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ALMOST DEFINITE MATRICES REVISITED

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Abstract. A real matrix A is called as an almost definite matrix if $\langle x, Ax\rangle = 0 \Longrightarrow Ax = 0$. This notion is revisited. Many basic properties of such matrices are established. Several characterizations for a matrix to be an almost definite matrix are presented. Comparisons of certain properties of almost definite matrices with similar properties for positive definite or positive semidefinite matrices are brought to the fore. Interconnections with matrix classes arising in the theory of linear complementarity problems are discussed briefly.

Key words. Almost definite matrix, range-symmetric matrix, Moore-Penrose inverse, group inverse, positive semidefinite matrix.

AMS subject classifications. 15A09, 15A39, 15B57.

1. Introduction. The central object of this article is the class of almost definite matrices with real entries. Let us recall that a complex square matrix A is called an almost definite matrix if $x^*Ax = 0 \Rightarrow Ax = 0$. In what follows, we give a brief survey of the literature. Almost definite matrices were studied first in [7], where a certain electromechanical system is shown to have a solution if and only if a specific matrix is almost definite. The authors of [14] study certain extensions of the work of [7]. Here, the authors provide a frame work in terms of generalized inverses of a certain partitioned matrix in which the notions of the fundamental bordered matrix of linear estimation of a Gauss-Markov model and the Duffin-Morley linear electromechanical system are shown to be related. In [13], the author presents suficient conditions for a complex matrix to be almost definite. This is done by considering a cartesian decomposition of the matrix concerned. In [12], the author extends some monotonicity type results known for positive semidefinite matrices to the case of almost definite matrices.

In the present work, we take a fresh look at almost definite matrices. We shall be concerned with real matrices. $A \in \mathbb{R}^{n\times n}$ is almost definite if $\langle x, Ax\rangle = 0 \Longrightarrow Ax = 0$, where $\langle \cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^n$. We prove many fundamental properties of such matrices. For instance, we show that for an almost definite matrix,

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a certain generalized inverse always exists (Lemma 3.1). The set of almost definite matrix is closed with respect to certain operations performed on them (Theorem 3.1). Statements on the entries of an almost definite matrix are proved (Theorem 3.2). For further results, we refer to the third section. In Section four, we prove new characterizations for almost definite matrices. In Section five, perhaps the most important section of this article, we collect certain results that hold for almost definite matrices. The point of view that is taken in this section is that these results are motivated by corresponding results for positive definite or positive semidefinite matrices. Among others, we would like to highlight Theorem 5.3 and Theorem 5.7 as stand out results. The final section shows how invertible almost definite matrices are related to certain matrix classes arising in the theory of linear complementarity problems.

2. Preliminary Notions. Let $\mathbb{R}^{m\times n}$ denote the set of all $m \times n$ matrices over the real numbers. For $A \in \mathbb{R}^{n\times n}$ let $S(A)$, $K(A)$, $R(A)$, $N(A)$, $\eta(A)$ and $rk(A)$ denote the symmetric part of A ($\frac{1}{2}(A+A^T)$), the skew-symmetric part of A ($\frac{1}{2}(A-A^T)$), the range space of A, the null space of A, the nullity of A and the rank of A. The Moore-penrose (generalized) inverse of a matrix $A \in \mathbb{R}^{m\times n}$ is the unique matrix $X \in \mathbb{R}^{n\times m}$ satisfying $A = AXA, X = XAX, (AX)^T = AX$ and $(XA)^T = XA$ and is denoted by $A^\dagger$. The group (generalized) inverse of a matrix $A \in \mathbb{R}^{n\times n}$, if it exists, is the unique matrix $X \in \mathbb{R}^{n\times n}$ satisfying $A = AXA, X = XAX$ and $AX = XA$ and is denoted by $A^\#$. A well known characterization for the existence of $A^\#$ is that $R(A) = R(A^2)$; equivalently, $N(A) = N(A^2)$. If A is nonsingular, then $A^{-1} = A^\dagger = A^\#$. Recall that $A \in \mathbb{R}^{n\times n}$ is called range-symmetric if $R(A) = R(A^T)$. If A is range-symmetric, then $A^\dagger = A^\#$. For more details on generalized inverses of matrices, we refer to [3].

$A \in \mathbb{R}^{n\times n}$ is called a positive semidefinite matrix if $x^TAx \geq 0$, for all $x \in \mathbb{R}^n$. $A \in \mathbb{R}^{n\times n}$ is called a negative semidefinite matrix if $-A$ is positive semidefinite. Let $\mathbb{R}^n_+$ denote the nonnegative orthant of $\mathbb{R}^n$, viz., $(x_i) = x \in \mathbb{R}^n_+$ means $x_i \geq 0$, $1 \leq i \leq n$.

In the rest of this section, we collect results that will be used in the sequel. The first result gives a formula for the Moore-Penrose inverse of a symmetric block matrix.

Theorem 2.1. *Let* $U = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$ *be a symmetric matrix. Suppose that* $R(B) \subseteq R(A)$ *and* $R(B^TA^\dagger) \subseteq R(F)$, *where* $F = D - B^TA^\dagger B$ *is the pseudo Schur complement of* A *in* U. *Then*

$$U^\dagger = \begin{pmatrix} A^\dagger + A^\dagger BF^\dagger B^TA^\dagger & -A^\dagger BF^\dagger \\ -F^\dagger B^TA^\dagger & F^\dagger \end{pmatrix}.$$

Proof. First, we observe that $B^TA^\dagger A = B^T$ (since $R(B) \subseteq R(A)$) and $A^\dagger B = A^\dagger BFF^\dagger$ (since $R(B^TA^\dagger) \subseteq R(F)$). Now, $R(B - BFF^\dagger) \subseteq N(A^\dagger) = N(A^T)$. Also,

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$R(B-BFF^\dagger)\subseteq R(B)\subseteq R(A)$ and so $BF^\dagger F=B$. We then have $DF^\dagger F=(F+B^TA^\dagger B)F^\dagger F=F+B^TA^\dagger BF^\dagger F=F+B^TA^\dagger B=D$. Also, $A^\dagger B+A^\dagger BF^\dagger B^TA^\dagger B-A^\dagger BF^\dagger D=A^\dagger B+A^\dagger BF^\dagger(B^TA^\dagger B-D)=A^\dagger B-A^\dagger BF^\dagger F=0$. If we set

$$X=\begin{pmatrix} A^\dagger+A^\dagger BF^\dagger B^TA^\dagger & -A^\dagger BF^\dagger \\ -F^\dagger B^TA^\dagger & F^\dagger \end{pmatrix}.$$

It then follows that

$$XU=\begin{pmatrix} A^\dagger+A^\dagger BF^\dagger B^TA^\dagger & -A^\dagger BF^\dagger \\ -F^\dagger B^TA^\dagger & F^\dagger \end{pmatrix}\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}=\begin{pmatrix} A^\dagger A & 0 \\ O & F^\dagger F \end{pmatrix}.$$

So, $(XU)^T=XU$. Also, $UXU=\begin{pmatrix} A & BF^\dagger F \\ B^T & DF^\dagger F \end{pmatrix}=\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}=U$. Further, $XUX=X$.

