 0$. Consequently, for s and t with large magnitude so that $d_2 s^2 - d_3 a_{12}a_{21}$ and $d_3 t^2 - d_4 a_{23}a_{32}$ are nonnegative and with appropriate signs so that the term $s(a_{21}a_{32} - a_{12}a_{23})$ in (5.1) and the term $t(a_{32}a_{43} - a_{23}a_{34})$ in (5.4) are nonnegative, we get all 3×3 principal minors of B nonnegative. Moreover, at least one of them can be made positive, because $d_i > 0$ for some i. Further, choosing $|t| = |s|$ we get

$$\det B = s^4 + p(s),$$

where $p(s)$ is a polynomial in s of degree at most 3. Hence, for large values of $|s|$, $\det B > 0$, and B is a P_0^+-matrix.

Next, let M be a partial P_0^+-matrix specifying the digraph $D_4(6,4)$. We show that for the following choices of the unspecified entries and some suitable choice of t and s, the completion

$$B = \begin{bmatrix} d_1 & a_{12} & t & -a_{41} \\ a_{21} & d_2 & a_{23} & s \\ -t & -a_{23} & d_3 & a_{34} \\ a_{41} & -s & a_{43} & d_4 \end{bmatrix}$$

of M is a P_0^+-matrix. Clearly, for any nonzero choices of t and s, all 2×2 principal minors of B are nonnegative and $B(1,3) > 0$, $B(2,4) > 0$. Now, 3×3 principal minors of B are

$$B(1,2,3) = d_3 B(1,2) + d_1 a_{23}^2 + d_2 t^2 - t(a_{12} + a_{21})a_{23} \tag{5.5}$$
$$B(1,2,4) = d_4 B(1,2) + d_2 a_{41}^2 + d_1 s^2 + s(a_{12} + a_{21})a_{41} \tag{5.6}$$
$$B(1,3,4) = d_1 B(3,4) + d_3 a_{41}^2 + d_4 t^2 + t(a_{34} + a_{43})a_{41} \tag{5.7}$$
$$B(2,3,4) = d_2 B(3,4) + d_4 a_{23}^2 + d_3 s^2 - s(a_{34} + a_{43})a_{23}. \tag{5.8}$$

Case 1: $a_{23} = 0$ or $a_{41} = 0$. If $a_{23} = 0$, then choose signs for s and t such that $s(a_{12} + a_{21})a_{41} \geq 0$ and $t(a_{34} + a_{43})a_{41} \geq 0$. Then, all 3×3 principal minors are nonnegative, and since $d_i > 0$ for some i, at least one of these minors is positive. Finally, for large values of $|s|$ and $|t|$, $\det B > 0$. The case when $a_{41} = 0$ is similar.

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Case 2: $a_{23} \neq 0$ and $a_{41} \neq 0$. We note that one or more of the diagonal entries of M are positive, since M is a partial P_0^+-matrix. If $d_1 > 0$, then we choose t such that

$$t(a_{34} + a_{43})a_{41} \geq 0$$
$$d_1 a_{23}^2 - t(a_{12} + a_{21})a_{23} > 0.$$

Further, we choose appropriate sign for s such that $s(a_{34} + a_{43})a_{23} \leq 0$. Then for large values of $|s|$ all 3×3 principal minors are nonnegative, and $B(1, 2, 3)$, $B(1, 2, 4)$ and $\det B$ are positive. We get similar results if one of d_2, d_3 and d_4 is positive instead of d_1.

Similarly, with suitable values of t and s, a partial P_0^+-matrix M specifying the digraph $D_4(6, 5)$ can be completed to a P_0^+-matrix

$$B = \begin{bmatrix} d_1 & a_{12} & t & -a_{41} \\ a_{21} & d_2 & -a_{32} & s \\ -t & a_{32} & d_3 & a_{34} \\ a_{41} & -s & a_{43} & d_4 \end{bmatrix}.$$

