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ON THE INVERSE OF A CLASS OF BIPARTITE GRAPHS WITH UNIQUE PERFECT MATCHINGS*

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Abstract. Let G be a simple, undirected graph and $G_{\rm w}$ be the positive weighted graph obtained from G by giving weights to its edges using the positive weight function w. A weighted graph $G_{\rm w}$ is said to be nonsingular if its adjacency matrix $A(G_{\rm w})$ is nonsingular. Let $\mathcal{G}$ denote the class of connected, unweighted, bipartite graphs G with a unique perfect matching $\mathcal{M}$ such that $G/\mathcal{M}$ (the graph obtained by contracting the matching edges in G) is bipartite. Similarly, let $\mathcal{G}_{\rm w}$ denote the class of connected, weighted, bipartite graphs $G_{\rm w}$ with a unique perfect matching such that the underlying unweighted graph $G \in \mathcal{G}$. These graphs are known to be nonsingular. In (*Inverses of trees*, Combinatorica, 5(1):33–39, 1985), Godsil showed that if $G \in \mathcal{G}$, then $A(G)^{-1}$ is signature similar to a nonnegative matrix, that is, there exists a diagonal matrix D with diagonal entries ± 1 such that $DA(G)^{-1}D$ is nonnegative. The graph associated to the matrix $DA(G)^{-1}D$ is called the inverse of G and it is denoted by G^+. The graph G^+ is an undirected, weighted, connected, bipartite graph with a unique perfect matching. Notice that unweighted trees which are nonsingular are contained inside the class $\mathcal{G}$.

In (*On reciprocal eigenvalue property of weighted trees*, Linear Algebra and its Applications, 438:3817–3828, 2013), Neumann and Pati have characterized graphs that occur as inverses of nonsingular, unweighted trees. We generalize this result and constructively characterize the class of weighted graphs which can occur as the inverse of any graph in $\mathcal{G}$. We also show that for a graph $G \in \mathcal{G}$, the inverse $G^+ \in \mathcal{G}$ if and only if $G \cong G^+$ (isomorphic).

A weighted graph $G_{\rm w}$ is said to have the property R if for each eigenvalue λ of $A(G_{\rm w})$, $1/\lambda$ is also an eigenvalue of $A(G_{\rm w})$. If further, the multiplicity of λ and $1/\lambda$ are the same, then $G_{\rm w}$ is said to have property SR. A characterization of the class of nonsingular, weighted trees $T_{\rm w}$ with at least 8 vertices that have property R was given in (*On reciprocal eigenvalue property of weighted trees*, Linear Algebra and its Applications, 438:3817–3828, 2013) under some restriction on the weights. It is natural to ask for such a characterization for the whole of $\mathcal{G}_{\rm w}$, possibly with some weaker restrictions on the weights. We supply such a characterization. In particular, for trees it settles an open problem raised in (*On reciprocal eigenvalue property of weighted trees*, Linear Algebra and its Applications, 438:3817–3828, 2013).

Key words. Adjacency matrix, Inverse graph, Property R, Property SR, Corona, Weighted graph.

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1. Introduction. We consider simple, undirected graphs. If G is a graph, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. By $G_{\rm w}$ we denote the weighted graph obtained from G by giving weights to its edges using the function ${\rm w} : E(G) \to (0, \infty)$. The unweighted graph G may be viewed as a weighted graph where each edge has weight 1. Let $G_{\rm w}$ be a weighted graph on vertices $1, \ldots, n$. The notations $i \sim j$ (resp. $i \nsim j$) mean 'i is adjacent j' (resp. i is not adjacent to j). The notation $[i, j]$ is used to denote an edge. The adjacency matrix $A(G_{\rm w})$ of $G_{\rm w}$ is the square symmetric matrix of size n whose (i, j)th entry a_{ij} is given by

$$a_{ij} = \begin{cases} {\rm w}([i,j]) & \text{if } i \sim j \\ 0 & \text{otherwise.} \end{cases}$$

We say λ is an eigenvalue of $G_{\rm w}$ if λ is an eigenvalue of $A(G_{\rm w})$. The spectral radius of $G_{\rm w}$ is denoted by $\rho(G_{\rm w})$. If $G_{\rm w}$ is connected, then $A(G_{\rm w})$ is irreducible and so $\rho(G_{\rm w})$ is simple with a positive eigenvector called a *Perron vector*. A weighted graph $G_{\rm w}$ is *singular* (resp. *nonsingular*) if $A(G_{\rm w})$ is singular (resp. nonsingular). A *perfect matching* in a graph G is a spanning forest whose components are paths on two vertices. Studying the properties of graphs by associating matrices with them is a vast area of research; see [2, 4, 5, 6, 11].

Definition 1.1. Let G and H be two graphs. A mapping $f : V(G) \to V(H)$ is an isomorphism of graphs if f is bijective and any two vertices u and v of G are adjacent if and only if $f(u)$ and $f(v)$ are adjacent in H. If an isomorphism exists between two graphs, then the graphs are called isomorphic and we write $G \cong H$.

Definition 1.2. [9] Let G be a connected, bipartite, unweighted graph with a unique perfect matching $\mathcal{M}$. An edge in $\mathcal{M}$ is called a *matching edge* and other edges are called *nonmatching edges*. By $G/\mathcal{M}$, we denote the graph obtained from G by contracting each matching edge to a single vertex. Let $\mathcal{G}$ denote the class of bipartite, connected, unweighted graphs G with a unique perfect matching $\mathcal{M}$ such that $G/\mathcal{M}$ is bipartite.

