Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 28, pp. 99-116, May 2015

ELA

BOUNDING THE CP-RANK BY GRAPH PARAMETERS*

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Abstract. The cp-rank of a graph G, $\mathrm{cpr}(G)$, is the maximum cp-rank of a completely positive matrix with graph G. One obvious lower bound on $\mathrm{cpr}(G)$ is the (edge-) clique covering number, $\mathrm{cc}(G)$, i.e., the minimal number of cliques needed to cover all of G's edges. It is shown here that for a connected graph G, $\mathrm{cpr}(G) = \mathrm{cc}(G)$ if and only if G is triangle free and not a tree. Another lower bound for $\mathrm{cpr}(G)$ is $\mathrm{tf}(G)$, the maximum size of a triangle free subgraph of G. We consider the question of when does the equality $\mathrm{cpr}(G) = \mathrm{tf}(G)$ hold.

Key words. Completely positive matrices, cp-rank, clique covering number, maximum size triangle free subgraph, outerplanar graph.

AMS subject classifications. 15B48, 05C50, 05C10

1. Introduction.

A square matrix A is *completely positive* if it has a factorization

$$A = BB^T, \quad B \geq 0, \tag{1.1}$$

where B is not necessarily square. The set of all $n \times n$ completely positive matrices is denoted by $\mathcal{CP}_n$. For $A \neq 0$, the minimal number of columns in such a B is the *cp-rank* of A, denoted here by $\mathrm{cpr}(A)$. The factorization (1.1) is a *cp-factorization* of A; if the number of columns of B is $\mathrm{cpr}(A)$, (1.1) is a *minimal cp-factorization*. For a matrix of order $n \leq 4$, $\mathrm{cpr}(A) \leq n$. In general, however, estimating the cp-rank of completely positive matrices is an open problem. A tight upper bound on the cp-rank of a rank r, $r \geq 2$, completely positive matrix (of any order) is known [1, 8]: $\frac{r(r+1)}{2} - 1$, see also [3, Section 3.2]. This yields the upper bound $\frac{n(n+1)}{2} - 1$ on the cp-ranks of $n \times n$ matrices, but this bound is not tight: in [14] it was shown that for $n \geq 5$ the least upper bound on the cp-ranks of $n \times n$ completely positive matrices is not greater than $\frac{n(n+1)}{2} - 4$.

It was conjectured by Drew, Johnson and Loewy in 1994 that $\mathrm{cpr}A \leq \lfloor \frac{n^2}{4} \rfloor$ for every $n \times n$ matrix, $n \geq 4$ [7]. This bound holds for $n = 5$ [10, 13]. However, recently this conjecture (the DJL conjecture) was disproved by Bomze, Schachinger

*Received by the editors on August 16, 2014. Accepted for publication on May 18, 2015. Handling Editor: Kevin vander Meulen.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 28, pp. 99-116, May 2015

and Ullrich, who presented counter examples for any $n \geq 7$, and showed that the correct tight upper bound is of the order $\frac{n^2}{2}$ [4, 5]. Finding an exact tight upper bound on the cp-ranks of $n \times n$ matrices of order $n \geq 6$ is still an open problem, and it is not known whether the DJL bound holds for $n = 6$.

We denote

$$p_n = \max\{\mathrm{cpr}(A)|A \text{ is an } n \times n \text{ completely positive matrix}\},$$

and for a (simple, undirected) graph G on n vertices the *cp-rank* of G, $\mathrm{cpr}(G)$, is defined by

$$\mathrm{cpr}(G) = \max\{\mathrm{cpr}(A)|A \text{ is completely positive and } G(A) = G\},$$

where, as usual, the graph $G(A)$ of an $n \times n$ symmetric matrix A is the simple graph on n vertices, such that for $i \neq j$, ij is an edge if and only if $a_{ij} \neq 0$. It can be shown that if G' is a subgraph of G, then $\mathrm{cpr}(G') \leq \mathrm{cpr}(G)$ (see Lemma 3.2(a) below). In particular, $p_n = \mathrm{cpr}(K_n)$, where K_n is the complete graph on n vertices. Thus $\mathrm{cpr}(K_n)$ is the tight upper bound on the cp-ranks of all $n \times n$ completely positive matrices, and for an $n \times n$ matrix A which is not positive, $\mathrm{cpr}(A) \leq \mathrm{cpr}(G(A))$ may provide a better estimate on $\mathrm{cpr}(A)$. Moreover, bounds of the cp-rank involving graphs can be used to estimate the cp-ranks of certain positive matrices, e.g. [13, Theorem 4.1]. Therefore it is useful to study the bounds on the cp-rank in terms of graphs. A result of this kind was obtained in [11], where graphs G on n vertices such that $\mathrm{cpr}(G) = n$ were characterized. (Note that the number of vertices of a graph G is a lower bound on $\mathrm{cpr}(G)$, since a nonsingular completely positive matrix with graph G always exists, and $\mathrm{cpr}(A) \geq \mathrm{rank}(A)$ for every completely positive matrix A [3, Proposition 3.2].) In this paper we consider the question of equality between $\mathrm{cpr}(G)$ and two other (purely graph theoretic) graph parameters: $\mathrm{cc}(G)$, the minimum number of cliques needed to cover G's edges, and $\mathrm{tf}(G)$, the maximum number of edges in a triangle free subgraph of G. Both these parameters are lower bounds on $\mathrm{cpr}(G)$ (see Sections 4 and 5). We will show that $\mathrm{cpr}(G) = \mathrm{cc}(G)$ if and only if G is a triangle free graph with no tree component. In the course of the proof we also show that $\mathrm{tf}(G) \geq \mathrm{cc}(G)$, i.e., the lower bound $\mathrm{tf}(G)$ on $\mathrm{cpr}(G)$ is tighter than $\mathrm{cc}(G)$. We then consider the question when is $\mathrm{cpr}(G) = \mathrm{tf}(G)$. We will show that a necessary condition for this equality to hold is that each component of G has a block on more than 3 vertices, and that $\mathrm{cpr}(G) = \mathrm{tf}(G)$ for graphs satisfying this necessary condition that belong to certain classes of graphs, including graphs with no odd cycle on 5 vertices or more and outerplanar graphs. The graph-associated bounds $\mathrm{tf}(G)$ and $\mathrm{cc}(G)$ on the cp-rank may be used to bound the cp-ranks of certain completely positive matrices that lie on the boundary of the completely positive cone (and may be positive!). This will be demonstrated in a separate paper on 6×6 completely positive matrices.