Finally, let M be a partial P_0^+-matrix specifying the digraph $D_4(6, 7)$. We show that for some choices of s, t and x, y, z, w the completion

$$B = \begin{bmatrix} d_1 & a_{12} & x & s \\ t & d_2 & a_{23} & z \\ y & a_{32} & d_3 & a_{34} \\ a_{41} & w & a_{43} & d_4 \end{bmatrix}$$

of M is a P_0^+-matrix. Clearly, for any choices of s, t, x, y, z, w such that $a_{12}t, a_{41}s,\ xy$ and zw are nonpositive and at least one of them negative, we have all 2×2 principal minors of B are nonnegative and at least one of them is positive. Now, the 3×3 principal minors of B are

$$B(1,2,3) = \big(d_1 B(2,3) - d_3 a_{12} t - d_2 xy\big) + a_{32} tx + a_{12} a_{23} y \tag{5.9}$$
$$B(1,3,4) = \big(d_1 B(3,4) - d_3 a_{41} s - d_4 xy\big) + a_{34} a_{41} x + a_{43} sy \tag{5.10}$$
$$B(1,2,4) = \big(d_4 B(1,2) - d_2 a_{41} s - d_1 zw\big) + a_{41} a_{12} z + stw \tag{5.11}$$
$$B(2,3,4) = \big(d_2 B(3,4) - d_4 a_{23} a_{32} - d_3 zw\big) + a_{32} a_{43} z + a_{23} a_{34} w. \tag{5.12}$$

Since the terms in the parentheses in the right sides of the above equations are all nonnegative under our choices of the unspecified entries, the 3×3 principal minors are nonnegative if

$$a_{32} tx + a_{12} a_{23} y \geq 0 \tag{5.13}$$
$$a_{34} a_{41} x + a_{43} sy \geq 0 \tag{5.14}$$
$$a_{41} a_{12} z + stw \geq 0 \tag{5.15}$$
$$a_{32} a_{43} z + a_{23} a_{34} w \geq 0. \tag{5.16}$$

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In view of Lemma 5.5, the above equations have solutions with $s = t = 0$, and arbitrary large values of $-xy$ and $-zw$ in all cases except when $a_{12}a_{23}a_{34}a_{41} > 0$ and $a_{32}a_{43} = 0$. In the latter case, putting $s = t = 0$, we have

$$B = \begin{bmatrix} d_1 & a_{12} & x & 0 \\ 0 & d_2 & a_{23} & z \\ y & 0 & d_3 & a_{34} \\ a_{41} & w & 0 & d_4 \end{bmatrix}.$$

If $d_1 > 0$, then a P_0^+-completion of M is obtained by setting $x = y = 0$ and choosing $-z = w$ such that $a_{23}a_{34}\big(d_1 w - a_{12}a_{41}\big) > 0$ in B. A P_0^+-completion of M can be obtained in a similar way in case some other d_i, instead of d_1, is positive. Note that the structure of the above matrix B is invariant under a cyclic permutation. This completes the proof. ∎

LEMMA 5.7. *The digraphs $D_4(5, 25)$ and $D_4(5, 27)$ have P_0^+-completion.*

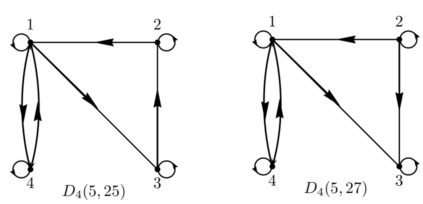

FIG. 5.3. *The digraphs have P_0^+-completion.*

Proof. Let M be a partial P_0^+-matrix specifying the digraph $D_4(5, 25)$. We show that for some choices of r, s, t, x, y, z and w the completion

$$B = \begin{bmatrix} d_1 & r & a_{13} & a_{14} \\ a_{21} & d_2 & s & x \\ t & a_{32} & d_3 & z \\ a_{41} & y & w & d_4 \end{bmatrix}$$

of M is a P_0^+-matrix.

Case 1: $a_{13} \neq 0$ or $a_{21} \neq 0$. We put

$$r = -a_{21},\ s = -a_{32},\ t = -a_{13},\ y = -x \ \text{ and } \ w = -z.$$

Then, all 2×2 principal minors of B are nonnegative and one of $B(1, 2)$ and $B(1, 3)$ is positive. Further, the 3×3 principal minors of B are

$$B(1, 2, 3) = d_1 B(2, 3) + d_3 a_{21}^2 + d_2 a_{13}^2 \tag{5.17}$$
$$B(1, 2, 4) = d_2 B(1, 4) + d_4 a_{21}^2 + d_1 x^2 - a_{21}(a_{14} + a_{41})x \tag{5.18}$$
$$B(1, 3, 4) = d_3 B(1, 4) + d_4 a_{13}^2 + d_1 z^2 + a_{13}(a_{14} + a_{41})z \tag{5.19}$$
$$B(2, 3, 4) = d_4 B(2, 3) + d_3 x^2 + d_2 z^2. \tag{5.20}$$