In [9], the author proved that if $G \in \mathcal{G}$, then $A(G)^{-1}$ is signature similar to a nonnegative matrix, that is, there exists a diagonal matrix D with diagonal entries ± 1 (called a signature matrix) such that $DA(G)^{-1}D$ is nonnegative. The weighted graph associated to the matrix $DA(G)^{-1}D$ is called the inverse of G and it is denoted by G^+. Note that the inverse G^+ of a graph $G \in \mathcal{G}$ is a bipartite graph with a unique perfect matching. The notion of the inverse of a graph was motivated by quantum chemistry. In [9], the author posed an open problem to *characterize the bipartite graphs G with a unique perfect matching such that G possesses an inverse.* In [1], Akbari and Kirkland characterized the bipartite, unicyclic graphs G with a unique perfect matching such that G possesses an inverse. In [15], Tifenbach and Kirkland supplied necessary and sufficient conditions for a bipartite graph with a

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unique perfect matching to possess an inverse. Note that, in both these documents it was not necessary that $G/\mathcal{M}$ is bipartite. This necessary and sufficient condition required some constructions involving the directed graphs and the undirected interval graphs. In [15], the authors discussed different characteristics of the inverses of the bipartite, unicyclic graphs with unique perfect matchings. The characteristics of the inverses of the graphs in $\mathcal{G}$ is completely unknown.

It is known that for a tree $G \in \mathcal{G}$, the inverse G^+ is an unweighted graph; see [7]. Consider a graph $G \in \mathcal{G}$ such that G^+ is an unweighted graph. Then it is not necessary that $G^+ \in \mathcal{G}$. For example, the path P_6 on 6 vertices is in $\mathcal{G}$, whereas its inverse P_6^+, is not in $\mathcal{G}$ even though P_6^+ is an unweighted graph. In [13, Theorem 2.6], the authors have characterized graphs which can occur as inverses of nonsingular unweighted trees.

In Section 2, we supply different characteristics of the inverses of the graphs in $\mathcal{G}$. We also characterize the graphs $G \in \mathcal{G}$ such that $G^+ \in \mathcal{G}$. It turns out that for a graph $G \in \mathcal{G}$, the inverse $G^+ \in \mathcal{G}$ if and only if $G \cong G^+$ (isomorphic). This adds to the earlier studies on self dual graphs, see Tifenbach [16]. We give a constructive characterization of the class of weighted graphs H_{w} that can occur as the inverse of some graph $G \in \mathcal{G}$, generalizing the result in [13].

Definition 1.3. Let G be an unweighted graph with a unique perfect matching. We shall consider weight functions w such that $\mathrm{w}(e) = 1$ for each matching edge e. Let $\mathcal{W}_G$ be the class of such weight functions on G.

The definition of the inverse of a weighted graph is same as that of an unweighted graph. It is also an interesting problem to characterize the weighted, bipartite graphs with unique perfect matchings which possess inverses. In Section 3, we show that for each graph $G_{\mathrm{w}} \in \mathcal{G}_{\mathrm{w}}$ with $\mathrm{w} \in \mathcal{W}_G$, the inverse G_{w}^+ always exists.

The following definition is taken from Frucht and Harary [8].

Definition 1.4. Let G_1 and G_2 be two graphs on disjoint sets of n and m vertices, respectively. The corona $G_1 \circ G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and n copies of G_2, and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2. The corona $G_1 \circ G_2$ has $n(m+1)$ vertices and $|E(G_1)| + n(|E(G_2)| + m)$ edges. The corona $G \circ K_1$ is sometimes called a *simple corona*, where K_p denotes the complete graph on p vertices. If T is a tree, then we call $T \circ K_1$ a *corona tree*. For an example, see the Figure 1.1.

A graph G is said to satisfy the *property SR* if $1/\lambda \in \sigma(G)$ whenever $\lambda \in \sigma(G)$ and both have the same multiplicity. Cvetkovic [7] and Godsil and McKay [10] have shown that a tree has property SR if and only if it is a corona tree.

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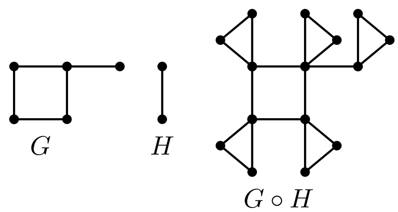

FIG. 1.1. *Corona of two graphs.*

A graph G is said to satisfy the *property R* if $1/\lambda \in \sigma(G)$ whenever $\lambda \in \sigma(G)$, where the multiplicity condition is relaxed. Barik, Neumann and Pati [3] have characterized the trees with property R. Interestingly, it turns out that a tree T has property R if and only if it has property SR.

A characterization of the class of nonsingular, weighted trees T_{w} with at least 8 vertices that have property R was given in [13] under the restriction that each matching edge has weight 1 and each nonmatching edge has weight at least 1.

PROPOSITION 1.5. *[13] Let T be a nonsingular tree on at least 8 vertices and $\mathrm{w} \in \mathcal{W}_T$ such that $\mathrm{w}(e) \geq 1$ for each edge in T. Then the weighted tree T_{w} has property R if and only if $T_{\mathrm{w}} = T'_{\mathrm{w}} \circ K_1$, where T'_{w} is an weighted tree with edge weights at least 1.*

In [13], the authors posed the question of *whether this result is true even when one allows the weights of the nonmatching edges to be any positive number.* In Section 3, we answer this question affirmatively. Moreover, we supply a characterization of the graphs G_{w} in $\{G_{\mathrm{w}} : G_{\mathrm{w}} \in \mathcal{G}_{\mathrm{w}}, \mathrm{w} \in \mathcal{W}_G\}$ which satisfy property R. It turns out that they must be simple corona of connected, bipartite, weighted graphs.