Finally,

$$UX=\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}\begin{pmatrix} A^\dagger+A^\dagger BF^\dagger B^TA^\dagger & -A^\dagger BF^\dagger \\ -F^\dagger B^TA^\dagger & F^\dagger \end{pmatrix}=\begin{pmatrix} AA^\dagger & 0 \\ 0 & FF^\dagger \end{pmatrix}$$

so that, $(UX)^T=UX$. Thus

$$U^\dagger=\begin{pmatrix} A^\dagger+A^\dagger BF^\dagger B^TA^\dagger & -A^\dagger BF^\dagger \\ -F^\dagger B^TA^\dagger & F^\dagger \end{pmatrix}.$$

□

REMARK 2.1. *If we drop the condition $R(B^TA^\dagger)\subseteq R(F)$ then the above formula for the Moore-Penrose inverse does not hold. Let $U=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then $R(B^TA^\dagger)\nsubseteq R(F)$ and $\begin{pmatrix} A^\dagger+A^\dagger BF^\dagger B^TA^\dagger & -A^\dagger BF^\dagger \\ -F^\dagger B^TA^\dagger & F^\dagger \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. But $U^\dagger=\frac{1}{4}U$.*

The formula for the Moore-Penrose inverse of a symmetric block matrix in terms of the pseudo Schur complement is given next. Its proof is similar to the previous result and is omitted.

THEOREM 2.2. *Let $U=\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$ be a symmetric matrix. Suppose that $R(B^T)\subseteq R(D)$ and $R(BD^\dagger)\subseteq R(G)$, where $G=A-BD^\dagger B^T$. Then*

$$U^\dagger=\begin{pmatrix} G^\dagger & -G^\dagger BD^\dagger \\ -D^\dagger B^TG^\dagger & D^\dagger+D^\dagger B^TG^\dagger BD^\dagger \end{pmatrix}.$$

The next result concerns range symmetry of a block matrix and a sufficient condition when the pseudo Schur complement is zero. While the former is proved in

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[11] (Real version of Theorem 3), the latter is stated in [4] (Real version of Corollary, pp.171).

Theorem 2.3. *Let* $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{n\times n}$ *with* $rk(U) = rk(A)$. *Then*
(*a*) U *is a range-symmetric matrix if and only if* A *is range-symmetric and* $CA^\dagger = (A^\dagger B)^T$.
(*b*) $F = 0$, *where* $F = D - CA^\dagger B$ *is the pseudo Schur complement of* A *in* U.

The following assertion on the Hadamard product is well known.

Theorem 2.4. *(Theorem 7.5.3, [8]) Let* $A, B \in \mathbb{C}^{n\times n}$.
(*a*) *If both* A *and* B *are positive semidefinite or negative semidefinite, then the Hadamard product* $A \circ B$ *is positive semidefinite.*
(*b*) *If* A *is positive semidefinite and* B *is negative semidefinite, then* $A \circ B$ *is negative semidefinite.*

We conclude this section with a lemma on linear equations.

Lemma 2.1. *(Corollary 2, [3]) Let* $A \in \mathbb{R}^{m\times n}$ *and* $b \in \mathbb{R}^m$. *Then the system* $Ax = b$ *is consistent if and only if* $AA^\dagger b = b$. *In such a case, the general solution is given by* $x = A^\dagger b + z$, *for some* $z \in N(A)$.

3. Basic Properties. In this section, we prove certain basic results on almost definite matrices. We show first, that if A is almost definite then $A^\#$ exists. We show that the set of almost definite matrices is closed with respect to the unary operations of the transpose of a matrix and Moore-Penrose inversion. These are included in Theorem 3.1. In Theorem 3.2, we make certain assertions on the entries of an almost definite matrix. We then briefly study a generalization of almost definiteness. We conclude the section with statements on the null space and the range space of the symmetric part of an almost definite matrix. This appears as Theorem 3.5.

First, we show that if A is almost definite, then $A^\#$ exists. In fact, we prove a little more.

Lemma 3.1. *Let* $A \in \mathbb{R}^{n\times n}$ *be almost definite. Then* A *is range symmetric.*

Proof. Let $A^T x = 0$. Then $\langle x, Ax\rangle = \langle x, A^T x\rangle = 0$ and by the almost definiteness of A it follows that $Ax = 0$. Hence $N(A^T) \subseteq N(A)$. A standard rank argument shows that these subspaces are equal. □

It now follows that if A is almost definite, then $A^\dagger = A^\#$. In the next result, we collect some preliminary results on almost definite matrices.

Theorem 3.1. *For* $A \in \mathbb{R}^{n\times n}$ *we have the following:*

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(a) *A is almost definite if and only if* A^T *is almost definite.*
(b) *A is almost definite if and only if* $A^\dagger$ *is almost definite.*
(c) *Let A be almost definite. Then* PAP^T *is almost definite for any* $P \in \mathbb{R}^{n\times n}$*. Conversely, let P be invertible. If* PAP^T *is almost definite, then A is almost definite.*
(d) *If A is almost definite, then so is its symmetric part* $S(A)$*. The converse is not true.*

Proof. In (a) and (b), it is clear that it suffices to prove just a one way implication.
(a): If $\langle x, A^T x\rangle = 0$, then $\langle Ax, x\rangle = 0$ and so by the almost definiteness of A we have $Ax = 0$. By Lemma 3.1, since A is range symmetric we then have $A^T x = 0$, showing that A^T is almost definite.
(b): Note that $R(A^\dagger) = R(A^T) = R(A)$. Let $\langle x, A^\dagger x\rangle = 0$. Set $y = A^\dagger x$, so that $x = Ay + z$ for some $z \in N(A^T)$, by Lemma 2.1. Then $y \in R(A)$ and so $0 = \langle x, A^\dagger x\rangle = \langle Ay + z, y\rangle = \langle Ay, y\rangle$. By the almost definiteness of A we have $Ay = 0$ so that $y \in N(A) \cap R(A)$. Thus $A^\dagger x = 0$, proving the almost definiteness of $A^\dagger$.
(c): Let A be almost definite and $\langle x, PAP^T x\rangle = 0$. Then $\langle P^T x, AP^T x\rangle = 0$ so that $AP^T x = 0$. Thus $PAP^T x = 0$, showing the almost definiteness of PAP^T. Conversely, suppose that PAP^T is almost definite, with P being invertible. Let $\langle y, Ay\rangle = 0$. There exists x such that $y = P^T x$. Thus, $0 = \langle P^T x, AP^T x\rangle = \langle x, PAP^T x\rangle$ and so by the almost definiteness of PAP^T, we then have $PAP^T x = 0$. Since P is invertible, we have $0 = AP^T x = Ay$, proving that A is almost definite.
(d): The proof follows from the observation that $\langle x, S(A)x\rangle = 2\langle x, Ax\rangle$. For the second part, let $A = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$. Then $S(A) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. It may be verified that $S(A)$ is almost definite, whereas A is not. $\square$

In the next result, we make certain assertions on the entries of an almost definite matrix. Since A is almost definite if and only A^T is almost definite, statements made about the columns of A have analogues for its rows.

