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ON THE CHUDNOVSKY-SEYMOUR-SULLIVAN CONJECTURE ON CYCLES IN TRIANGLE-FREE DIGRAPHS*

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In memory of David A. Gregory

Abstract. For a simple digraph G without directed triangles or digons, let $\beta(G)$ be the size of the smallest subset $X \subseteq E(G)$ such that $G \setminus X$ has no directed cycles, and let $\gamma(G)$ be the number of unordered pairs of nonadjacent vertices in G. In 2008, Chudnovsky, Seymour, and Sullivan showed that $\beta(G) \leq \gamma(G)$, and conjectured that $\beta(G) \leq \gamma(G)/2$. Recently, Dunkum, Hamburger, and Pór proved that $\beta(G) \leq 0.88\gamma(G)$. In this note, we prove that $\beta(G) \leq 0.8616\gamma(G)$.

Key words. Digraph, triangle free digraph, cycle, in-degree, out-degree.

AMS subject classifications. 05C20, 05C35, 05C38.

1. Introduction. We will follow the notation from [2, 4]. All digraphs $G = (V, E)$ considered in this note are finite and simple. A digraph G is called 3-*free* if G has no directed cycle of length at most three. A digraph is *acyclic* if it has no directed cycles. For a digraph G, let $\beta(G)$ denote the minimum cardinality of a set $X \subset E(G)$ such that $G \setminus X$ is acyclic, and let $\gamma(G)$ be the number of missing edges of G (that is, the number of unordered pairs of nonadjacent vertices.) In 2008, Chudnovsky, Seymour, and Sullivan [2] made the following conjecture.

Conjecture 1.1 (Chudnovsky, Seymour, and Sullivan). *If G is a 3-free digraph, then*

$$\beta(G) \leq \frac{1}{2}\gamma(G).$$

In support of the above conjecture, Chudnovsky, Seymour, and Sullivan [2] showed that $\beta(G) \leq \gamma(G)$. Recently, Dunkum, Hamburger, and Pór [4] improved the result

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to $\beta(G) \leq 0.88\gamma(G)$. Conjecture 1.1 is closely related to the following special case of a conjecture by Caccetta and Häggkvist [1].

CONJECTURE 1.2 (Caccetta and Häggkvist). *Any digraph on n vertices with minimum out-degree at least $n/3$ contains a directed triangle.*

Conjecture 1.2 is still open. In fact, the following weaker conjecture is also open even if a similar in-degree condition is added.

CONJECTURE 1.3. *Any digraph on n vertices with both minimum out-degree and minimum in-degree at least $n/3$ contains a directed triangle.*

Conjecture 1.3 is from folklore, and some partial results of the conjecture can be found in [3, 11, 6]. Chudnovsky, Seymour, and Sullivan [2] commented that proving Conjecture 1.1 may provide some useful information towards proving Conjecture 1.2. To see this, their partial result ($\beta(G) \leq \gamma(G)$) on Conjecture 1.1 has been applied by Hamburger, Haxell, and Kostochka [6] to improve a result of Shen [11] on Conjecture 1.2. Recently, the same partial result was also applied by Hladký, Král', and Norine [7] who used the theory of flag algebras to prove the currently best result in this direction, namely, any digraph on n vertices with minimum out-degree at least $0.3465n$ contains a directed triangle.

In this note, we prove that $\beta(G) \leq 0.8616\gamma(G)$. We mention that this result has been cited by [8, 9, 10] after we uploaded our paper on arXiv in 2009. Lichiardopol [10] has applied our result to prove the currently best partial result on Conjecture 1.3: for $\beta \geq 0.343545$, any digraph of order n with both minimum out-degree and minimum in-degree at least βn contains a directed triangle. Liang and Xu have extended the research to 4- and 5-free digraphs [8] and to the general m-free digraphs [9]. Fox, Keevash, and Sudakov [5] proved that every m-free digraph satisfies $\beta(G) \leq c\gamma(G)/m^2$, where c is an absolute constant.