2. Inverses of a class of graphs. We shall require a combinatorial description of the inverse of a connected, bipartite graph with a unique perfect matching. The description involves the term 'alternating path' which we define now.

DEFINITION 2.1. Consider a graph G with a unique perfect matching $\mathcal{M}$. A path $P = [u_1, u_2, \ldots, u_{2k}]$ is called an *alternating path* if the edges $[u_i, u_{i+1}] \in \mathcal{M}$ for each $i = 1, 3, \ldots, 2k-1$ and the other edges are nonmatching edges.

The following description of the inverse of the adjacency matrix of a connected, bipartite graph with a unique perfect matching is a restatement of results contained in [1, 3]. We follow the convention that a sum over an empty set is zero.

LEMMA 2.2. *Let G be a connected, bipartite graph with a unique perfect matching.*

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Let $B = [b_{ij}]$, where

$$b_{ij} = \sum_{P(i,j) \in \mathcal{P}_G} (-1)^{(\|P(i,j)\|-1)/2},$$

where $\mathcal{P}_G$ is the set of alternating paths in G, $P(i,j)$ means an i-j-path and $\|P(i,j)\|$ denotes the number of edges in $P(i,j)$. Then $B = A(G)^{-1}$.

Remark 2.3. Let $G \in \mathcal{G}$. As G and $G/\mathcal{M}$ are bipartite, each cycle in G must contain an even number of matching edges. It is clear that, if $G \in \mathcal{G}$ and a path from i to j contains an odd (resp. even) number of nonmatching edges, then each path from i to j must contain an odd (resp. even) number of nonmatching edges.

The following results will be used in the sequel.

Lemma 2.4. *[9] Let $G \in \mathcal{G}$. Then G^+ exists. Furthermore, under a permutation similarity $A(G^+)$ dominates $A(G)$ entrywise.*

Recall that $\mathcal{P}_G$ is the set of alternating paths in G.

Corollary 2.5. *Let $G \in \mathcal{G}$. Then $|\mathcal{P}_G| \geq |E(G^+)| \geq |E(G)|$.*

Proof. If $[i,j] \in E(G^+)$, then there is an i-j-alternating path in G and in view of Remark 2.3, we get the first inequality. The second one follows by Lemma 2.4. □

Using Lemma 2.2 and Remark 2.3, one gets the following conclusion.

Lemma 2.6. *Let $G \in \mathcal{G}$ and $[i,j] \in E(G^+)$. Then the weight of the edge $[i,j]$ in G^+ is the total number of alternating i-j-paths in G. Hence G^+ is an unweighted graph if and only if the number of alternating i-j-paths is at most one.*

Let $\mathcal{G}_u$ be the class of graphs $G \in \mathcal{G}$ such that G^+ is unweighted. Considering $G = P_6$, we know that there are graphs in $\mathcal{G}_u$ such that G^+ is not isomorphic to G. In fact, G^+ is not even in $\mathcal{G}$. **Question**: Are there graphs in $\mathcal{G}_u$ for which $G^+ \in \mathcal{G}$ but G^+ is not isomorphic to G?

Theorem 2.7. *Let $G \in \mathcal{G}_u$. Then the following are equivalent.*

1) $|\mathcal{P}_G| = |E(G)|$.
2) $G \cong G^+$.
3) $G^+ \in \mathcal{G}$.
4) $G = G_1 \circ K_1$ *for some connected bipartite graph* G_1.

Proof. 1)⇒2). It follows that $|E(G^+)| = |E(G)|$. As $G \in \mathcal{G}_u$ using Lemmas 2.2, 2.4 and 2.6 we see that $G \cong G^+$. The proofs of 2)⇒3) and 4)⇒1) are trivial.

3)⇒4). Suppose that G is not a corona. Then there exists a matching edge $[u_i, u_i']$ which is not a leaf. This can be extended to an alternating path $[u_k, u_k', u_i, u_i', u_m, u_m']$

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in G. Hence the cycle $[u_k, u_i', u_i, u_m', u_k] \in G^+$. As this cycle contains only one matching edge, it cannot be in $\mathcal{G}$, a contradiction.□

REMARK 2.8. The equivalence of items 2. and 4. in Theorem 2.7 has previously been observed in [14].

LEMMA 2.9. *[13] Let G be a bipartite, connected graph with a unique perfect matching. Then G has at least two pendant (degree one) vertices.*

The following theorem gives some structural information on connected, bipartite graphs with unique perfect matchings.

THEOREM 2.10. *Let G be a connected, bipartite graph with a unique perfect matching $\mathcal{M}$. Then G has a pendant vertex v such that $G-v-v'$ is connected, where v' is the vertex adjacent to v.*

Proof. In view of Lemma 2.9, take a pendant vertex u. Let u' be adjacent to u. If $G-u-u'$ is connected, we have nothing to prove. So, assume that $G-u-u'$ is not connected. Let C be a component of $G-u-u'$.