Theorem 3.2. *Let A be almost definite. Then the following hold:*
(a) *All the column entries corresponding to any zero diagonal entry of A are zero. In particular, if A is invertible, then all its diagonal entries are non-zero.*
(b) *No non-zero column of A can be perpendicular to the corresponding column of* $AA^\#$*.*
(c) *If the sum of all the entries of A is zero, then each column sum of A is zero.*

Proof. (a): Let e^k be the kth column of the identity matrix. Let $b^k = Ae^k$ be a column of A such that $b_{kk} = 0$, where b_{kk} is the kth coordinate of b^k. Then $\langle Ae^k, e^k\rangle = 0$. By the almost definiteness of A, it now follows that $b^k = 0$. The second part follows from the fact that an invertible matrix cannot have a zero column.
(b): First observe that by Lemma 3.1, the group inverse $A^\#$ exists equals $A^\dagger$ and so by (b) of Theorem 3.1, it is almost definite. As in (a), set $A = (b^1, b^2, \ldots, b^n)$. Then each

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$b^k \in R(A)$ and $AA^{\#} = A^{\#}A = (A^{\#}b^1, A^{\#}b^2, \ldots, A^{\#}b^n)$. Note that the kth column of $AA^{\#}$ is $AA^{\#}e^k = A^{\#}Ae^k = A^{\#}b^k$. Let $0 \neq b^k$ be such that $\langle b^k, A^{\#}b^k\rangle = 0$. So, $A^{\#}b^k = 0$ so that $b^k \in N(A^{\#}) = N(A)$. However since the subspaces $R(A)$ and $N(A)$ are orthogonal complementary, this means that $b^k = 0$, a contradiction.
(c): Let $e \in \mathbb{R}^n$ be the vector with all entries 1. If the sum of all the entries of A is zero, then $\langle Ae, e\rangle = 0$. We then have $Ae = 0$ so that each row sum is zero. Since $N(A) = N(A^T)$, it follows that each column sum is zero. □

In what follows, we take a brief digression to consider another class of matrices which includes almost definite matrices as a subclass. We present a sufficient condition that guarantees when a matrix in this new class turns out to be almost definite. The precise definition is given next.

Definition 3.1. *$A \in \mathbb{R}^{n\times n}$ is called pseudo almost definite if*

$$\langle x, Ax\rangle = 0,\ x \in R(A^T) \Longrightarrow x = 0.$$

Unlike an almost definite matrix, a pseudo almost definite matrix is not necessarily range symmetric. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then it may be verified that A is pseudo almost definite matrix which is not range symmetric. Next, we prove a relationship between pseudo almost definite matrices and almost definite matrices.

Theorem 3.3. *Any almost definite $A \in \mathbb{R}^{n\times n}$ is pseudo almost definite. The converse is true if A is range symmetric.*

Proof. Let A be almost definite, $\langle x, Ax\rangle = 0$ and $x \in R(A^T)$. Then $Ax = 0$ and so $x = 0$, showing that A is pseudo almost definite. Conversely, suppose that A is pseudo almost definite and range symmetric. Let $\langle x, Ax\rangle = 0$. Consider the decomposition $x = x^1 + x^2$, where $x^1 \in R(A^T)$ and $x^2 \in N(A)$. Then $0 = \langle x, Ax\rangle = \langle (x^1+x^2), A(x^1+x^2)\rangle = \langle x^1, Ax^1\rangle + \langle x^2, Ax^1\rangle$, where we have used the fact that $Ax^2 = 0$. Since A is range symmetric, the subspaces $R(A)$ and $N(A)$ are orthogonal complementary and so $\langle x^2, Ax^1\rangle = 0$. Thus $\langle x^1, Ax^1\rangle = 0$ with $x^1 \in R(A^T)$. The pseudo almost definiteness of A then implies that $x^1 = 0$. We then have $x = x^2 \in N(A)$ and so $Ax = 0$, showing that A is almost definite. □

It is easy to see that if a matrix A is positive definite, then it is invertible and that the inverse is also positive definite. A not so well known result states that if a matrix is positive semidefinite, then its group inverse exists and is also positive semidefinite. This is a consequence of Theorem 2 [10], where it is shown that a positive semidefinite matrix is range symmetric. Let us also recall that in Theorem 3.1, the existence of the group inverse of a matrix is shown, if it is almost definite. In the next result, we show that if a matrix is pseudo almost definite or its symmetric part is almost

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definite, then its group inverse exists.

THEOREM 3.4. *Let $A \in \mathbb{R}^{n\times n}$. Suppose that A is either pseudo almost definite or $S(A)$ is almost definite. Then $A^{\#}$ exists.*

Proof. Suppose that A is pseudo almost definite. We show that $(A^T)^{\#}$ exists. It then follows that $A^{\#}$ exists, since we have the formula $(A^T)^{\#} = (A^{\#})^T$. Set $B = A^T$. We show that $N(B^2) = N(B)$. It suffices to show that $N(B^2) \subseteq N(B)$. Let $y = Bx$ and suppose that $By = 0$ so that $B^2x = 0$. Then $0 = \langle y, By\rangle = \langle y, A^Ty\rangle = \langle y, Ay\rangle$. Also $y = Bx = A^Tx \in R(A^T)$. Since A is pseudo almost definite, we then have $y = 0$ so that $Bx = 0$. Thus $N(B^2) = N(B)$, as required.

Next, let $S(A)$ be almost definite. Let $A^2x = 0$. We show that $Ax = 0$. Set $y = Ax$, so that $Ay = 0$. So $\langle y, S(A)y\rangle = \frac{1}{2}\langle y, (A + A^T)y\rangle = 0$, so that $S(A)y = 0$, since $S(A)$ is almost definite. This means that $Ay+A^Ty = 0$. Then $0 = A^Ty = A^TAx$ so that $Ax = 0$. This shows that $A^{\#}$ exists. □

In the last part of this section, we prove certain miscellaneous results for almost definite matrices.