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The paper is organized as follows: in Section 2 we discuss terminology and background, and in Section 3 some basic facts about the cp-rank and minimal cp-factorizations. In Section 4 we characterize all graphs for which $\mathrm{cpr}(G) = \mathrm{cc}(G)$, and in Section 5 we discuss $\mathrm{tf}(G)$ and graphs which satisfy $\mathrm{cpr}(G) = \mathrm{tf}(G)$.

2. Terminology and Background. Our graph notations and terminology mostly follow [6], but we mention a few of these here. We consider only simple graphs. Given a graph $G = (V, E)$, we refer to G's vertex set V also as $V(G)$, and to its edge set E as $E(G)$. The *size* of G is the number of its edges $|E(G)|$. A graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$ (notation: $G' \subseteq G$). The subgraph G' is *induced* if E' consists of all the edges of G which have both ends in V'. A *clique* of G is the vertex set of a complete subgraph. For $e \in E(G)$, $G - e$ denotes the graph obtained from G by deleting the edge e, and for $G' \subseteq G$ and $e \in E(G) \setminus E(G')$, $G' + e$ denotes the graph obtained from G' by adding the edge e and its vertices. For $v \in V(G)$, $G - v$ denotes the graph obtained from G by deleting the vertex v and its incident edges. More generally, for $U \subseteq V(G)$, $G - U$ denotes the subgraph of G induced on the complement of U, $V(G) \setminus U$. For $U \subseteq V(G)$, $G[U]$ denotes the induced subgraph of G with vertex set U. The union of two graphs $G_1 \cup G_2$ is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Similarly $G_1 \cap G_2$ has vertex set $V(G_1) \cap V(G_2)$ and edge set $E(G_1) \cap E(G_2)$. For a vertex v, $d(v)$ denotes the *degree* of v, i.e., the number of edges incident with v.

A *cut-vertex* is a vertex whose deletion disconnects a (connected) component of G. A graph is 2-*connected* if it has at least 3 vertices, and no cut-vertex. A *block* of G is a connected subgraph that does not have a cut-vertex, and is maximal with respect to this property (so each block is either an isolated vertex, or an edge or a 2-connected subgraph of G).

The complete graph on n vertices is denoted by K_n, and the complete bipartite graph with partition classes of sizes m and k by $K_{m,k}$. A cycle on n vertices is denoted by C_n. The graph obtained by adding to C_{2k} a chord from every even vertex to the subsequent even vertex (including vertex $2k$ to 2, and assuming the vertices of C_{2k} are labeled consecutively), is denoted by S_{2k}. The graph on n vertices consisting of $n-2$ triangles sharing a common base is denoted by T_n.

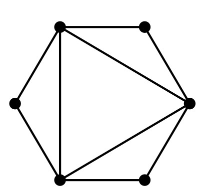

Figure 1: S_6

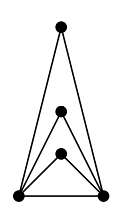

Figure 2: T_5

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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A completely positive matrix A such that $G(A) = G$ is called a *cp-realization of* G. A matrix which is both positive semidefinite and (entrywise) nonnegative is called *doubly nonnegative*. Every completely positive matrix is doubly nonnegative, but for matrices of order 5 or more the reverse implication does not hold. A graph G is *completely positive* if every doubly nonnegative matrix with this graph is completely positive. The following characterization of completely positive graphs was obtained in a series of papers by Maxfield and Minc, Berman and Hershkowitz, Berman and Grone, Kogan and Berman and Ando, see [3] and the reference therein.

Proposition 2.1. *Let G be a graph. Then the following are equivalent:*

(a) *G is completely positive.*
(b) *G contains no odd cycle of length 5 or more.*
(c) *Each block of G either has at most 4 vertices, or is bipartite, or is a T_k for some k.*

We refer to an odd cycle of length 5 or more as a *long odd cycle*.