Claim. *The component C has a pendant vertex v which is also a pendant vertex in G.* To see the claim, note that C has pendant vertex (by Lemma 2.9), say v. If v is not pendant vertex of G, then $[u', v] \in E(G)$. As C has a perfect matching $\mathcal{M}$, let $[v, v'] \in \mathcal{M}$. Note that $v' \nsim u'$, otherwise we have an odd cycle, which is not possible. Thus, $v' \sim v_1$ for some $v_1 \in C$. Again, as C has a perfect matching $\mathcal{M}$, let $[v_1, v_1'] \in \mathcal{M}$. Note again that $v_1' \nsim u'$, otherwise we have an odd cycle, which is not possible. Continue the process. As C is finite, we must have a repetition of a vertex, say w, for the first time. Our walk so far must look like $[v, v', v_1, v_1', \ldots, v_{j-1}', v_j, w]$ or $[v, v', v_1, v_1', \ldots, v_{j-1}', w]$. As each vertex on $[v, v', v_1, v_1', \ldots, v_{j-1}']$ have already been matched, our walk so far cannot look like $[v, v', v_1, v_1', \ldots, v_{j-1}', v_j, w]$. So our walk so far looks like $[v, v', v_1, v_1', \ldots, v_{j-1}', w]$. If $w \in \{v, v_1, \ldots, v_{j-2}\}$, then we have an alternating cycle giving us more than one perfect matching. If $w \in \{v', v_1', \ldots, v_{j-2}'\}$, then we have an odd cycle, which is not possible. Thus, the claim is justified.

Now we continue the main proof. Select a pendant vertex v of G which is in C. Let v be adjacent to v'. If $G-v-v'$ is connected, then we have nothing to prove. So, suppose that $G-v-v'$ is disconnected. Consider a component C' which does not contain u'. Then C' is a strict subgraph of C with less number of vertices. So C' has a pendant vertex which is also a pendant vertex in G. As G is finite, this process cannot be continued indefinitely, so we finally get a pendant vertex w of G which is adjacent to w' and $G-w-w'$ has a component which is just an edge $[x, x']$. Assume (in view of Lemma 2.9) that x is a pendant vertex of G. Then $d_G(x') = 2$

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and $G - x - x'$ is connected. $\square$

Proposition 2.11. *Let $G \in \mathcal{G}$. Take a pendant vertex u which is adjacent to u' such that $G - u - u'$ is connected. Let $N_G(u') = \{v_1, v_2, \ldots, v_m\}$. Then each path from v_i to v_j contains an even number of matching edges.*

Proof. Suppose that there is a path $P(v_i, v_j)$ from v_i to v_j contains an odd number of matching edges. Then the cycle $\Gamma = [u', P(v_i, v_j), u']$ has an odd number of matching edges. This contradicts the fact that $G \in \mathcal{G}$. $\square$

Lemma 2.12. *Let $G \in \mathcal{G}$ have $n \geq 4$ vertices and let $H = G^+$. Then there exist vertices $u', u, v_1, v_2, \ldots, v_m \in H$ such that*

1. *u' is pendant in H;*
2. *$u \sim u'$ and for each $i = 1, \ldots, m$, we have $u' \nsim v_i$ in H;*
3. *$N_H(u) = \cup_{i=1}^m N_H(v_i) \cup \{u'\}$; and*
4. *for each $x \in \cup_{i=1}^m N_H(v_i)$ we have $\mathrm{w}_H([u, x]) = \sum_{v_i \sim x} \mathrm{w}_H([v_i, x])$.*

Proof. We use Proposition 2.11 and select $u, u', v_1, v_2, \ldots, v_m \in G$ as described there. Using Lemma 2.2, we see that u' is pendant, $u' \sim u$ and for each $i = 1, \ldots, m$, $u' \nsim v_i$ in H.

Let $x \in \cup_{i=1}^m N_H(v_i)$. Then there is an alternating path from some v_i to x in G. Hence an alternating path from u to x exists in G. Using Remark 2.3, we see that $[u, x] \in E(H)$.

Conversely, if $x \sim_H u$ and $x \neq u'$, then there is an alternating path P from u to x in G. As u is pendant and $u \sim_G u'$, the third vertex on P must be some v_i. Then $P - u - u'$ gives an alternating path from v_i to x in G. Thus $x \sim_H v_i$.

The final assertion follows from the fact that $\mathrm{w}_H([u, x])$ is the number of alternating paths from u to x in G and when we delete u and u' from each such path we get an alternating path from some v_i to x in G. $\square$

We have the following extension of Theorem 2.2 of [13].

Theorem 2.13. *Let $\mathcal{F}_1 = \{P_2\}$ and $\mathcal{F}_2 = \{P_4\}$. Notice that graphs in $\mathcal{F}_i$ are precisely the inverses of graphs of order $2i$ in $\mathcal{G}$. Assume that we have constructed the graph class $\mathcal{F}_{k-1}$ which consists of the graphs which can occur as the inverse of some graph of order $2(k-1)$ in $\mathcal{G}$. Now we construct $\mathcal{F}_k$ in the following manner.*

1. *Take a graph $H_{k-1} \in \mathcal{F}_{k-1}$. Take a disjoint union of $[u, u']$ with H_{k-1}.*
2. *Choose a set of vertices $S = \{v_1, v_2, \ldots, v_m\}$ in H_{k-1} such that $dist_{H_{k-1}}(v_i, v_j)$ is even and no path from v_i to v_j in H_{k-1}^+ contains an odd number of matching edges for $i, j = 1, 2, \ldots, m$.*
3. *For each $x \in \cup_{i=1}^m N_{H_{k-1}}(v_i)$, add the edge $[u, x]$ and put the weight $\mathrm{w}([u, x]) =$*

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$\sum_{x\sim v_i} \mathrm{w}_{H_{k-1}}([x,v_i])$ *and the weight* $\mathrm{w}([u,u']) = 1$.