REMARK 3.1. *Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$. Then A and B are almost definite matrix but $A+B$ is not almost definite. However, we have the following. Let A be almost definite and B be skew-symmetric matrix with $N(A) \subseteq N(B)$. Then $A + B$ is almost definite. For, let $x \in \mathbb{R}^n$ be such that $\langle x, (A + B)x\rangle = 0$. Now $0 = \langle x, (A + B)x\rangle = \langle x, Ax\rangle$, since B is skew-symmetric. Then $Ax = 0$, since A almost definite. Thus $(A + B)x = 0$, since $N(A) \subseteq N(B)$.*

A matrix $A \in \mathbb{C}^{n\times n}$ is called *almost positive definite* (see [12] and the references cited therein) if A is almost definite and satisfies the inequality $Re\langle x, Ax\rangle \geq 0$ for all $x \in \mathbb{C}^n$. We recall a couple of results from [12], whose generalizations we shall be studying next. Among other results, it is shown that if $A - B$ is positive semi definite and B is almost positive definite then A is almost positive definite. It is also demonstrated that if $A - B$ is almost positive definite, B is range hermitian and $N(A) \subseteq N(B)$, then $N(A) = N(H(B))$ if and only if $R(A) = R(H(B))$. Here, range hermitianness of B means that $R(B) = R(B^*)$ and the hermitian part $H(B)$ of a matrix B is defined by $\frac{1}{2}(B + B^*)$. The following observation was also made: If $A \in \mathbb{R}^{n\times n}$ is range symmetric, then $N(A) \subseteq N(S(A))$. A stronger conclusion could be drawn if A is almost definite. We summarize these in the next result. We continue to restrict our attention to real matrices.

THEOREM 3.5. *The following hold:*
(a) Let A be almost definite. Then $N(A) = N(S(A))$ and $R(A) = R(S(A))$.
(b) Let $A - B$ be almost definite, B be range symmetric and $N(A) \subseteq N(B)$. Then A

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is range symmetric. If, in addition, $R(A) \subseteq R(S(B))$, then $N(A) = N(S(B))$.

Proof. (a): Let $Ax = 0$. Then $A^T x = 0$ so that $x \in N(S(A))$. So, $N(A) \subseteq N(S(A))$. Note that this inclusion is true if A is just range symmetric. On the other hand, let $(A + A^T)x = 0$. Then $0 = \langle x, (A + A^T)x\rangle = 2\langle x, Ax\rangle$ and so $Ax = 0$. This shows that $N(S(A)) \subseteq N(A)$. The second equality follows by taking orthogonal complements and by using the range symmetry of A.
(b): Since B is range symmetric, as remarked in the proof of (a), we have $N(B) \subseteq N(S(B))$. So, $N(A) \subseteq N(S(B))$. The inclusion $R(A) \subseteq R(S(B))$ yields $N(S(B)) \subseteq N(A)$, where we have used the range symmetry of A and the symmetry of $S(B)$. $\square$

4. Characterizations. In this section, we present certain necessary and sufficient conditions for almost definiteness. In the first result, Theorem 4.1, equivalent conditions are given for the symmetric part $S(A)$ of a matrix A to be almost definite. In Theorem 4.2, we characterize the almost definiteness of A. The third assertion, viz., Theorem 4.3 characterizes symmetric almost definite matrices and the fourth result, Theorem 4.5 concerns inferences on the factors of a full-rank factorization. The last result, Theorem 4.6, deals with the almost definiteness of $I - A^T A$.

We begin with the following assertion, motivated by Theorem 1, [9].

THEOREM 4.1. *Let $A \in \mathbb{R}^{n\times n}$. Then the following statements are equivalent:*
(a) *$S(A)$ is almost definite.*
(b) *$rk[S(A)X] \leq rk(X^T AX)$ for all $X \in \mathbb{R}^{n\times n}$.*
(c) *$x^T Ax = 0 \Rightarrow S(A)x = 0$ for all $x \in \mathbb{R}^n$.*

Proof. $(a) \Rightarrow (b)$: Let $X \in \mathbb{R}^{n\times n}$ and $u \in N(X^T AX)$. Then $X^T AXu = 0$ so that $u^T X^T AXu = 0$. So, $\langle u, X^T S(A)Xu\rangle = 0$. Thus $S(A)Xu = 0$, since $S(A)$ is almost definite. Thus $N(X^T AX) \subseteq N(S(A)X)$, which in turn gives $rk[S(A)X] = n - \eta(S(A)X) \leq n - \eta(X^T AX) \leq rk(X^T AX)$, as required.
$(b) \Rightarrow (c)$: Let $x \in \mathbb{R}^n$ be such that $x^T Ax = 0$. Let $X = (x, 0, 0, \dots, 0) \in \mathbb{R}^{n\times n}$. Then $X^T AX = 0$. Thus $rk(S(A)X) = 0$ so that $S(A)x = 0$.
$(c) \Rightarrow (a)$: Let $x \in \mathbb{R}^n$ such that $x^T S(A)x = 0$. Then $\langle x, Ax\rangle = \frac{1}{2}(\langle x, Ax\rangle + \langle x, A^T x\rangle) = \frac{1}{2}\langle x, S(A)x\rangle = 0$. Thus $S(A)x = 0$, showing that $S(A)$ is almost definite. $\square$

Next, almost definiteness of a matrix is characterized. This is motivated by Corollary 2, [9].

THEOREM 4.2. *Let $A \in \mathbb{R}^{n\times n}$. Then the following statements are equivalent:*
(a) *$N(AX) = N(X^T AX)$ for all $X \in \mathbb{R}^{n\times n}$.*
(b) *$rk(AX) = rk(X^T AX)$ for all $X \in \mathbb{R}^{n\times n}$.*
(c) *A is almost definite.*

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Proof. $(a) \Rightarrow (b)$: This follows from a standard argument using the rank and the nullity of a matrix.
$(b) \Rightarrow (c)$: Let $x \in \mathbb{R}^n$ such that $x^T Ax = 0$. Let $X = (x, 0, 0, \dots, 0) \in \mathbb{R}^{n\times n}$. Then $X^T AX = 0$ so that $rk(AX) = 0$ and so $AX = 0$. Thus $Ax = 0$, showing that A is almost definite.
$(c) \Rightarrow (a)$: Let $X \in \mathbb{R}^{n\times n}$ and $u \in N(X^T AX)$. Then $X^T AXu = 0$ so that $u^T X^T AXu = 0$. Thus $AXu = 0$, since A is almost definite. So, $N(X^T AX) \subseteq N(AX)$. Thus $N(AX) = N(X^T AX)$, for all $X \in \mathbb{R}^{n\times n}$. ◻

For the next result, we use the following notation: Let $A \in \mathbb{R}^{n\times n}$. If either $\langle x, Ax\rangle \geq 0$ for all $x \in \mathbb{R}^n$ or $\langle x, Ax\rangle \leq 0$ for all $x \in \mathbb{R}^n$, we say that $\pm A$ is positive semidefinite. We have the following result:

Theorem 4.3. *Let A be a symmetric matrix. Then A is almost definite if and only if $\pm A$ is positive semidefinite.*