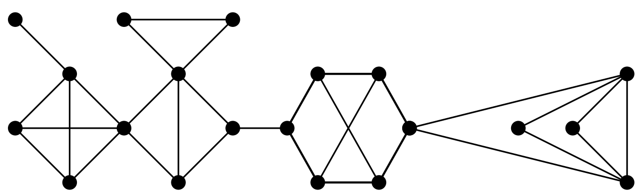

Figure 3: A graph with no long odd cycle

We also recall here the known characterizations of graphs G for which $\mathrm{cpr}(G)$ is minimal (i.e., equal to $|V(G)|$), and of graphs G which have the property that every cp-realization A of G has minimal cp-rank (i.e., $\mathrm{cpr}(A) = \mathrm{rank}(A)$).

Proposition 2.2. [11] *The following are equivalent for a connected graph G on n vertices:*

(a) $\mathrm{cpr}(G) = n$.
(b) $|V(G')| \geq |E(G')|$ *for every triangle free subgraph G' of G.*
(c) *Each block of G is either a K_4 or a subgraph of S_{2k} for some $k \geq 3$, and at most one block has more than 3 vertices.*

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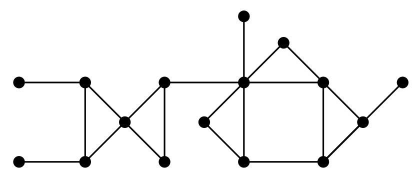

Figure 4: A graph G with $\mathrm{cpr}(G) = |V(G)|$

Proposition 2.3. [11] *The following are equivalent for a connected graph G on n vertices:*

(a) $\mathrm{cpr}(A) = \mathrm{rank}(A)$ *for every cp-realization A of G.*
(b) *Each block of G is either an edge or an odd cycle, and at most one block has more than 3 vertices.*

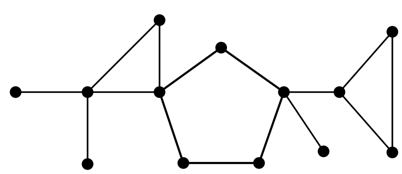

Figure 5: A graph G s.t. $\mathrm{cpr}(A) = \mathrm{rank}(A)$ for every cp-realization A of G

In particular, $\mathrm{cpr}(A) = \mathrm{rank}(A)$ for every completely positive matrix A of order at most 3. Propositions 2.2 and 2.3 rely on the following earlier results:

Proposition 2.4. *If G is a connected triangle free graph on $n \geq 4$ vertices and A is a CP matrix realization of G, then:*

(a) [2] *If G is a tree,* $\mathrm{cpr} A = \mathrm{rank}\, A$.
(b) [7] *If G is not a tree,* $\mathrm{cpr} A = |E(G)|$.

Some additional notations: the nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$, and for $\mathbf{x} \in \mathbb{R}^n_+$, the *support* of $\mathbf{x}$ is $\mathrm{supp}\, \mathbf{x} = \{1 \leq i \leq n \,|\, x_i \neq 0\}$. We often use the fact that when $B = [\mathbf{b}_1|\dots|\mathbf{b}_k]$, (1.1) is equivalent to

$$A = \sum_{i=1}^{k} \mathbf{b}_i \mathbf{b}_i^T, \quad \mathbf{b}_i \in \mathbb{R}^n_+ . \tag{2.1}$$

The sum (2.1) is called a *cp-decomposition* of A (a *minimal cp-decomposition* if $\mathrm{cpr}(A) = k$). The vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$ denote the standard basis vectors in $\mathbb{R}^n$ (i.e., every entry of $\mathbf{e}_i$ is zero, except for entry i, which is 1).

We denote by $\mathbb{R}^{m\times n}$ the space of all $m \times n$ real matrices, and by $\mathbb{R}^{m\times n}_+$ the cone of nonnegative matrices in $\mathbb{R}^{m\times n}$. The matrix E_{ij} denotes the $n \times n$ matrix with all entries zero except for the ij entry, which is equal to 1. Other matrix notations: I_n

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and J_n are the identity matrix of order $n \times n$ and the all ones matrix of that order, respectively. For $A \in \mathbb{R}^{n\times n}$ and $\alpha \subseteq \{1,\dots,n\}$, $A[\alpha]$ denotes the principal submatrix of A on rows and columns α. We abbreviate $A[\{i_1,\dots,i_k\}]$ as $A[i_1,\dots,i_k]$. For $M \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times n}$, $M \oplus N$ is the direct sum of M and N.

3. Basic results on cpr(G) and minimal cp-factorization. In this section we collect some basic properties of the parameter $\operatorname{cpr}(G)$, and some technical lemmas about it and about minimal cp-factorizations (some cp-rank results depend on the existence of special minimal cp-factorizations). We begin with the following observation:

LEMMA 3.1. *Let $G = G_1 \cup G_2$, where $G_i = (V_i, E_i)$, $i = 1, 2$, are induced subgraphs of G. Let A be a cp-realization of G. Then A can be represented as $A = A_1 + A_2$, where each A_i is a completely positive matrix, which is zero except for $A_i[V_i]$, $G(A_i[V_i]) \subseteq G_i$, $i = 1, 2$, and*

$$\operatorname{cpr}(A) = \operatorname{cpr}(A_1) + \operatorname{cpr}(A_2).$$

If $|V_1 \cap V_2| \geq 1$ and all the blocks of G_1 are edges and triangles, then $A_1[V_1]$ can be chosen to be singular.