Then $\mathcal{F}_k$ *consists of the graphs of order* $2k$ *which occur as the inverses of the graphs in* $\mathcal{G}$.

Before we supply a proof let us illustrate the theorem by constructing one element from $\mathcal{F}_3$ and one element from $\mathcal{F}_4$.

Note that $H_2 = P_4 = [x,x',y,y'] \in \mathcal{F}_2$ and $H_2^+ = [x',x,y',y] \in \mathcal{G}$. We take a disjoint union of H_2 and $[u,u']$. We cannot choose more than one vertices from H_2 which satisfy Condition 2 in the hypothesis in Theorem 2.13. So, let us choose $S = \{y\}$. Now, $N_{H_2}(y) = \{x',y'\}$. Now we add the edges $[u,x']$ and $[u,y']$. Since the weights of the edges $[x',y]$ and $[y,y']$ are 1 in H_2, using Condition 3, we put $\mathrm{w}([u,x']) = \mathrm{w}([u,y']) = 1$. This graph H_3 is shown in Figure 2.1. By using Lemma 2.2, we see that $H_3^+ = P_6 = [x',x,y',y,u',u]$.

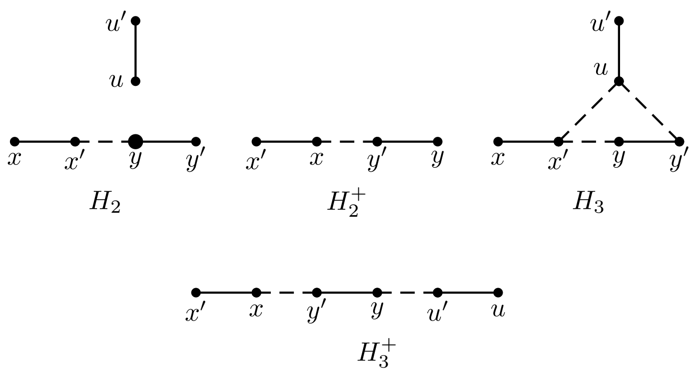

FIG. 2.1. *Construction of a graph* H_3 *in* $\mathcal{F}_3$.

Now, let us take a disjoint union of H_3 and $[u_1,u_1']$. Take $S = \{x,u\}$. Then these vertices have even distance among them in H_3 and no path among them in H_3^+ contain an odd number of matching edges. This satisfy Condition 2 in the hypothesis of Theorem 2.13. Notice that $N_{H_3}(x) = \{x'\}$ and $N_{H_3}(u) = \{u',x',y'\}$. Now we add the edges $[u_1,x'],[u_1,y']$ and $[u_1,u']$. Using Condition 3, we put $\mathrm{w}([u_1,x']) = 2$ because x' is in $N_{H_3}(x)\cap N_{H_3}(u)$. Similarly, we put $\mathrm{w}([u_1,y']) = 1$ and $\mathrm{w}([u_1,u']) = 1$. The new graph H_4 and its inverse H_4^+ are shown in Figure 2.2.

Now we prove the theorem.

Proof. Let $G \in \mathcal{G}$ of order $2k$ and $G^+ = H$. By using Proposition 2.11 and Lemma 2.12, there exist vertices $u',u,v_1,v_2,\ldots,v_m \in G$ such that

1. u is pendant and $u \sim u'$ in G;
2. $N_G(u') = \{v_1,v_2,\ldots,v_m\}$;
3. u' is pendant in H;

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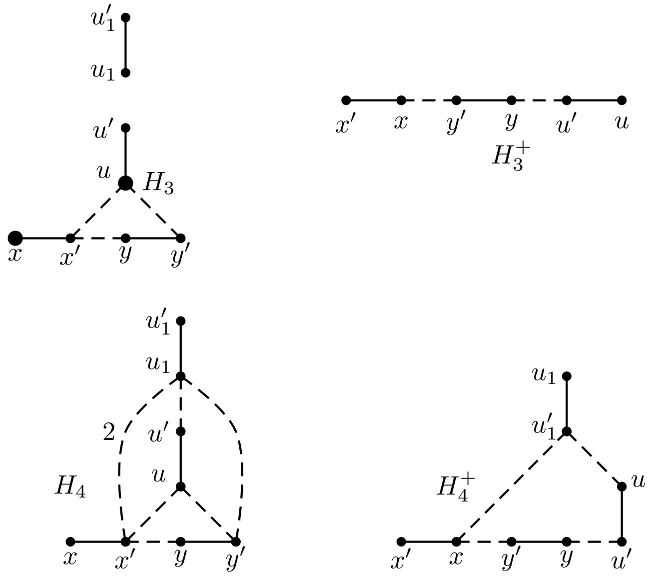

FIG. 2.2. *Construction of a graph H_4 in $\mathcal{F}_4$.*

4. $u \sim u'$ and for each $i = 1, \ldots, m$, we have $u' \nsim v_i$ in H;
5. $N_H(u) = \cup_{i=1}^m N_H(v_i) \cup \{u'\}$; and
6. for each $x \in \cup_{i=1}^m N_H(v_i)$ we have $\mathrm{w}_H([u, x]) = \sum_{v_i \sim x} \mathrm{w}_H([v_i, x])$.