Proof. **Necessity**: The proof is by contradiction. Suppose that A is symmetric and almost definite. Assume that there exist $x, y \in \mathbb{R}^n$ such that

$$\langle x, Ax\rangle < 0 \text{ and } \langle y, Ay\rangle > 0$$

By the continuity of the function $\phi(u) := \langle u, Au\rangle, u \in \mathbb{R}^n$, it follows that there exists $z \in \mathbb{R}^n$ such that $z = \lambda x+(1-\lambda)y$ and $\langle z, Az\rangle = 0$. By the almost definiteness of A we then have $Az = 0$, so that $Ax = \alpha Ay$, for some $\alpha < 0$. So, $0 > \langle x, Ax\rangle = \alpha\langle x, Ay\rangle$ so that $\langle x, Ay\rangle > 0$. On the ther hand, we have $0 > -\langle y, Ay\rangle = -\frac{1}{\alpha}\langle y, Ax\rangle = -\frac{1}{\alpha}\langle x, Ay\rangle$ (using the symmetry of A) so that $\langle x, Ay\rangle < 0$, a contradiction. Thus $\pm A$ is positive semidefinite.
Sufficiency: Suppose that A is positive semidefinite, without loss of generality. Since A is symmetric, there exists a diagonal matrix D with nonnegative diagonal entries such that $A = UDU^T$, where U is an orthogonal matrix. It is easy to see that D is almost definite and so by (c) of Theorem 3.1, it follows that A is almost definite. ◻

Before proceeding with the other characterizations, let us present another sufficient condition for $\pm A$ to be positive semidefinite.

Theorem 4.4. *Let $A \in \mathbb{R}^{n\times n}$ be an invertible almost definite matrix. Then $\pm A$ is positive definite.*

Proof. Suppose that A is an invertible almost definite matrix and there exist $x, y \in \mathbb{R}^n$ such that $\langle Ax, x\rangle < 0$ and $\langle Ay, y\rangle > 0$. Then there exists $z \in \mathbb{R}^n$ such that $z = \lambda x + (1 - \lambda)y$ and $\langle Az, z\rangle = 0$. By the almost definiteness of A and by its invertiblility, we have $z = 0$, so that $x = \alpha y$, for some $\alpha < 0$. Then $0 > \langle Ax, x\rangle = \alpha^2\langle Ay, y\rangle$ so that $\langle Ay, y\rangle < 0$, a contradiction. ◻

It is well known that any non-zero matrix $A \in \mathbb{C}^{n\times n}$ of rank r can be factorized

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as $A = BC$, where $B \in \mathbb{C}^{n\times r}$ and $C \in \mathbb{C}^{r\times n}$ with $rank(B) = rank(C) = r$ [3]. Such a factorization is referred to as a full-rank factorization due to the reason that the factors B and C have full-rank. Let $A = BC$ be a full-rank factorization of A. The following result yields information about the factors B and C, given that A is almost definite. This characterization is already known (Theorem 5.1, [5]) and is listed among several other statements. It is included with a proof, for the sake of completeness.

THEOREM 4.5. *Let $A \in \mathbb{R}^{n\times n}$ and $A = BC$ be a full-rank factorization. Set $S = \{x \in \mathbb{R}^n : \langle x, Ax\rangle = 0\}$. Then A is almost definite if and only if $S = N(C) = N(B^T)$.*

Proof. Let us note that the factors B and C satisfy the equalities $R(B) = R(A)$ and $N(C) = N(A)$. The first formula in turn yields, $N(B^T) = N(A^T)$. Let A be almost definite. Then $S = N(A) = N(C)$ and $N(B^T) = N(A^T) = N(A) = S$. Conversely, suppose that $\langle x, Ax\rangle = 0$ so that $x \in S$. Then $Cx = 0$ and so $Ax = 0$, proving the almost definiteness of A. □

In the next result, the notation $\| x \|$ stands for the Euclidean vector norm (or the 2-norm) of $x \in \mathbb{R}^n$ and for $A \in \mathbb{R}^{n\times n}$, $\| A \|$ denotes the matrix norm induced by the Euclidean vector norm on $\mathbb{R}^n$. The next result is an analogue of a well known result which states that $I - A^TA$ is positive definite if and only if A is a contraction ($\| A \| < 1$).

THEOREM 4.6. *Let $A \in \mathbb{R}^{n\times n}$. Then $I - A^TA$ is almost definite if and only if either $\| A \| \leq 1$ or A^{-1} exists and $\| A^{-1} \| \leq 1$.*

Proof. **Necessity**: Let $I - A^TA$ be almost definite. Then by Theorem 4.3, it follows that $\pm(I - A^TA)$ is positive semidefinite. First, let $I - A^TA$ be positive semidefinite so that for all $x \in \mathbb{R}^n$, $\langle x, x\rangle - \langle x, A^TAx\rangle \geq 0$. Then $\| Ax \| \leq \| x \|$ so that $\| A \| \leq 1$, proving the first part. On the other hand if $I - A^TA$ is negative semidefinite then $\| x \| \leq \| Ax \|$. Now, if $Ax = 0$, then $x = 0$ and so A^{-1} exists. It is easy to see that $\| A^{-1} \| \leq 1$.
Sufficiency: Let $\| A \| \leq 1$. Then for all $x \in \mathbb{R}^n$, $\| Ax \| \leq \| x \|$ so that $\langle x, x\rangle - \langle x, A^TAx\rangle \geq 0$. Thus $I - A^TA$ is positive semidefinite. On ther other hand, if A^{-1} exists and $\| A^{-1} \| \leq 1$, then for all $x \in \mathbb{R}^n$ we have $\| A^{-1}x \| \leq \| x \|$ so that $\| Ax \| \geq \| x \|$. Thus $\langle x, x\rangle - \langle x, A^TAx\rangle \leq 0$. Thus $\pm(I - A^TA)$ is positive semidefinite. Since $I - A^TA$ is symmetric, by Theorem 4.3, $I - A^TA$ is almost definite. □

In the next result, we consider almost definiteness of a block matrix. A more detailed study on such matrices is carried out in the next section (Theorem 5.3).

THEOREM 4.7. *Let $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{R}^{n\times n}$ with $rk(U) = rk(A)$. Then U is*

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almost definite if and only if A is almost definite with $CA^{\dagger} = (A^{\dagger}B)^T$.

Proof. Suppose that U is almost definite. Then A is also almost definite and U is range-symmetric. By Theorem 2.3, $CA^{\dagger} = (A^{\dagger}B)^T$.