Proof. Let $A = BB^T$, $B = [\mathbf{b}_1|\dots|\mathbf{b}_k] \in \mathbb{R}_+^{n\times k}$, be a minimal cp-factorization of A. Define

$$\Omega_1 = \{1 \leq i \leq k \mid \operatorname{supp} \mathbf{b}_i \subseteq V_1\}, \quad \text{and} \quad \Omega_2 = \{1,\dots,k\} \setminus \Omega_1.$$

Let $A_i = \sum_{j\in\Omega_i} \mathbf{b}_j \mathbf{b}_j^T$, $i = 1, 2$. Then A_1 and A_2 are completely positive and $G(A_i[V_i]) \subseteq G_i$, $i = 1, 2$. Clearly, $\operatorname{cpr}(A) = |\Omega_1| + |\Omega_2|$, and by the minimality of the cp-factorization of A, $\operatorname{cpr}(A_i) = |\Omega_i|$, $i = 1, 2$. Thus

$$\operatorname{cpr}(A) = |\Omega_1| + |\Omega_2| = \operatorname{cpr}(A_1) + \operatorname{cpr}(A_2). \tag{3.1}$$

If $|V_1 \cap V_2| \geq 1$ and all of the blocks of G_1 are triangles and edges, then A_1 defined above satisfies $\operatorname{cpr}(A_1) = \operatorname{rank}(A_1)$ by Proposition 2.3. If $A_1[V_1]$ is nonsingular, let $m \in V_1 \cap V_2$, and let $\delta > 0$ be the maximal positive number such that $Q_1 = A_1 - \delta E_{mm}$ is positive semidefinite. Then Q_1 is doubly nonnegative, $\operatorname{rank}(Q_1) = \operatorname{rank}(A_1) - 1$ and $G(Q_1) = G(A_1) \subseteq G_1$. By Proposition 2.1, Q_1 is completely positive, and by Proposition 2.3,

$$\operatorname{cpr}(Q_1) = \operatorname{rank}(Q_1) = \operatorname{rank}(A_1) - 1 = \operatorname{cpr}(A_1) - 1.$$

Let $Q_2 = A_2 + \delta E_{mm}$. Then $A = Q_1 + Q_2$, $\operatorname{cpr}(Q_2) \leq \operatorname{cpr}(A_2) + 1$, $G(Q_i) = G(A_i) \subseteq G_i$, $i = 1, 2$. This yields

$$\operatorname{cpr}(A) \leq \operatorname{cpr}(Q_1) + \operatorname{cpr}(Q_2) \leq \operatorname{cpr}(A_1) - 1 + \operatorname{cpr}(A_2) + 1 = \operatorname{cpr}(A_1) + \operatorname{cpr}(A_2) = \operatorname{cpr}(A).$$

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Thus the middle inequalities are actually equalities, and $A = Q_1 + Q_2$ satisfies the last assertion. ☐

Note that Lemma 3.1 includes the case that G has a cut-vertex ($G_1 \cap G_2$ is a single vertex). Some basic facts on $\mathrm{cpr}(G)$ are stated in the next lemma.

Lemma 3.2. *Let G be a graph on n vertices.*

(a) *If G' is a subgraph of G, then* $\mathrm{cpr}(G') \le \mathrm{cpr}(G)$.
(b) *If $G = G_1 \cup G_2$, where G_1 and G_2 are induced subgraphs of G, then* $\mathrm{cpr}(G) \le \mathrm{cpr}(G_1) + \mathrm{cpr}(G_2)$.
(c) *If G is the disjoint union of graphs G_1 and G_2, then* $\mathrm{cpr}(G) = \mathrm{cpr}(G_1) + \mathrm{cpr}(G_2)$.

Proof. Part (a) was proved in [11], and we include a short proof here for completeness: Suppose $|V(G)| = n$ and $|V(G')| = m$. Every cp-realization A' of G' can be extended to a cp-realization A_ε of G for every $\varepsilon > 0$, by setting

$$(A_\varepsilon)_{ij} = \begin{cases} (A')_{ij} & \text{if } ij \in E(G') \\ \varepsilon & \text{if } ij \in E(G) \setminus E(G') \\ (A')_{ii} + (n-m)\varepsilon & \text{if } j = i \text{ and } i \in V(G') \\ n\varepsilon & \text{if } j = i \text{ and } i \in V(G) \setminus V(G') \end{cases}$$

Then A_ε is completely positive as a sum of the completely positive matrix $A' \oplus 0$ and a diagonally dominant nonnegative matrix. For every $\varepsilon > 0$, $\mathrm{cpr}(A_\varepsilon) \le \mathrm{cpr}(G)$, and since $\lim_{\varepsilon \to 0+} A_\varepsilon = A' \oplus 0$, we get that for every cp-realization A' of G', $\mathrm{cpr}(A') \le \liminf_{\varepsilon \to 0+} \mathrm{cpr}(A_\varepsilon) \le \mathrm{cpr}(G)$. Thus $\mathrm{cpr}(G') \le \mathrm{cpr}(G)$.

Part (b) follows by combining part (a) and Lemma 3.1.

Part (c): If $A = A' \oplus A''$, where both A' and A'' are completely positive, then $\mathrm{cpr}(A) = \mathrm{cpr}(A') + \mathrm{cpr}(A'')$ (see, e.g., [13, Proposition 2.2]). ☐

The bound on $\mathrm{cpr}(G)$ in part (b) of Lemma 3.2 may be tightened when $G_1 \cap G_2$ is a cut-vertex of G, and G_1 is very small:

Lemma 3.3. *Let $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2)$ is a single vertex v, and G_1 is connected and has m vertices, $m = 2$ or 3, Then* $\mathrm{cpr}(G) = m - 1 + \mathrm{cpr}(G_2)$.