2. Proof of the main result. In this section, we follow the ideas in [2, 4] for partitioning the vertex set of a digraph. For each vertex v in G, let $A(v)$ and $B(v)$ be the set of out-neighbors and the set of in-neighbors of v, respectively. Then there are no edges from $A(v)$ to $B(v)$; or else, G would contain a directed triangle. Let $g(v)$ be the number of missing edges between $A(v)$ and $B(v)$. Denote $C(v) := V - A(v) - B(v) - \{v\}$. Dunkum, Hamburger, and Pór [4] partitioned V into V_1, V_2, $\{v\}$ such that $V_1 = B(v) \cup C_{B(v)}$ and $V_2 = A(v) \cup C_{A(v)}$, where $C_{A(v)} \cup C_{B(v)}$ forms a certain partition of $C(v)$. Given such a partition $V_1 \cup V_2 \cup \{v\}$ of V, let $G[V_1]$ and $G[V_2]$ be the subgraphs induced by V_1 and by V_2, respectively. The edges which are missing outside of $G[V_1]$ and $G[V_2]$ are denoted as *missing edges*. Note that removing the set of edges from V_2 to V_1 destroys all directed cycles outside of $G[V_1]$ and $G[V_2]$.

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 28, pp. 117-123, September 2015

Thus the edges from V_2 to V_1 are called *decycling edges*. An easy induction argument [2, 4] shows that, for any real μ with $0 \leq \mu \leq 1$, if the number of missing edges is at least $(1+\mu)$ times the number of decycling edges, then $\gamma(G) \geq (1+\mu)\beta(G)$ (see the proof of Theorem 2.5). The following two lemmas are due to Dunkum, Hamburger, and Pór [4].

LEMMA 2.1 ([4]). *If*

$$2\gamma(G) + \frac{1}{2}\sum_{v\in V(G)} \binom{|C(v)|}{2} + \frac{1-\mu}{4}\sum_{v\in V(G)} t(v) \geq \mu \sum_{v\in V(G)} g(v),$$

then for some vertex v there exists a partition $V_1, V_2, \{v\}$ where the number of missing edges is at least $(1+\mu)$ times the number of decycling edges.

LEMMA 2.2 ([4]). *If*

$$g(v) \geq |C(v)|^2(1+\mu)\left(\frac{1+\mu+\sqrt{(1+\mu)^2+1+\mu}}{2} + \frac{1}{4}\right)$$

for a vertex v, then there exists a partition V_1, V_2, $\{v\}$ where the number of missing edges is at least $(1+\mu)$ times the number of decycling edges.

Let $e(v)$ be the number edges from $C_{A(v)}$ to $C_{B(v)}$. The next lemma is a modification of Lemma 2.2. The proof of Lemma 2.3 is quite similar to the proof of Lemma 2.2 in [4]. To make the note self-contained, we include a proof.

LEMMA 2.3. *If*

$$g(v) \geq |C(v)|^2(1+\mu)\left(\frac{1+\mu+\sqrt{(1+\mu)^2+\frac{4(1+\mu)e(v)}{|C(v)|^2}}}{2} + \frac{e(v)}{|C(v)|^2}\right)$$

for a vertex v, then there exists a partition V_1, V_2, $\{v\}$ where the number of missing edges is at least $(1+\mu)$ times the number of decycling edges.

Proof. Following the ideas in [4], we partition the vertex set of G into V_1, V_2, $\{v\}$ as follows. First let $B(v) \subseteq V_1$ and $A(v) \subseteq V_2$. Second, for any $u \in C(v)$, let $k_v(u)$ (resp. $l_v(u)$) be the number of vertices $w \in A(v)$ (resp. $w \in B(v)$) with $wu \in E(G)$ (resp. $uw \in E(G)$), and further let $u \in V_1$ if $l_v(u) > k_v(u)$ and let $u \in V_2$ otherwise. Denote the two subsets of $C(v)$ by $C_{A(v)}$ and $C_{B(v)}$; that is, $C_{A(v)} = C(v) \cap V_2$ and $C_{B(v)} = C(v) \cap V_1$. Denote $m_v(u) := \min\{k_v(u), l_v(u)\}$ and $M := \sum_{v\in C(v)} m_v(u)$.

For each $u \in C(v)$, there are $k_v(u)$ and $l_v(u)$ edges from $A(v)$ to v and from v to $B(v)$, respectively. Denote the two sets by $K_v(u) \subseteq A(v)$ and $L_v(u) \subseteq B(v)$. Any edge from $L_v(u)$ to $K_v(u)$ would form a directed triangle together with v. Thus