Conversely, Suppose that A is almost definite and $CA^{\dagger} = (A^{\dagger}B)^T$. The condition $rk(U) = rk(A)$ gives $D = CA^{\dagger}B$, since the pseudo Schur complement is zero (by Theorem 2.3). It may be verified that $U = P \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} P^T$, where $P = \begin{pmatrix} I & 0 \\ CA^{\dagger} & I \end{pmatrix}$ is invertible. Since the block matrix $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ is almost definite, by (c) Theorem 3.1, we conclude that U is almost definite. ∎

5. Comparisons with Positive Semidefinite Matrices. The objective of this section is to prove certain results for almost definite matrices that are analogous to these of positive definite or positive semidefinite matrices. This perspective does not seem to have been taken in the earlier works. Before getting to the precise details, let us briefly outline the important results that have been proved in this section. In Theorem 5.3, we consider a block almost definite matrix and draw several conclusions. Among these, the almost definiteness of the principal diagonal blocks and the pseudo Schur complements are proved. In Theorem 5.4, we obtain an extension of a similar result for positive semidefinite matrices proved in Proposition of [1]. Finally, an analogue of a certain result for the Hadamard product of a positive definite matrix and its inverse [2] is obtained in Theorem 5.7. To summarize, we reiterate that each of the results of this section could be thought of an analogue of a corresponding result for positive semidefinite or positive definite matrices.

First, we start with a simple result.

THEOREM 5.1. *(Real version of Theorem 1, [15]) Let A, $B \in \mathbb{R}^{n\times n}$ such that $R(I - A^TB) \subseteq R(I - A^TA)$. Then*
$(I - B^TB) - (I - B^TA)(I - A^TA)^{\dagger}(I - A^TB) = -(A - B)^T(I - AA^T)(A - B)$.

THEOREM 5.2. *Let A, $B \in \mathbb{R}^{n\times n}$ such that, $I - AA^T$ is almost definite and $R(I - A^TB) \subseteq R(I - A^TA)$. Then the matrix $(I - B^TA)(I - A^TA)^{\dagger}(I - A^TB) - (I - B^TB)$ is almost definite.*

Proof. Let $I - AA^T$ be almost definite. Now by Theorem 5.1, we have $(I - B^TB) - (I - B^TA)(I - A^TA)^{\dagger}(I - A^TB) = -(A - B)^T(I - AA^T)^{\dagger}(A - B)$. By (c) Theorem 3.1, $(A - B)^T(I - AA^T)^{\dagger}(A - B)$ is almost definite so its negative is also almost definite. The conclusion now follows. ∎

We turn our attention to certain inheritance properties of almost definite matrices.

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Let us recall that if $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a symmetric positive semidefinite matrix, then the principal diagonal blocks A and D are also positive semidefinite. It is also known that if A is invertible, then the Schur complement $F = D - CA^{-1}B$ is also positive semidefinite. In the next result, we obtain similar results for almost definite matrices, among other results.

THEOREM 5.3. *Let A and D be square matrices and $U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be an almost definite block matrix. We have the following:*
(a) $N(A) \subseteq N(C), R(B) \subseteq R(A)$ and the principal diagonal blocks, A and D are almost definite.
(b) The matrix $U_P = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$ is almost definite. (The subscript P signifies that U_P is permutationally equivalent to U). We also have the inclusions $R(C) \subseteq R(D)$ and $N(D) \subseteq N(B)$.
(c) The pseudo Schur complement $F = D - CA^{\dagger}B$ and the complementary pseudo Schur complement are almost definite.
(d) Let A, D be symmetric and $C = B^T$. Then the matrices $V_1 = \begin{pmatrix} BD^{\dagger}B^T & B \\ B^T & D \end{pmatrix}$ and $V_2 = \begin{pmatrix} A & B \\ B^T & B^TA^{\dagger}B \end{pmatrix}$ are almost definite.
(e) If $A = 0$ or $D = 0$, then $B = 0$ and $C = 0$.

Proof. (a): Let $Ax = 0$ and set $z = \begin{pmatrix} x \\ 0 \end{pmatrix}$. Then $\langle z, Uz \rangle = 0$ so that $\begin{pmatrix} Ax \\ Cx \end{pmatrix} = Uz = 0$. This shows that $Cx = 0$ and so $N(A) \subseteq N(C)$. Since U is almost definite if and only if so is U^T (by (a) of Theorem 3.1), it now follows that $N(A^T) \subseteq N(B^T)$, which in turn is the same as $R(B) \subseteq R(A)$, proving the second inclusion.

If $\langle x, Ax \rangle = 0$, then by setting z as above, it follows that $\langle z, Uz \rangle = \langle x, Ax \rangle = 0$. By the almost definiteness of U we have $Ax = 0$, proving the almost definiteness of A. The proof for D is entirely similar and is skipped.

(b): Let $P = \begin{pmatrix} 0 & I_1 \\ I_2 & 0 \end{pmatrix}$, where I_1 and I_2 are identity matrices of appropriate orders. Then $U_P = PUP^T$. Thus U_P is almost definite, since U is almost definite. The inclusions $R(C) \subseteq R(D)$ and $N(D) \subseteq N(B)$ follow from (a).

(c): Let us now show that the pseudo Schur complement F is almost definite. Let $\langle x, Fx \rangle = 0$. Set $w = \begin{pmatrix} -A^{\dagger}Bx \\ x \end{pmatrix}$. Then $Uw = \begin{pmatrix} 0 \\ Fx \end{pmatrix}$, so that

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$\langle w, Uw\rangle = \langle x, Fx\rangle = 0$. Here, we have used the fact that $AA^\dagger B = B$, since $R(B) \subseteq R(A)$, by (a). By the almost definiteness of U, we then have $Uw = 0$. This means that $Fx = 0$, showing that F is almost definite. The proof for the almost definiteness of G follows, by appealing to (b).

(d): Let $z = \begin{pmatrix} x \\ y \end{pmatrix}$. Then

$$\langle z, V_1 z\rangle = \langle x, BD^\dagger B^T x\rangle + \langle y, B^T x\rangle + \langle x, By\rangle + \langle y, Dy\rangle.$$

Let $w = \begin{pmatrix} 0 \\ D^\dagger B^T x + y \end{pmatrix}$. Then

$$\langle w, Uw\rangle = \langle x, BD^\dagger DD^\dagger B^T x\rangle + \langle y, DD^\dagger B^T x\rangle + \langle x, BD^\dagger Dy\rangle + \langle y, Dy\rangle$$

By (b), we have $R(B^T) \subseteq R(D)$ (using the symmetry of D). Again, since D is symmetric, it commutes with its Moore-Penrose inverse $D^\dagger$. We then have $V_1 z = Uw$. If $\langle z, V_1 z\rangle = 0$, then

$$\langle w, Uw\rangle = \langle x, BD^\dagger B^T x\rangle + \langle y, B^T x\rangle + \langle x, By\rangle + \langle y, Dy\rangle = \langle z, V_1 z\rangle = 0.$$

By the almost definiteness of U, it then follows that $V_1 z = Uw = 0$, showing the almost definiteness of V_1. Similarly we can show that V_2 is almost definite.