Proof. Without loss of generality assume $V_1 = V(G_1) = \{1, \ldots, m\}$ and $V_2 = V(G_2) = \{m, \ldots, n\}$. Let A be a cp-realization of G such that $\mathrm{cpr}(A) = \mathrm{cpr}(G)$. By Lemma 3.1, $A = A_1 + A_2$, $\mathrm{cpr}(A) = \mathrm{cpr}(A_1) + \mathrm{cpr}(A_2)$, where each A_i is completely positive and zero except for $A_i[V_i]$, $G(A_i[V_i])) \subseteq G_i$, and $A_1[V_1]$ is singular. Since $m \le 3$, by Proposition 2.3, $\mathrm{cpr}(A_1) = \mathrm{rank}(A_1) \le m - 1$ and thus

$$\mathrm{cpr}(G) = \mathrm{cpr}(A) = \mathrm{cpr}(A_1) + \mathrm{cpr}(A_2) \le m - 1 + \mathrm{cpr}(G_1).$$

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To prove the reverse inequality, let Q_1 be zero except for its leading $m \times m$ principal submatrix. If $m = 2$ let that leading 2×2 submatrix be J_2, and if $m = 3$ let the leading 3×3 submatrix be

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ (if } G_1 \text{ is a triangle) or } \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \text{ (if } G_1 \text{ is a path).}$$

In all these cases, Q_1 is completely positive and $\mathrm{cpr}(Q_1) = \mathrm{rank}(Q_1) = m - 1$. Let Q_2 have zero $m - 1$ first rows and columns with $Q_2[m, \dots, n]$ a cp-realization of G_2 such that $\mathrm{cpr}(Q_2) = \mathrm{cpr}(G_2)$. Let $A = Q_1 + Q_2$. Decompose A as in Lemma 3.1: $A = A_1 + A_2$, where $\mathrm{cpr}(A) = \mathrm{cpr}(A_1) + \mathrm{cpr}(A_2)$ and $A_1[V_1]$ is singular. Then A_1 is equal to Q_1 in all entries except possibly the mm entry. Since $A_1[V_1]$ is singular, $\mathrm{rank}(A_1) \leq m - 1$. Since the first $m - 1$ rows of A_1 are equal to the corresponding rows of Q_1, and are linearly independent, $\mathrm{rank}(A_1) = m - 1$ and $A_1 = Q_1$. Thus $A_2 = Q_2$. We therefore have $\mathrm{cpr}(A_1) = m - 1$, $\mathrm{cpr}(A_2) = \mathrm{cpr}(G_2)$, yielding that

$$\mathrm{cpr}(G) \geq \mathrm{cpr}(A) = \mathrm{cpr}(A_1) + \mathrm{cpr}(A_2) = m - 1 + \mathrm{cpr}(G_2). \square$$

The following known result follows similarly from Lemma 3.1.

LEMMA 3.4. [3, Lemma 3.3] *Suppose a graph G has a non-isolated vertex v with $d(v) \leq 2$. Then*

$$\mathrm{cpr}(G) \leq d(v) + \mathrm{cpr}(G - v).$$

The remaining results in this section consider the existence of special minimal cp-factorizations. The following useful observation will be used:

LEMMA 3.5. [10, Observation 1] *Let $\mathbf{b}, \mathbf{d} \in \mathbb{R}^n_+$ such that* $\mathrm{supp}\, \mathbf{b} \subseteq \mathrm{supp}\, \mathbf{d}$. *Then there exist vectors $\tilde{\mathbf{b}}, \tilde{\mathbf{d}} \in \mathbb{R}^n_+$ such that $\tilde{\mathbf{b}}\tilde{\mathbf{b}}^T + \tilde{\mathbf{d}}\tilde{\mathbf{d}}^T = \mathbf{b}\mathbf{b}^T + \mathbf{d}\mathbf{d}^T$, $\mathrm{supp}\, \tilde{\mathbf{d}} = \mathrm{supp}\, \mathbf{d}$, $\mathrm{supp}\, \tilde{\mathbf{b}} \subseteq \mathrm{supp}\, \tilde{\mathbf{d}}$, $\mathrm{supp}\, \mathbf{d} \setminus \mathrm{supp}\, \mathbf{b} \subseteq \mathrm{supp}\, \tilde{\mathbf{b}}$, and for at least one $i \in \mathrm{supp}\, \mathbf{b}$, $i \notin \mathrm{supp}\, \tilde{\mathbf{b}}$.*

We first consider minimal cp-factorizations of nonsingular cp-realizations of trees.