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Volume 28, pp. 117-123, September 2015

these $k_v(u)l_v(u)$ edges between $K_v(u)$ and $L_v(u)$ are missing. Each missing edge between $A(v)$ and $B(v)$ can be counted with multiplicity at most $|C(v)|$ in the sum $\sum_{u\in C(v)} k_v(u)l_v(u)$. This yields a lower bound for the number of missing edges $g(v)$ between $A(v)$ and $B(v)$:

$$g(v) \geq \frac{1}{|C(v)|}\sum_{u\in C(v)} k_v(u)l_v(u) \geq \frac{1}{|C(v)|}\sum_{u\in C(v)} m_v^2(u) \geq \left(\frac{\sum\limits_{u\in C(v)} m_v(u)}{|C(v)|}\right)^2.$$

Recall that $M = \sum_{u\in C(v)} m_v(u)$. Thus

$$g(v) \geq \frac{M^2}{|C(v)|^2}. \tag{2.1}$$

To count the number of decycling edges, we see that there are three types of decycling edges: edges from $A(v)$ to $C_{B(v)}$, edges from $C_{A(v)}$ to $B(v)$, and edges from $C_{A(v)}$ to $C_{B(v)}$. The number of decycling edges of the first two types is M. Recall that $e(v)$ is the number of edges from $C_{A(v)}$ to $C_{B(v)}$. So the total number of decycling edges is $M + e(v)$. If

$$M \leq \frac{1+\mu+\sqrt{(1+\mu)^2 + \frac{4(1+\mu)e(v)}{|C(v)|^2}}}{2}|C(v)|^2,$$

then $g(v)$ is at least

$$|C(v)|^2(1+\mu)\left(\frac{1+\mu+\sqrt{(1+\mu)^2 + \frac{4(1+\mu)e(v)}{|C(v)|^2}}}{2} + \frac{e(v)}{|C(v)|^2}\right) \geq (1+\mu)(M+e(v))$$

and we are done. Now we may suppose

$$M \geq \frac{1+\mu+\sqrt{(1+\mu)^2 + \frac{4(1+\mu)e(v)}{|C(v)|^2}}}{2}|C(v)|^2,$$

which implies

$$\frac{M^2}{|C(v)|^4} - \frac{(1+\mu)M}{|C(v)|^2} - \frac{(1+\mu)e(v)}{|C(v)|^2} \geq 0. \tag{2.2}$$

By (2.1) and (2.2),

$$g(v) \geq \frac{M^2}{|C(v)|^2} \geq (1+\mu)(M+e(v)),$$

from which Lemma 2.3 follows. $\square$

Theorem 2.4. *Let μ be a positive real satisfying the four inequalities:*

Electronic Journal of Linear Algebra ISSN 1081-3810
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(I) $4\mu^2 + 5\mu - 1 \leq 0$,
(II) $24\mu^4 + 49\mu^3 + 8\mu^2 - 19\mu + 2 \leq 0$,
(III) $8\mu^3 + 20\mu^2 + 13\mu - 5 \leq 0$, *and*
(IV) $32\mu^4 - 8\mu^3 - 159\mu^2 - 130\mu + 25 \geq 0$.

Then there exists a vertex v and a partition V_1, V_2, $\{v\}$ where the number of missing edges is at least $(1+\mu)$ times the number of decycling edges.

Proof. Since $2\gamma(G) = \sum_{v\in V(G)} |C(v)|$, by Lemmas 2.1, we may assume that

$$\sum_{v\in V(G)} |C(v)| + \frac{1}{2}\sum_{v\in V(G)} \binom{|C(v)|}{2} + \frac{1-\mu}{4}\sum_{v\in V(G)} t(v) < \mu \sum_{v\in V(G)} g(v).$$

Thus

$$\frac{1}{4}\sum_{v\in V(G)} |C(v)|^2 + \frac{1-\mu}{4}\sum_{v\in V(G)} t(v) < \mu \sum_{v\in V(G)} g(v),$$

which implies that there exists some vertex v such that

$$\frac{1}{4}|C(v)|^2 + \frac{1-\mu}{4}t(v) < \mu g(v). \tag{2.3}$$

By Lemma 2.3, we may also assume that

$$g(v) < |C(v)|^2(1+\mu)\left(\frac{1+\mu+\sqrt{(1+\mu)^2 + \frac{4(1+\mu)e(v)}{|C(v)|^2}}}{2} + \frac{e(v)}{|C(v)|^2}\right) \tag{2.4}$$

Combining (2.3) with (2.4),

$$\frac{1}{4}|C(v)|^2 + \frac{1-\mu}{4}t(v) < |C(v)|^2\mu(1+\mu)\left(\frac{1+\mu+\sqrt{(1+\mu)^2 + \frac{4(1+\mu)e(v)}{|C(v)|^2}}}{2} + \frac{e(v)}{|C(v)|^2}\right)$$