It is clear that $G-u-u' \in \mathcal{G}$. By the hypothesis there is a graph $H' \in \mathcal{F}_{k-1}$ such that $(G - u - u')^+ = H'$. Then the graph $H - u - u' = H'$ and $H \in \mathcal{F}_k$. Therefore the inverses of the graphs $G \in \mathcal{G}$ of order $2k$ lie in $\mathcal{F}_k$.

Now we show that each $H_k \in \mathcal{F}_k$ is the inverse of some graph $G \in \mathcal{G}$ or order $2k$. Let $H_k \in \mathcal{F}_k$. Then H_k is constructed in the following manner.

1. Take a graph $H_{k-1} \in \mathcal{F}_{k-1}$. Take a disjoint union of $[u, u']$ with H_{k-1};
2. Choose a set of vertices $S = \{v_1, v_2, \ldots, v_m\}$ in H_{k-1} such that $dist_{H_{k-1}}(v_i, v_j)$ is even and no path from v_i to v_j in H_{k-1}^+ contains an odd number of matching edges for $i, j = 1, 2, \ldots, m$;
3. For each $x \in \cup_{i=1}^m N_{H_{k-1}}(v_i)$, add the edge $[u, x]$ and put the weight $\mathrm{w}_{H_k}([u, x]) = \sum_{x \sim v_i} \mathrm{w}_{H_{k-1}}([x, v_i])$ and the weight $\mathrm{w}_{H_k}([u, u']) = 1$.

By the hypothesis there is a graph $G_{k-1} \in \mathcal{G}$ such that $G_{k-1}^+ = H_{k-1}$. Now take the disjoint union of G_{k-1} and $[u, u']$. Adding the edges $[u', v_1], \ldots, [u', v_m]$ we get a new graph G_k. It is clear that $G_k \in \mathcal{G}$. By Lemma 2.12, $G_K^+ = H_k$. Therefore each graph $H_k \in \mathcal{F}_k$ is the inverse of some graph in $\mathcal{G}$ of order $2k$. Hence we conclude that the class $\mathcal{F}_k$ consists of the graphs of order $2k$ which occur as the inverses of the graphs in $\mathcal{G}$. The proof is complete.$\square$

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3. Property R of graphs in $\mathcal{G}_{\mathrm{w}}$ with $\mathrm{w} \in \mathcal{W}_G$**.** Let $G_{\mathrm{w}} \in \mathcal{G}_{\mathrm{w}}$, that is, G_{w} is a weighted, connected, bipartite graph with a unique perfect matching such that the underlying unweighted graph $G \in \mathcal{G}$. Recall that $\mathcal{W}_G$ be the class of weight functions w on G such that $\mathrm{w}(e) = 1$ for each matching edge e. A characterization of the class of the nonsingular, weighted trees T_{w} with at least 8 vertices, having property R was given in [13] under the restriction that $\mathrm{w} \in \mathcal{W}_T$ and $\mathrm{w}(e) \geq 1$ for each edge.

Let $G_{\mathrm{w}} \in \mathcal{G}_{\mathrm{w}}$ with $\mathrm{w} \in \mathcal{W}_G$. One would expect G_{w}^+ to exist. Indeed, it is true. In this section, we first prove this. Then we characterize the class of graphs in $\mathcal{G}_{\mathrm{w}}$ with $\mathrm{w} \in \mathcal{W}_G$ which have property R. This characterization gives the answer to the question which was posed in [13]. We shall use the following result which is a restatement of a result in [13].

Lemma 3.1. *Let G_{w} be a weighted, connected, bipartite, graph with a unique perfect matching $\mathcal{M}$. Define $g(e) = -\mathrm{w}(e)$, if $e \notin \mathcal{M}$, and $g(e) = \frac{1}{\mathrm{w}(e)}$, if $e \in \mathcal{M}$. Let $B = [b_{ij}]$, where*

$$b_{ij} = \sum_{P(i,j) \in \mathcal{P}_{G_{\mathrm{w}}}} \prod_{e \in P(i,j)} g(e),$$

where $\mathcal{P}_{G_{\mathrm{w}}}$ is the set of alternating paths in G_{w} and $P(i,j)$ means an i-j-path. Then $B = A(G_{\mathrm{w}})^{-1}$.

The following is an extension of Theorem 2.2 of [9] to the weighted case. If P is a path we use $\mathrm{w}(P)$ to mean the weight of P, which is the product of the weights of the edges on P.

Theorem 3.2. *Let $G_{\mathrm{w}} \in \mathcal{G}_{\mathrm{w}}$ with $\mathrm{w} \in \mathcal{W}_G$. Then G_{w}^+ exists. Take the permutation matrix $P = [p_{ij}]$ given by the matching, that is, $p_{ij} = 1$ if $[i,j]$ is a matching edge and 0, otherwise. Then $P^{-1}A(G_{\mathrm{w}}^+)P \geq A(G_{\mathrm{w}})$.*

Proof. First, we define a signature matrix D in the following way. Put $d_{11} = 1$. For $i \neq 1$, put $d_{ii} = 1$ if any path (hence each path, by Remark 2.3) from 1 to i contains an even number of nonmatching edges; otherwise we put $d_{ii} = -1$.