(e): In particular, the diagonals of A and D are zero. By Theorem 3.2 (a), it follows that $B = 0$ and $C = 0$. $\square$

We have the following consequence of Theorem 5.3.

Corollary 5.1. *Let $U \in \mathbb{R}^{n\times n}$ with rank r. Then U is almost definite if and only if every principal submatrix of rank r is almost definite.*

Proof. Let A be a principal submatrix of U of rank r. Then there exists a permutation matrix P such that $V = PUP^T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ for some matrices B, C and D, with D being a square matrix. Since U is almost definite, by (c) Theorem 3.1, V is almost definite. By (a) Theorem 5.3, it now follows that A is almost definite. The converse is easy to see. $\square$

To motivate the next result, let us recall the result of [1] (Proposition 1). For a block matrix M, for the sake of convenience, we denote its subblocks with the subscript M. In this notation, let $U = \begin{pmatrix} A_U & B_U \\ B_U^T & D_U \end{pmatrix}$ be positive semidefinite, where A_U and D_U are symmetric matrices. Let $V = U^\dagger = \begin{pmatrix} A_V & B_V \\ B_V^T & D_V \end{pmatrix}$ be partitioned

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conformably with U. If $rk(U) = rk(A_U) + rk(D_U)$, then $A_V - A_U^\dagger$ is positive semidefinite. An analogue of such a condition is true for almost definite matrices. However, we provide a different sufficient condition. This is given next.

Theorem 5.4. *Let* $U = \begin{pmatrix} A_U & B_U \\ B_U^T & D_U \end{pmatrix}$ *be an almost definite matrix, where* A_U *and* D_U *are symmetric matrices. Let* $R(B_U^T A_U^\dagger) \subseteq R(F_U)$, *where* $F_U = D_U - B_U^T A_U^\dagger B_U$ *is the pseudo Schur complement of* A_U *in* U. *Suppose that* $U^\dagger = V = \begin{pmatrix} A_V & B_V \\ B_V^T & D_V \end{pmatrix}$. *Then* $A_V - A_U^\dagger$ *is almost definite.*

Proof. Let U be almost definite. Then $R(B_U) \subseteq R(A_U)$, by (a) Theorem 5.3. By Theorem 2.1,

$$U^\dagger = \begin{pmatrix} A_U^\dagger + A_U^\dagger B_U F_U^\dagger B_U^T A_U^\dagger & -A_U^\dagger B_U F_U^\dagger \\ -F_U^\dagger B_U^T A_U^\dagger & F_U^\dagger \end{pmatrix}.$$

Comparing the two expressions for $U^\dagger$, we have, $A_V - A_U^\dagger = A_U^\dagger B_U F_U^\dagger (A_U^\dagger B_U)^T$. The almost definiteness of U implies that F_U and $F_U^\dagger$ are almost definite (by (c) of Theorem 5.3 and (b) of Theorem 3.1). By (c) of Theorem 3.1, it now follows that $A_V - A_U^\dagger$ is almost definite. ☐

Using a certain unitary equivalence of a range symmetric matrix, we prove the next result.

Theorem 5.5. *Let* $A \in \mathbb{R}^{n\times n}$ *with* $r = rank(A)$. *Then* A *is almost definite if and only if there exists an invertible almost definite matrix* $C \in \mathbb{R}^{r\times r}$ *such that* $V^T C V = A$, *where* $V \in \mathbb{R}^{r\times n}$ *satisfies* $VV^T = I$.

Proof. Since A is range symmetric, we have the orthogonal direct sum decomposition $\mathbb{R}^n = R(A) \oplus N(A)$ [3]. This means that there exists an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ and an invertible $C \in \mathbb{R}^{r\times r}$ such that

$$A = Q \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} Q^T.$$

By (c) Theorem 3.1 and (a) Theorem 5.3, it follows that C is almost definite. Let Q be partitioned as $[Q_1, Q_2]$, where $Q_1 \in \mathbb{R}^{n\times r}$ and $Q_2 \in \mathbb{R}^{r\times n}$. Set $V = Q_1^T$. Then $V^T C V = A$. Since Q is an orthogonal matrix, we have $VV^T = Q_1^T Q_1 = I$.

Conversely, let $A = V^T C V$ with $V \in \mathbb{R}^{r\times n}$ satisfying $VV^T = I$. Let $\langle x, Ax\rangle = 0$. Then $\langle x, V^T C V x\rangle = 0$ so that $\langle Vx, CVx\rangle = 0$. By the almost definiteness of C it then follows that $Vx = 0$, which in turn implies that $Ax = 0$, showing the almost definiteness of A. ☐

In the last part of this section, we consider the Hadamard product of almost

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definite matrices. The Schur (or Hadamard) product of two matrices $A = (a_{ij})$, $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ is denoted by $A \circ B$ and defined as $A \circ B = (a_{ij}b_{ij}) \in \mathbb{R}^{n\times n}$. To motivate our result, let us recall that in [2], it was proved that if A is a symmetric positive definite matrix then $A \circ A^{-1} \geq I$. In other words, $A \circ A^{-1} - I \geq 0$. Here, for $A, B \in \mathbb{C}^{n\times n}$ $A \geq B$ denotes that $A - B$ is hermitian positive semidefinite. We obtain an analogue in Theorem 5.7 for symmetric almost definite matrices.

THEOREM 5.6. *Let $A, B \in \mathbb{R}^{n\times n}$ be two symmetric almost definite matrices. Then $A \circ B$ is almost definite.*

Proof. Suppose that $A, B \in \mathbb{R}^{n\times n}$ are two symmetric almost definite matrices. By Theorem 4.3, $\pm A$ and $\pm B$ are positive semidefinite. Then by Theorem 2.4, $\pm(A \circ B)$ is positive semidefinite. Thus by theorem 4.3, $A \circ B$ is almost definite. $\square$

THEOREM 5.7. *Let $U, V \in \mathbb{R}^{n\times n}$ be two symmetric almost definite matrices, and suppose that U, $U^\dagger$, V and $V^\dagger$ are conformally partitioned as follows:*

$$U = \begin{pmatrix} A_U & B_U \\ B_U^T & D_U \end{pmatrix}, U^\dagger = \begin{pmatrix} \tilde{A}_U & \tilde{B}_U \\ \tilde{B}_U^T & \tilde{D}_U \end{pmatrix}$$

$$V = \begin{pmatrix} A_V & B_V \\ B_V^T & D_V \end{pmatrix}, V^\dagger = \begin{pmatrix} \tilde{A}_V & \tilde{B}_V \\ \tilde{B}_V^T & \tilde{D}_V \end{pmatrix}.$$

Let $R(B_U^T A_U^\dagger) \subseteq R(F)$, $R(B_V D_V^\dagger) \subseteq R(G_V)$, where $F_U = D_U - B_U^T A_U^\dagger B_U$ and $G_V = A_V - B_V D_V^\dagger B_V^T$. Then

$$U \circ V - \begin{pmatrix} A_U \circ \tilde{A}_V^\dagger & 0 \\ 0 & \tilde{D}_U^\dagger \circ D_V \end{pmatrix}$$

is almost definite. In particular the matrix,

$$U \circ U^\dagger - \begin{pmatrix} A_U \circ A_U^\dagger & 0 \\ 0 & \tilde{D}_U \circ D_U^\dagger \end{pmatrix}$$

is almost definite.