LEMMA 3.6. *Let G be a tree on n vertices, and let A be a nonsingular completely positive matrix whose graph is G. Then for every vertex $1 \leq i \leq n$ there exists a minimal cp-factorization $A = BB^T$ where $B \in \mathbb{R}^{n \times n}_+$, the supports of $n - 1$ of the columns of B are the $n - 1$ edges of G, and the support of one column is $\{i\}$.*

Proof. By relabeling the vertices we may assume for convenience that $i = n$. Let $\delta > 0$ be the maximal such that $A_0 = A - \delta E_{nn}$ is positive semidefinite. Then

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A publication of the International Linear Algebra Society
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A_0 is a singular doubly nonnegative matrix whose graph is G. Thus $\mathrm{cpr}A_0 = n-1$ (Proposition 2.4(a) and the paragraph preceding that proposition). Let $A_0 = B_0B_0^T$ where $B_0 \in \mathbb{R}_+^{n\times(n-1)}$. Since the support of each column of B_0 is a clique in $G(A_0)$, which is triangle free and has $n-1$ edges, the $n-1$ columns of B_0 are supported by the $n-1$ edges of G. Thus $A = BB^T$, where $B = [B_0\ \sqrt{\delta}\mathbf{e}_n] \in \mathbb{R}_+^{n\times n}$. □

For 2×2 matrices this implies

Corollary 3.7. *Let $A \in \mathcal{CP}_2$. Then for every $1 \le i \le 2$ there exists a minimal cp-factorization of A, $A = BB^T$, in which i is in the support of at most one column of B.*

Proof. If A is singular, $\mathrm{rank}(A) = 1$ and thus $\mathrm{cpr}(A) = 1$, which implies the result. If A is nonsingular, either A is diagonal, in which case the result is obvious, or $G(A)$ is a tree. In the latter case, assume without loss of generality that $i = 1$, and apply Lemma 3.6 for the vertex 2. □

For 3×3 matrices we have the following:

Lemma 3.8. *Let $A \in \mathcal{CP}_3$. Then*

(a) *For every $1 \le i < j \le 3$ such that $a_{ij} > 0$, there exists a minimal cp-factorization of A, $A = BB^T$, in which there is exactly one column of B whose support contains $\{i, j\}$.*
(b) *For every $1 \le i \le 3$ such that $a_{ii} > 0$, there exists a minimal cp-factorization of A, $A = BB^T$, in which at least one column and at most two columns have i in their support.*

Proof. By simultaneously permuting the rows and columns of A we may assume in (a) that $i = 1$ and $j = 2$, and in (b) that $i = 1$. If A has a zero diagonal entry, then it has a zero row (and column), so A is a direct sum of $A' \in \mathcal{CP}_2$ and a 1×1 zero matrix. In that case, (b) is trivially true (since $p_2 = 2$), and (a) easily follows from Lemma 3.6 if A' is nonsingular. Otherwise, A' is singular, $\mathrm{rank}\,A' = 1$ then $\mathrm{cpr}A = \mathrm{cpr}A' = 1$, and again (a) holds.

So suppose $A \in \mathcal{CP}_3$ and all the diagonal entries of A are nonzero. Then in every cp-factorization 1 is in the support of at least one column. By [3, Corollary 2.13], either $A = LL^T$, where L is a lower triangular nonnegative matrix , or $A = UU^T$, where U is an upper triangular nonnegative matrix. In the first case, 1 is in the support of exactly one column of L (the first), i.e. (b) holds. Also, $\{1, 2\}$ is contained in the support at most one column of L (again, the first column). If $a_{12} > 0$, it is also true that $\{1, 2\}$ is contained in the support of at least one column of L. This proves (a) and (b) in this case.

In the second case, if both the supports of the second and last column of U contain

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A publication of the International Linear Algebra Society
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$\{1,2\}$, then U has the following pattern:

$$\begin{bmatrix} * & + & + \\ 0 & + & + \\ 0 & 0 & + \end{bmatrix},$$

where $*$ is either positive or zero ($u_{33} > 0$ since A has no zero row). By Lemma 3.5 the last two columns in U may be replaced by two columns with the pattern

$$\begin{matrix} * & + \\ * & + \\ + & + \end{matrix},$$

where at least one of the $*$'s is zero. The nonnegative matrix $\bar{U}$ resulting from this replacement satisfies $\bar{U}\bar{U}^T = UU^T = A$, and the supports of its first two columns do not contain $\{1,2\}$, which proves (a). As for (b), if only two columns in U have a nonzero first entry, the claim is obviously true. Otherwise, U has the following pattern

$$\begin{bmatrix} + & + & + \\ 0 & * & * \\ 0 & 0 & + \end{bmatrix}.$$

Using Lemma 3.5 on columns 1 and 3 of U we get a nonnegative matrix $\bar{U}$ in which the first entry on the first column is 0, implying (b). ∎

4. Graphs with $\mathrm{cpr}(G) = \mathrm{cc}(G)$. Let A be a cp-realization of a graph G with $\mathrm{cpr}(A) = \mathrm{cpr}(G)$. If $A = \sum_{i=1}^k \mathbf{b}_i\mathbf{b}_i^T$, $k = \mathrm{cpr}(G)$, is a minimal cp-decomposition, then the graphs $G(\mathbf{b}_i\mathbf{b}_i^T)$, $i = 1, \ldots, k$, are complete subgraphs of $G = G(A)$ covering the whole graph, and thus $\mathrm{cc}(G) \le \mathrm{cpr}(G)$. In this section we characterize all graphs for which $\mathrm{cpr}(G) = \mathrm{cc}(G)$. If G is a disjoint union of $G_1 \cup G_2$, then clearly $\mathrm{cc}(G) = \mathrm{cc}(G_1) + \mathrm{cc}(G_2)$. Combined with Lemma 3.2(c), this means that it suffices to consider connected graphs. Note that a graph G consisting of a single vertex is considered to be a (trivial) tree with zero edges, and has $\mathrm{cpr}(G) = 1$.