Assume that $DA(G_{\mathrm{w}})^{-1}D \not\geq 0$, that is, there exist i and j such that $d_{ii}A(G_{\mathrm{w}})^{-1}_{i,j}d_{jj} < 0$. First suppose that $A(G_{\mathrm{w}})^{-1}_{i,j} < 0$. Then $d_{ii} = d_{jj}$. So the parities of the number of nonmatching edges on any 1-i-path and any 1-j-path are the same. Hence, any i-j-path must contain an even number of nonmatching edges. In that case, $A(G_{\mathrm{w}})^{-1}_{i,j}$ must be nonnegative, by Lemma 3.1. A similar contradiction is obtained if $A(G_{\mathrm{w}})^{-1}_{i,j} > 0$. Hence $DA(G_{\mathrm{w}})^{-1}D \geq 0$, that is , G_{w}^+ exists.

To prove the next statement, let $\mathcal{M} = \{[u_k, u_k'] : k = 1, \ldots, 2n\}$. Now define a map $f : V(G_{\mathrm{w}}) \to V(G_{\mathrm{w}})$ such that $f(u_k) = u_k'$ and $f(u_k') = u_k$. Using the description of $A(G_{\mathrm{w}})^{-1}$ from Lemma 3.1, we see that, for each match-

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ing edge $[u_k, u'_k] \in E(G_{\rm w})$, the edge $[f(u_k), f(u'_k)] = [u'_k, u_k] \in E(G^+_{\rm w})$. In fact, $A(G^+_{\rm w})_{u'_k,u_k} = 1$. Thus $A(G^+_{\rm w})_{f(u_k),f(u'_k)} = A(G_{\rm w})_{u_k,u'_k}$.

For any nonmatching edge $[u, v] \in E(G_{\rm w})$, we have an alternating path $P^* = [u', u, v, v']$ of length 3 from u' to v'. Hence, each alternating path from $f(u) = u'$ to $f(v) = v'$ contains an odd number of nonmatching edges. Using of the description of $A(G_{\rm w})^{-1}$ from Lemma 3.1, we see that $[u', v'] = [f(u), f(v)] \in E(G^+_{\rm w})$ and

$$\begin{aligned} A(G^+_{\rm w})_{u',v'} &= \left| \sum_{P(u',v')\in\mathcal{P}_{G_{\rm w}}} \prod_{e\in P(u',v')} g(e) \right| = \sum_{P(u',v')\in\mathcal{P}_{G_{\rm w}}} {\rm w}(P(u', v')) \\ &\geq {\rm w}(P^*) = {\rm w}([u, v]) = A(G_{\rm w})_{u,v}. \end{aligned}$$

It follows that $P^{-1}A(G^+_{\rm w})P \geq A(G_{\rm w})$.☐

The following is another crucial observation.

LEMMA 3.3. *Let $G_{\rm w} \in \mathcal{G}_{\rm w}$ with ${\rm w} \in \mathcal{W}_G$. Let ρ be the spectral radius of $G_{\rm w}$. Then $1/\rho$ is the smallest positive eigenvalue of $G_{\rm w}$ if and only if $G_{\rm w} \cong G^+_{\rm w}$.*

Proof. By using Theorem 3.2, we see that $P^{-1}A(G^+_{\rm w})P \geq A(G_{\rm w})$, where P is the permutation matrix given by the matching. As $1/\rho$ is the smallest positive eigenvalue of $G_{\rm w}$, we see that ρ is the spectral radius of G^+. As G^+ is connected, using Perron-Frobenius theory [13, sec 8.1], we get $P^{-1}A(G^+_{\rm w})P = A(G_{\rm w})$. Hence $G_{\rm w} \cong G^+_{\rm w}$. The converse is straight forward.☐

LEMMA 3.4. *Let $G_{\rm w} \in \mathcal{G}_{\rm w}$ with ${\rm w} \in \mathcal{W}_G$, be such that $G_{\rm w} \cong G^+_{\rm w}$. Then $G_{\rm w}$ is a simple corona of a weighted, connected, bipartite graph $G'_{\rm w}$, where edge weights can be any positive real number..*

Proof. Let $G_{\rm w} \in \mathcal{G}_{\rm w}$ be such that $G_{\rm w} \cong G^+_{\rm w}$. Hence by Remark 2.3, each cycle in $G^+_{\rm w}$ has an even number of nonmatching edges. Let $\mathcal{M} = \{[u_k, u'_k] : k = 1, \ldots, 2n\}$.

Suppose that there is a matching edge, say $[u_i, u'_i]$, such that neither of u_i and u'_i is a pendant vertex. Then we can find an alternating path $[v', v, u'_i, u_i, w', w]$ of length 5. In view of Lemma 3.1 and Remark 2.3, we see that $[v', u_i, u'_i, w, v']$ is a cycle in $G^+_{\rm w}$ and it contains an odd number of nonmatching edges. We have a contradiction.

Hence, each matching edge must have a pendant endvertex. Noting that the weight of each matching edge is 1, we see that $G_{\rm w}$ is a simple corona $H_{\rm w} \circ K_1$, $H_{\rm w}$ is an weighted, connected, bipartite graph.☐

The following known result will be required in the sequel.

LEMMA 3.5. *[13] Let $G'_{\rm w}$ be a weighted, bipartite graph. Let $G_{\rm w}$ be obtained from $G'_{\rm w}$ by adding a new pendant vertex to each vertex of $G'_{\rm w}$ and by taking the weight of the new edges to be 1. Then $G_{\rm w}$ has property SR.*

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The following is an extension of Theorem 4.6 in [13]. It is our main result of this section. It answers the question raised in [13] affirmatively.