Proof. Let U and V be almost definite. Then $R(B_U) \subseteq R(A_U)$, by (a) Theorem 5.3. Now by Theorem 2.1, $F_U = D_U - B_U^T A_U^\dagger B_U = \tilde{D}_U^\dagger$ so that $D_U - \tilde{D}_U^\dagger = B_U^T A_U^\dagger B_U$. Similarly, by Theorem 2.2, $A_V - \tilde{A}_V^\dagger = B_V D_V^\dagger B_V^T$. By (d) Theorem 5.3,

$$N = \begin{pmatrix} A_U & B_U \\ B_U^T & D_U - \tilde{D}_U^\dagger \end{pmatrix} \text{ and } P = \begin{pmatrix} A_V - \tilde{A}_V^\dagger & B_V \\ B_V^T & D_V \end{pmatrix}$$

are almost definite. Thus by Theorem 5.6, $N \circ P$ is almost definite. Hence

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$$U \circ V - \begin{pmatrix} A_U \circ \tilde{A}_V^\dagger & 0 \\ 0 & \tilde{D}_U^\dagger \circ D_V \end{pmatrix}$$

is almost definite. To prove the second part, set $V = U^\dagger$. □

6. Relationships with Other Classes of Matrices. We conclude the discussion of almost definite matrices by studying the relationship between almost definite matrices and certain other classes of matrices, relevant especially in connection with linear complementarity problems. Let us recall the following classes of matrices. For more details on these we refer to the excellent book [6].

For a given $q \in \mathbb{R}^n$ and a matrix $A \in \mathbb{R}^{n\times n}$, the linear complementarity problem, abbreviated $LCP(q, A)$, is to find a $z \in \mathbb{R}^n_+$ such that $q + Az \geq 0$ and $z^T(q + Az) = 0$. A vector z is called a feasible solution for $LCP(q, A)$, if $z \geq 0$ and $q + Az \geq 0$. In such a case, we say that $LCP(q, A)$ is feasible. The solution set of LCP(q, A) is denoted by SOL(q, A). $A \in \mathbb{R}^{n\times n}$ is called a Q-matrix if LCP(q, A) has a solution for all $q \in \mathbb{R}^n$. For instance, it is known that a nonnegative matrix A is a Q-matrix if and only all its diagonal entries are positive. Let us turn to the next class of matrices. A is called a P-matrix if all its principal minors are positive. It can be shown that any P-matrix is a Q-matrix. One of the most fundamental results in the theory of linear complementarity problems states that A is a P-matrix if and only if $LCP(q, A)$ has a unique solution for all $q \in \mathbb{R}^n$. It is also very well known that for a matrix A whose off-diagonal entries are nonpositive, these two notions are equivalent. We consider a more general class next. A is called a P_0-matrix if all the principal minors of A are nonnegative. A close relationship between a P-matrix and a P_0-matrix exists. It is the statement that A is a P_0-matrix if and only for every $\epsilon > 0$, the matrix $A + \epsilon I$ is a P-matrix. $A \in \mathbb{R}^{n\times n}$ is called an R_0-matrix if SOL$(0, A) = \{0\}$. To state a result for an R_0-matrix, let us mention that A is an R_0-matrix if and only if the solution set of $LCP(q, A)$ is bounded for all $q \in \mathbb{R}^n$. Let us consider another class, a more general class then the previous one. $A \in \mathbb{R}^{n\times n}$ is called an R-matrix if for every $\alpha \geq 0$ and for some $q > 0$ the problem $LCP(\alpha q, A)$ has only one solution (namely, zero). It is a fact that if A is both a P_0-matrix and an R_0-matrix, then A is an R-matrix. It is also known that any R-matrix is a Q-matrix. $A \in \mathbb{R}^{n\times n}$ is called an S-matrix if there exists $z > 0$ such that $Az > 0$. It can be proved that any P-matrix is an S-matrix. It is rather well known that A is an S-matrix if and only if $LCP(q, A)$ is feasible for all $q \in \mathbb{R}^n$. $A \in \mathbb{R}^{n\times n}$ is called a copositive matrix if $x^T Ax \geq 0$, for all $x \in \mathbb{R}^n_+$. $A \in \mathbb{R}^{n\times n}$ is called a copositive-star matrix if A is copositive and for all $x \in \mathbb{R}^n_+$ with $x^T Ax = 0$, $Ax \geq 0$ implies $A^T x \leq 0$. A result for copositive-star matrices is given in the next result.

Theorem 6.1. *(Corollary 3.8.14, [6]) Let $A \in \mathbb{R}^{n\times n}$ be a copositive-star matrix. Then the following statements are equivalent:*

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(*a*) *A is an S-matrix.*
(*b*) *A is an R_0-matrix.*
(*c*) *A is a Q-matrix.*

THEOREM 6.2. *(Theorem 3.9.22, [6]) Let $A \in \mathbb{R}^{n\times n}$ be a P_0-matrix. Then the following statements are equivalent:*
(*a*) *A is an R_0-matrix.*
(*b*) *A is an R-matrix.*
(*c*) *A is a Q-matrix.*

We have the following result for certain subclasses of almost definite matrices.

THEOREM 6.3. *Let $A \in \mathbb{R}^{n\times n}$ be an invertible almost definite matrix. Then the following hold:*
(*a*) *If A is a copositive matrix, then A is an S-matrix and a Q-matrix, as well.*
(*b*) *If A is a P_0-matrix, then A is both an R-matrix and a Q-matrix.*

Proof. We prove that A is an R_0-matrix. Let $\langle x, Ax\rangle = 0$. Then we have in fact, $Ax = 0$, since A is almost definite and so $x = 0$, since A is invertible. Hence, the inequalities $x \geq 0, Ax \geq 0$ together with the complementarity condition $\langle x, Ax\rangle = 0$, imply that $x = 0$ and so A is an R_0-matrix.
(a): Let A be a copositive matrix. Let $x \geq 0, Ax \geq 0$ and $\langle x, Ax\rangle = 0$. Then $Ax = 0$ and since A is range symmetric, it follows that $A^T x = 0$. This proves that A is a copositive-star matrix. By Theorem 6.1, it now follows that A is an S-matrix and also a Q-matrix.
(b): As A is an R_0-matrix, the proof follows by appealing to Theorem 6.2. □

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