THEOREM 4.1. *A connected graph G satisfies* $\mathrm{cpr}(G) = \mathrm{cc}(G)$ *if and only if G is triangle free, but not a tree.*

Proof. Let G have n vertices. If G is triangle free, then the maximal cliques of G are G's edges, and $\mathrm{cc}(G) = |E(G)|$. By Proposition 2.2 $\mathrm{cpr}A = n$. Thus $\mathrm{cpr}(G) = n > n-1 = |E(G)| = \mathrm{cc}(G)$. If G is triangle free and not a tree, then $n \ge 4$, and Proposition 2.4(b) implies that $\mathrm{cpr}(G) = |E(G)|$, and thus $\mathrm{cpr}(G) = \mathrm{cc}(G)$.

To show that equality does not hold for graphs which are not triangle free, let $\mathrm{tc}(G)$ be the minimal number of triangles and edges of G needed to cover all of the

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edges of G. Lehel and Tuza proved in [9] that $\mathrm{tc}(G) \leq \mathrm{tf}(G)$, and equality holds if and only if G is triangle free. In addition, it is obvious that $\mathrm{cc}(G) \leq \mathrm{tc}(G)$ holds for every graph G. That is, for every graph G

$$\mathrm{cc}(G) \leq \mathrm{tc}(G) \leq \mathrm{tf}(G),$$

and if G is not triangle free, the right inequality is strict. Hence if G is not triangle free, there exists a triangle free subgraph G' of G such that $\mathrm{tf}(G) = |E(G')| > \mathrm{cc}(G)$, implying that

$$\mathrm{cpr}(G) \geq \mathrm{cpr}(G') = |E(G')| > \mathrm{cc}(G). \ \square$$

In the course of the proof of Theorem 4.1 we have also shown that among the two lower bounds on $\mathrm{cpr}(G)$, $\mathrm{cc}(G)$ and $\mathrm{tf}(G)$, the latter is tighter, i.e., we have proved:

THEOREM 4.2. *For every connected graph G,* $\mathrm{tf}(G) \geq \mathrm{cc}(G)$.

5. Graphs with $\mathbf{cpr}(G) = \mathbf{tf}(G)$. Let G' be a triangle free subgraph of a maximum size of a graph G. Then $\mathrm{tf}(G) = |E(G')|$, and by Proposition 2.4, $\mathrm{cpr}(G') \geq |E(G')|$. Since $\mathrm{cpr}(G') \leq \mathrm{cpr}(G)$ we get that $\mathrm{tf}(G) \leq \mathrm{cpr}(G)$. By Turan's Theorem, the number of edges in a triangle free graph on n vertices is at most $\lfloor \frac{n^2}{4} \rfloor$ (and this bound is attained when the graph is complete bipartite, with the independent bipartition sets being as balanced as possible). Thus $\mathrm{tf}(G) \leq \lfloor \frac{n^2}{4} \rfloor$ for every graph on n vertices. The examples disproving the DJL conjecture in [4, 5] show that for $n \geq 7$, $\mathrm{cpr}(K_n) = p_n > \lfloor \frac{n^2}{4} \rfloor = \mathrm{tf}(K_n)$. That is, $\mathrm{cpr}(G)$ may be strictly greater than $\mathrm{tf}(G)$. We now consider the question when does equality hold between these two parameters.

We begin with some trivial observations regarding $\mathrm{tf}(G)$.

LEMMA 5.1. *Let G be a graph, $G = G_1 \cup G_2$.*

(a) *If the union is disjoint, then* $\mathrm{tf}(G) = \mathrm{tf}(G_1) + \mathrm{tf}(G_2)$.
(b) *If $V(G_1) \cap V(G_2)$ is an independent set of G, then* $\mathrm{tf}(G) = \mathrm{tf}(G_1) + \mathrm{tf}(G_2)$.
(c) *If $V(G_1) \cap V(G_2)$ is a single vertex and G_1 is a K_m, $m = 2$ or 3, then* $\mathrm{tf}(G) = m - 1 + \mathrm{tf}(G_2)$.

Proof. Parts (a) and (b) are trivial. Part (c) is a special case of part (b), using the fact that $\mathrm{tf}(K_m) = m - 1$ when $m = 2$ or 3. $\square$

The proofs of the next two theorems (Theorems 5.3 and 5.4) are implicit in Propositions 2.1 and 2.2, and rely also on the following observation.

LEMMA 5.2. *If G' is a triangle free subgraph of a connected graph G, then there exists a spanning triangle free subgraph of G which has $|E(G')| + |V(G)| - |V(G')|$ edges.*

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Proof. If $V(G') \subsetneq V(G)$, then there exists an edge $e = xy$, $x \in V(G')$ and $y \in V(G) \setminus V(G')$, since G is connected. Then $G' + e$ is a triangle free subgraph of G. This step can be repeated $|V(G)| - |V(G')|$ times until a spanning triangle free subgraph is obtained. ☐

In particular, any triangle free subgraph of G of maximal size is necessarily spanning. Since every connected graph on n vertices has a spanning tree, $\mathrm{tf}(G) \geq n-1$ for every connected graph G on n vertices. The next theorem characterizes the connected graphs for which equality holds.