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For nonzero x and z satisfying

$$a_{21}(a_{14}+a_{41})x \leq 0 \quad \text{and} \quad a_{13}(a_{14}+a_{41})z \geq 0, \tag{5.21}$$

all 3×3 principal minors of B are nonnegative, and since $d_i > 0$ for some i, at least one of these minors is positive. Now, breaking the determinant along the first column of B we get

$$\det B = \det \begin{bmatrix} d_1 & -a_{21} & a_{13} & a_{14} \\ a_{21} & d_2 & -a_{32} & x \\ -a_{13} & a_{32} & d_3 & z \\ -a_{14} & -x & -z & d_4 \end{bmatrix} + \det \begin{bmatrix} 0 & -a_{21} & a_{13} & a_{14} \\ 0 & d_2 & -a_{32} & x \\ 0 & a_{32} & d_3 & z \\ a_{41}+a_{14} & -x & -z & d_4 \end{bmatrix}.$$

The terms in the first determinant not involving the diagonal entries d_i are given by the determinant of the skew-symmetric matrix

$$\begin{bmatrix} 0 & -a_{21} & a_{13} & a_{14} \\ a_{21} & 0 & -a_{32} & x \\ -a_{13} & a_{32} & 0 & z \\ -a_{14} & -x & -z & 0 \end{bmatrix},$$

which equals $(a_{13}x + a_{21}z + a_{14}a_{32})^2$. Further, the only terms of total degree more than 1 in x and z involving d_i in the first determinant are $d_1d_2z^2$ and $d_1d_3x^2$. The second determinant does not contribute any term with total degree more than 1 in x and z. Therefore, we get

$$\det B = (a_{13}x + a_{21}z + a_{14}a_{32})^2 + d_1d_2z^2 + d_1d_3x^2 + p(x,z),$$

where $p(x,z)$ is a polynomial in x and z with total degree at most 1. Since either $a_{13} \neq 0$ or $a_{21} \neq 0$, for x and z with large magnitudes we have $\det B > 0$.

Case 2: $a_{13} = a_{21} = 0$. In this case, the 3×3 principal minors of

$$B = \begin{bmatrix} d_1 & r & 0 & a_{14} \\ 0 & d_2 & s & x \\ t & a_{32} & d_3 & z \\ a_{41} & y & w & d_4 \end{bmatrix}$$

are given by

$$B(1,2,3) = d_1B(2,3) + rst \tag{5.22}$$
$$B(1,2,4) = d_2B(1,4) + x(a_{41}r - d_1y) \tag{5.23}$$
$$B(1,3,4) = d_3B(1,4) + w(a_{14}t - d_1z) \tag{5.24}$$
$$B(2,3,4) = d_4B(2,3) - d_3xy - d_2zw + a_{32}xw + syz. \tag{5.25}$$

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If $a_{14} \neq 0$ then we put

$$r = x = z = 0,\ t = s,\ w = \frac{a_{14}s}{|a_{14}s|},\ y = \frac{-|s|}{a_{14}},$$

where the nonzero s is to be chosen such that $a_{32}s \leq 0$. Then, all 3×3 principal minors are nonnegative and $B(1,3,4) > 0$. Since $\det B$ is a monic polynomial in $|s|$ of degree 3, B is a P_0^+-matrix for sufficiently large values of $|s|$. If $a_{41} \neq 0$, then we put

$$t = y = w = 0,\ r = z,\ x = \frac{a_{41}r}{|a_{41}r|},\ s = \frac{-|r|}{a_{41}},$$

where the nonzero s is to be chosen such that $a_{32}s \leq 0$. Then, B is a P_0^+-matrix for sufficiently large values of $|r|$. Finally, if $a_{14} = a_{41} = 0$, then we put $y = z = 0,\ x = t,\ r = st,\ w = -st$ with $s \neq 0$ such that $a_{32}s \leq 0$. Then, B is a P_0^+-matrix for large values of $|s|$.