Since $e(v) \leq t(v)$, we obtain

$$\frac{1}{4} < \mu(1+\mu)\frac{1+\mu+\sqrt{(1+\mu)^2 + \frac{4(1+\mu)e(v)}{|C(v)|^2}}}{2} + \frac{4\mu^2+5\mu-1}{4}\cdot\frac{t(v)}{|C(v)|^2}. \tag{2.5}$$

The proof is now broken into two cases:

Case 1: $t(v) \geq |C(v)|^2/4$. Recall that $4\mu^2 + 5\mu - 1 \leq 0$. Since

$$e(v) \leq |C_{A(v)}| \cdot |C_{B(v)}| = |C_{A(v)}| \cdot (|C(v)| - |C_{A(v)}|) \leq |C(v)|^2/4,$$

(2.5) implies that

$$\frac{1}{4} < \mu(1+\mu)\frac{1+\mu+\sqrt{(1+\mu)^2+1+\mu}}{2} + \frac{4\mu^2+5\mu-1}{16}. \tag{2.6}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 28, pp. 117-123, September 2015

ELA

Case 2: $t(v) \le |C(v)|^2/4$. Since $e(v) \le t(v)$, (2.5) implies that

$$\frac{1}{4} < \mu(1+\mu)\frac{1+\mu+\sqrt{(1+\mu)^2+\frac{4(1+\mu)t(v)}{|C(v)|^2}}}{2} + \frac{4\mu^2+5\mu-1}{4}\cdot\frac{t(v)}{|C(v)|^2}.$$

Define

$$f(x) = \mu(1+\mu)\frac{1+\mu+\sqrt{(1+\mu)^2+4(1+\mu)x}}{2} + \frac{(4\mu^2+5\mu-1)x}{4},$$

where $0 \le x = t(v)/|C(v)|^2 \le 1/4$. Taking the derivative of $f(x)$,

$$f'(x) = \frac{\mu(1+\mu)^2}{\sqrt{(1+\mu)^2+4(1+\mu)x}} + \frac{4\mu^2+5\mu-1}{4} \ge \frac{\mu(1+\mu)^2}{\sqrt{(1+\mu)^2+1+\mu}} + \frac{4\mu^2+5\mu-1}{4}.$$

It is easy to check that when $4\mu^2+5\mu-1 \le 0$ we have

$$\frac{\mu(1+\mu)^2}{\sqrt{(1+\mu)^2+1+\mu}} + \frac{4\mu^2+5\mu-1}{4} \ge 0 \text{ iff } 24\mu^4+49\mu^3+8\mu^2-19\mu+2 \le 0.$$

Thus $f'(x) \ge 0$, which implies that $f(x)$ is increasing. Thus

$$\frac{1}{4} < f(x) \le f\left(\frac{1}{4}\right) = \mu(1+\mu)\frac{1+\mu+\sqrt{(1+\mu)^2+1+\mu}}{2} + \frac{4\mu^2+5\mu-1}{16}.$$

By combining the above two cases, we always have (2.6). Furthermore it is easy to check that, when $8\mu^3+20\mu^2+13\mu-5 \le 0$, (2.6) is equivalent to $32\mu^4-8\mu^3-159\mu^2-130\mu+25 < 0$, a contradiction. ∎

Theorem 2.5. *If G is a 3-free digraph, then $\beta(G) < 0.8616\gamma(G)$.*

Proof. We prove the theorem by induction on the number of vertices of G. Set $\mu = 0.16065$. Then $1/(1+\mu) < 0.8616$ and μ satisfies all four inequalities in Theorem 2.4. By Theorem 2.4 there exists a vertex v and a partition V_1, V_2, $\{v\}$ where the number of missing edges, denoted ρ, is at least $(1+\mu)$ times the number of decycling edges, denoted τ; that is, $\tau < \rho/(1+\mu)$. Obviously $\beta(G) \le \beta(G[V_1]) + \beta(G[V_2]) + \tau$. By induction hypothesis, $\beta(G[V_1]) < 0.8616\gamma(G[V_1])$ and $\beta(G[V_2]) < 0.8616\gamma(G[V_2])$. Putting all these together yields

$$\begin{aligned}\beta(G) &\le \beta(G[V_1]) + \beta(G[V_2]) + \tau \\ &< 0.8616\gamma(G[V_1]) + 0.8616\gamma(G[V_2]) + \rho/(1+\mu) \le 0.8616\gamma(G).\end{aligned}$$

∎

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