THEOREM 3.6. *Let* $G \in \mathcal{G}_{\mathrm{w}}$ *with* $\mathrm{w} \in \mathcal{W}_G$. *Then the following are equivalent.*

1. G_{w} *has property SR.*
2. G_{w} *has property R.*
3. $G_{\mathrm{w}} \cong G^+_{\mathrm{w}}$.
4. $G_{\mathrm{w}} = G'_{\mathrm{w}} \circ K_1$ *where* G'_{w} *is a weighted, connected, bipartite graph where edge weights can be any positive real number.*

Proof. 1)⇒2). It follows from the definition of property R.

2)⇒3). Let ρ be the spectral radius of G_{w}. Since G_{w} has property R, so $1/\rho$ is the smallest positive eigenvalue of G_{w}. Then by using Lemma 3.3, we get $G_{\mathrm{w}} \cong G^+_{\mathrm{w}}$.

3)⇒4). The proof follows by using Lemma 3.4

4)⇒1). The proof follows by using Lemma 3.5.□

4. Conclusion. The notion of the graph inverse was defined by C. D. Godsil in [9] for bipartite graphs with a unique perfect matching. In the same article, author supplied a class of graphs (which we denoted by $\mathcal{G}$, see, Definition 1.2) which possess inverses. In this article, we have extended this notion of graph inverse to weighted graphs keeping the weights of the matching edges 1. The characteristics of the inverses of the graphs $G \in \mathcal{G}$ have never been discussed in general. In this article, we have supplied some characteristics of the inverses of the graphs $G \in \mathcal{G}$. In [13], the graphs which can occur as the inverse of some nonsingular trees have been characterized. Generalizing their result, we have supplied a constructive characterization of the inverses of the graphs $G \in \mathcal{G}$. In [14], authors showed that for any graph $G \in \mathcal{G}$, we have $G \cong G^+$ if and only if G is a simple corona of a connected, bipartite graph. It is clear that, if G is a simple corona, then $G^+ \in \mathcal{G}$. So, one naturally wonders whether the simple coronas of bipartite, connected graphs are the only class of graphs $G \in \mathcal{G}$ such that $G^+ \in \mathcal{G}$. We have answered this question affirmatively in Theorem 2.7 and we have seen that the class of graphs $G \in \mathcal{G}$ such that $G^+ \in \mathcal{G}$ is same as the class of graphs $G \in \mathcal{G}$ such that $G \cong G^+$. We also have shown that for any graph $G_{\mathrm{w}} \in \mathcal{G}_{\mathrm{w}}$ with $\mathrm{w} \in \mathcal{W}_G$, the inverse G^+_{w} exists. If we consider the weight function $\mathrm{w} \equiv 1$, then we get Theorem 2.2 of [9]. As an application of these results, we have characterized the class of graphs $G_{\mathrm{w}} \in \mathcal{G}_{\mathrm{w}}$ with $\mathrm{w} \in \mathcal{W}_G$ such that G_{w} has property R (resp. property SR). This characterization gives the answer to an open problem which was posed in [13].

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REFERENCES

[1] S. Akbari and S.J. Kirkland. *On Unimodular graphs.* Linear Algebra and its Applications, 421:3–15, 2007.
[2] R. B. Bapat. *Graphs and Matrices.* TRIM Series-58, Hindustan Book Agency, New Delhi, 2010.
[3] S. Barik, M. Neumann and S. Pati. *On nonsingular trees and reciprocal eigenvalue property.* Linear and Multilinear Algebra, 54:453–465, 2006.
[4] N. Biggs. *Algebraic graph theory.* Cambridge University Press, 1993.
[5] Richard A. Brualdi and Herbert J. Ryser. *Combinatorial Matrix Theory.* Cambridge University Press, 1991.
[6] D. M. Cvetkovic, M. Doob and H. Sachs. *Spectra of Graphs.* Academic Press, NewYork, 1980.
[7] Dragos M. Cvetkovic, Ivan Gutman and Slobodan K. Simic. *On self pseudo-inverse graphs.* Univ. Beogard. Publ. Elektrotehn. Fak., No. 602–633:111–117, 1978.
[8] R. Frucht and F. Harary. *On the corona of two graphs.* Aequationes Mathematicae, 4:322–325, 1970.
[9] C. D. Godsil. *Inverses of trees.* Combinatorica, 5(1):33–39, 1985.
[10] C. D. Godsil and B. D. Mckay. *A new graph product and its spectrum.* Bulletin of Australian Mathematical Society, 18:21–28, 1978.
[11] C. Godsil and G. Royal. *Algebraic Graph Theory.* Springer-Verlag, New York, 2001.
[12] Roger A. Horn and Charles R. Johnson. *Matrix Analysis.* Cambridge University Press, 1990.
[13] M. Neumann and S. Pati. *On reciprocal eigenvalue property of weighted trees.* Linear Algebra and its Applications, 438:3817–3828, 2013.
[14] R. Simion and D. S. Cao. *Solution to a problem of C. D. Godsil regarding bipartite graphs with unique perfect matching.* Combinatorica, 9(1):85–89, 1989.
[15] R. M. Tifenbach, S. J. Kirkland. *Directed intervals and the dual of a graph.* Linear Algebra and its Applications, 431:792–807, 2009.
[16] R. M. Tifenbach. *Strongly self-dual graphs.* Linear Algebra and its Applications, 435:3151–3167, 2011.