Similarly, with suitable values of r, x, y, w, z, t and s, any partial P_0^+-matrix

$$M = \begin{bmatrix} d_1 & r & a_{13} & a_{14} \\ a_{21} & d_2 & a_{23} & x \\ t & s & d_3 & z \\ a_{41} & y & w & d_4 \end{bmatrix}$$

specifying the digraph $D_4(5,27)$ can be completed to a P_0^+-matrix. □

Lemma 5.8. *The asymmetric digraphs $D_4(5,n)$, $n = 30, 32, 33, 35, 38$, have P_0^+-completion.*

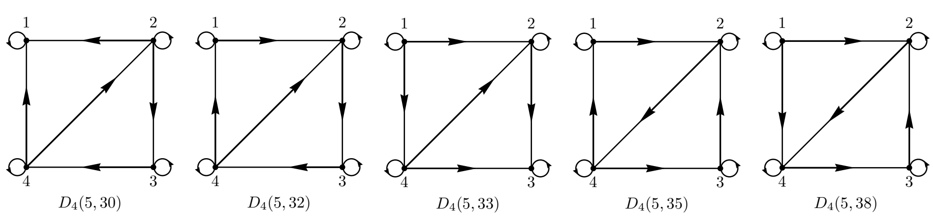

Fig. 5.4. *The digraphs have P_0^+-completion.*

Proof. We prove the result for $D_4(5,38)$; the proofs for the other digraphs are similar. Consider a partial P_0^+-matrix

$$M = \begin{bmatrix} d_1 & a_{12} & x & a_{14} \\ y & d_2 & r & a_{24} \\ s & a_{32} & d_3 & z \\ u & w & a_{43} & d_4 \end{bmatrix}$$

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specifying the digraph $D_4(5,38)$, where x,y,r,s,z,u,w are unspecified entries. Now, for a completion B of M the 3×3 principal minors of B are given by

$$B(1,2,3) = d_1B(2,3) + a_{12}sr - d_3ya_{12} + xya_{32} - d_2xs \tag{5.26}$$
$$B(1,2,4) = d_1B(2,4) - a_{12}yd_4 + a_{14}yw + a_{12}a_{24}u - d_2a_{14}u \tag{5.27}$$
$$B(1,3,4) = d_1B(3,4) + xzu - sxd_4 + a_{14}sa_{43} - a_{14}ud_3 \tag{5.28}$$
$$B(2,3,4) = d_2B(3,4) + rzw - ra_{32}d_4 + a_{24}a_{32}a_{43} - a_{24}wd_3. \tag{5.29}$$

Case 1: $a_{24}\neq 0$ or $a_{12}a_{43}+a_{14}a_{32}\neq 0$. In that case, we put

$$y=-a_{12},\ r=-a_{32},\ u=-a_{14},\ w=-a_{24},\ z=-a_{43},\ x=-s=t.$$

Then, all 2×2 principal minors are nonnegative and for $t\neq 0$, $B(1,3)>0$. Further, it can be easily seen that each principal minor of order 3×3 is nonnegative. Now, M being a partial P_0^+-matrix, at least one of the d_i is positive. Moreover, in case $(a_{14}a_{32}+a_{43}a_{12})\neq 0$, at least one of the $a_{12}a_{43}$ and $a_{14}a_{32}$ is nonzero. It can be easily verified that in each of the three cases, viz. $a_{24}\neq 0$, $a_{12}\neq 0\neq a_{43}$ and $a_{14}\neq 0\neq a_{32}$, at least one 3×3 principal minor of B is positive, depending on i with $d_i>0$. Further, from the condition $a_{24}\neq 0$ or $a_{12}a_{43}+a_{14}a_{32}\neq 0$, for any nonzero t we have

$$\begin{aligned}\det B = {} & (a_{14}a_{32}+a_{43}a_{12}+ta_{24})^2 + d_1d_2a_{43}^2 + d_1d_3a_{24}^2 + d_1d_4a_{32}^2\\ & + d_2d_3a_{14}^2 + d_2d_4t^2 + d_3d_4a_{12}^2 + d_1d_2d_3d_4 > 0.\end{aligned}$$

Case 2: $a_{24}=0, a_{12}a_{43}+a_{14}a_{32}=0$. This case is divided into the following subcases.

Subcase I: $\big(a_{32}\neq 0$ or $a_{14}\neq 0\big)$ and $\big(a_{12}\neq 0$ or $a_{43}\neq 0\big)$. In this case, we consider a completion B of M by putting $r=s=u=0$ in M. We can choose nonzero y,w and z with appropriate signs such that $a_{12}y\leq 0$, $a_{43}z\leq 0$ and $a_{14}yw\geq 0$. Then, each of the 2×2 principal minors of B becomes nonnegative. Since at least one of a_{12} and a_{43} is nonzero, we have either $B(1,2)>0$ or $B(3,4)>0$. Now, the equation $a_{12}a_{43}+a_{14}a_{32}=0$ implies that the sign of each nonzero term in the equation is opposite of the sign of the product of the other three, if the product is nonzero. So, we can always choose an appropriate sign for x such that $ya_{32}x\geq 0$ and $-xyzw>0$. Then, each of the 3×3 principal minors is nonnegative. Further, at least one of $a_{14}yw$ and $ya_{32}x$ is positive, since a_{14} or a_{32} is nonzero, which yields either $B(1,2,3)$ or $B(1,2,4)$ is positive. Finally, we choose $|x|=|y|=|z|=|w|=t>0$. Then, for sufficiently large values of t we have $\det B=t^4+p(t)>0$, where $p(t)$ is a polynomial of degree at most 3.

Subcase II: $(a_{32}=0$ and $a_{14}=0)$ and $(a_{12}\neq 0$ or $a_{43}\neq 0)$. In this case, consider a completion B of M obtained by putting $s=u=0$ in M. Now, we choose appropriate signs for $y,\ z$ and r such that $a_{12}y\leq 0$, $a_{43}z\leq 0$ and $rwz>0$. Since at least one of a_{12} and a_{43} is nonzero, each 2×2 principal minor of B is nonnegative, and either

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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$B(1,2) > 0$ or $B(3,4) > 0$. Moreover, each of the 3×3 principal minors of B is nonnegative and $B(2,3,4) > 0$. Finally, we choose appropriate sign for x and put $|x| = |y| = |z| = |w| = t > 0$. Then, for large values of t we have $\det B = t^4 + p(t) > 0$, where $p(t)$ is a polynomial of degree at most 3.

Subcase III: $a_{24} = 0$; ($a_{32} \neq 0$ or $a_{14} \neq 0$) and ($a_{12} = 0$ and $a_{43} = 0$). Consider a completion B of M obtained by putting, $u = r = 0$ and $s = -x = t$. Then, all 2×2 principal minors of B are nonnegative and for large values of t, $B(1,3) > 0$. Next, we choose appropriate signs for t and w such that $-tya_{32} \geq 0$ and $a_{14}yw \geq 0$. Since $a_{32} \neq 0$ or $a_{14} \neq 0$, each of the 3×3 principal minors is nonnegative, and either $B(1,2,3)$ or $B(1,2,4)$ is positive. Finally, we choose appropriate sign for z and put $|x| = |y| = |z| = |w| = t > 0$. Then, $\det B > 0$ for large values of t.

Subcase IV: We are left with the case when all specified off-diagonal entries are zero. In this case, a P_0^+-completion of M is obtained by putting $x = r = z = w = -y = -s = t > 0$.

Combining all cases, we see that the partial P_0^+-matrix M can be completed to a P_0^+-matrix, and therefore, the digraph $D_4(5,38)$ has P_0^+-completion. ▯

THEOREM 5.9. *For $1 \leq p \leq 4$, the digraph $D_p(q,n)$ has P_0^+-completion if and only if it is one of the digraphs listed below:*

$p = 1$		
$p = 2$;	$q = 0, 2$	
$p = 3$;	$q = 0, 1, 6$	
	$q = 2$;	$n = 2$
$p = 4$;	$q = 0, 1, 12$	
	$q = 2$;	$n = 1$–5
	$q = 3$;	$n = 1$–$7, 9, 10, 12, 13$
	$q = 4$;	$n = 1$–$9, 11, 16$–$20, 22$–26
	$q = 5$;	$n = 1$–$3, 7$–$10, 25, 27, 30, 32, 33, 35, 38$
	$q = 6$;	$n = 3$–$5, 7$.

Proof. It is clear that $D_p(q,n)$ has P_0^+-completion if $q = 0$ or it is a complete digraph.

Case: $p = 3$. It follows from Example 3.2 that $D_3(2,2)$ (i.e., D_3 in Figure 3.2) has P_0^+-completion. Further, $D_3(1,1)$ is a spanning subdigraph of $D_3(2,2)$ and has P_0-completion. Therefore, in view of Theorem 3.3, $D_3(1,1)$ has P_0^+-completion. Rest of the digraphs of order 3 appear in the lists in Lemma 5.1 and Lemma 5.2 and do not

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A publication of the International Linear Algebra Society
Volume 29, pp. 120-143, October 2015

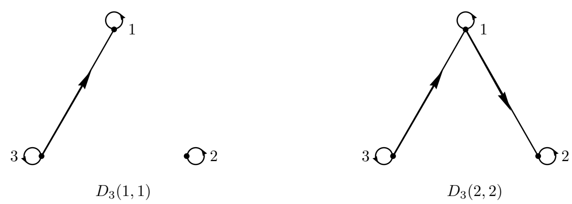

FIG. 5.5. *The digraphs have P_0^+-completion.*

have P_0^+-completion.

Case: $p = 4$. The digraphs $D_4(q,n), q = 5, n = 1$–3; $q = 4, n = 1$–9; $q = 3, n = 1$–7; $q = 2, n = 1$–5; $q = 1$, have P_0-completion (see [2]) and each of them is a spanning subdigraph of the digraph $D_4(6,3)$. Therefore, in view of Theorem 3.3, they have P_0^+-completion, since $D_4(6,3)$ has P_0^+-completion (see Lemma 5.6).

The digraphs $D_4(q,n), q = 5, n = 7, 9$; $q = 4, n = 16$–18, have P_0-completion and each of them is a spanning subdigraph of $D_4(6,4)$. Therefore, they have P_0^+-completion, since $D_4(6,4)$ has P_0^+-completion (see Lemma 5.6).

The digraphs $D_4(q,n), q = 5, n = 8, 10$; $q = 4, n = 19$, have P_0-completion and each of them is a spanning subdigraph of $D_4(6,5)$. Therefore, they have P_0^+-completion, since $D_4(6,5)$ has P_0^+-completion (see Lemma 5.6).

The digraphs $D_4(q,n), q = 4, n = 11, 20, 24$; $q = 3, n = 9, 10, 12$, have P_0-completion and each of them is a spanning subdigraph of $D_4(5,25)$. Therefore, they have P_0^+-completion, since $D_4(5,25)$ has P_0^+-completion (see Lemma 5.7).

Finally, the digraphs $D_4(q,n), q = 4, n = 22, 26$; $q = 3, n = 13$, have P_0-completion and each of them is a spanning subdigraph of $D_4(5,27)$. Therefore, they have P_0^+-completion, since $D_4(5,27)$ has P_0^+-completion (see Lemma 5.7).

The proof is complete, as the rest of the digraphs of order 4 appear in the lists in Lemmas 5.1–5.8. □

6. Comparison with P-, P_0- and Q-matrix completion problems. Although a P_0^+-matrix is a Q-matrix as well as a P_0-matrix, a digraph which has both Q-completion and P_0-completion may not have P_0^+-completion. The digraph $D_4(7,3)$ in Figure 6.1 is an example of a digraph having Q-completion as well as P_0-completions, but not P_0^+-completion.

We have observed that for all digraphs we have considered including the digraphs of order at most 4, if a digraph D has P_0^+-completion, then D has both P_0-completion and Q-completion. However, we do not know whether the result is true in general.

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A publication of the International Linear Algebra Society
Volume 29, pp. 120-143, October 2015

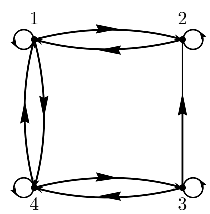

FIG. 6.1. *P_0^+-completion vs. Q- and P_0-completion*

The following result shows that the completion problems for the classes of P- and P_0^+-matrices are related. The effective argument used in the proof was applied for comparing a large number of pairs of completion problems by L. Hogben [9].

THEOREM 6.1. *Any digraph that has P_0^+-completion also has P-completion.*

Proof. Let D be a digraph which has P_0^+-completion and M be a partial P-matrix specifying D. Then, all fully specified principal minors of M are positive. Since determinant of a matrix is a continuous function of its entries, there is $\epsilon > 0$ such that the partial matrix M_0 obtained from M by decreasing the specified diagonal entries by ϵ is a partial P-matrix. Since a partial P-matrix is a partial P_0^+-matrix, M_0 is a partial P_0^+-matrix specifying D. Consequently, M_0 has a P_0^+-completion B_0. We now have a P-completion of M, namely, $B = B_0 + \epsilon I$, where I is the identity matrix. $\square$

The converse of Theorem 6.1 is not true. For example, the digraph $D_3(2,1)$ does not have P_0^+-completion (see Lemma 5.2). However, $D_3(2,1)$ has P-completion, since any digraph of order 3 has P-completion (see [10]).

Acknowledgment: We thank the referee for helpful comments and suggestions. We express our gratitude to Professor Leslie Hogben for her inspiration to pursue this research work. The article is dedicated to Professor R.B. Bapat.

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Electronic Journal of Linear Algebra ISSN 1081-3810
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