THEOREM 5.3. *A connected graph G on n vertices has $\mathrm{tf}(G) = n-1$ if and only if each block of G is either an edge or a triangle.*

Proof. By Lemma 5.2, if $\mathrm{tf}(G) = |V(G)| - 1$, then every triangle free subgraph G' of G is a tree. In particular, no block of G has more than 3 vertices. Conversely, suppose G is a connected graph on n vertices with blocks $G_1, \ldots, G_k$, and $|V(G_i)| = n_i$, $1 \leq i \leq k$, $n_i \in \{2, 3\}$. Then

$$n = \sum_{i=1}^{k} n_i - (k-1) = 1 + \sum_{i=1}^{k} (n_i - 1) = 1 + \sum_{i=1}^{k} \mathrm{tf}(G_i) = 1 + \mathrm{tf}(G),$$

since $\mathrm{tf}(K_m) = m - 1$ if $m = 2$ or $m = 3$. ☐

The next theorem characterizes the graphs G with $\mathrm{tf}(G) = |V(G)|$.

THEOREM 5.4. *A connected graph G on n vertices has $\mathrm{tf}(G) = n$ if and only if G has exactly one block on more than 3 vertices, and this block is a K_4 or a subgraph of S_{2k} for some $k \geq 3$.*

Proof. By Lemma 5.2, if $\mathrm{tf}(G) = |V(G)|$, G cannot contain a triangle free subgraph G' with more edges than vertices, otherwise G' could be extended to a spanning triangle free subgraph of size greater than n by adding $|V(G)| - |V(G')|$ edges, which would imply that $\mathrm{tf}(G) > |V(G)|$. Thus, by the equivalence of part (b) and (c) in Proposition 2.2, each block of G is either a K_4, or a subgraph of S_{2k} for some $k \geq 3$, with at most one block on more than 3 vertices. On the other hand, Theorem 5.3 guarantees the existence of at least one such block.

The converse also follows from Proposition 2.2 and Theorem 5.3: If G has the block structure described in the theorem, then $\mathrm{tf}(G) > n-1$ by Theorem 5.3 and, by Proposition 2.2, any spanning triangle free subgraph G' of G of maximal size has at most $|V(G')| = n$ edges. Thus $\mathrm{tf}(G) \leq n$. ☐

We can now consider the question which graphs satisfy the equality $\mathrm{tf}(G) = \mathrm{cpr}(G)$. Note that by Lemmas 3.2(c) and 5.1(a), a graph satisfies this equality if and only if each of its components does. It therefore suffices to consider connected graphs.

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From Lemma 5.1(c) and Lemma 3.3 we can deduce the following:

LEMMA 5.5. *Let G be a connected graph. If each block G' of G that has more than 3 vertices satisfies* $\mathrm{cpr}(G') = \mathrm{tf}(G')$*, and G has at least one such block, then* $\mathrm{cpr}(G) = \mathrm{tf}(G)$*.*

Proof. The proof uses induction on the number of blocks. For G consisting of one block there is nothing to prove. Assume that G satisfies the assumptions and has $k \geq 2$ blocks, and that the claim holds for graphs with less than k blocks.

Suppose $G = G_1 \cup G_2$, where G_1 and G_2 are connected and $G_1 \cap G_2$ is a single vertex, G_1 is either a K_2 or a K_3, and G_2 has $k-1$ blocks, at least one of which has more than 3 vertices. In that case,

$$\mathrm{tf}(G) = m - 1 + \mathrm{tf}(G_2) = m - 1 + \mathrm{cpr}(G_2) = \mathrm{cpr}(G)$$

by Lemma 5.1(c), the induction hypothesis, and Lemma 3.3.

Otherwise, $G = G_1 \cup G_2$, where G_1 and G_2 are connected and $G_1 \cap G_2$ is a single vertex, G_1 has more than 3 vertices, and G_2 has $k-1$ blocks, at least one of which on more than 3 vertices. By Lemma 3.2(b), the induction hypothesis, and Lemma 5.1(b),

$$\mathrm{tf}(G) \leq \mathrm{cpr}(G) \leq \mathrm{cpr}(G_1) + \mathrm{cpr}(G_2) = \mathrm{tf}(G_1) + \mathrm{tf}(G_2) = \mathrm{tf}(G),$$

implying that $\mathrm{cpr}(G) = \mathrm{tf}(G)$. □

A graph G such that $\mathrm{tf}(G) = |V(G)| - 1$ clearly does not satisfy $\mathrm{tf}(G) = \mathrm{cpr}(G)$. Here are some graphs that satisfy $\mathrm{tf}(G) = \mathrm{cpr}(G)$:

THEOREM 5.6. *If a connected graph G is one of the following graphs, then* $\mathrm{cpr}(G) = \mathrm{tf}(G)$*.*

(a) *A triangle free graph, which is not a tree.*
(b) *A graph that has no long odd cycle, which has a cycle on more than* 3 *vertices.*
(c) *A graph that has exactly one block on more than 3 vertices, and this block is a K_4 or a subgraph of S_{2k} for some $k \geq 3$.*

Proof. In case (a), $\mathrm{tf}(G) = |E(G)|$ by definition, and $\mathrm{cpr}(G) = |E(G)|$ by Proposition 2.4.

In case (b), by the structure of a graph with no long odd cycle (Proposition 2.1), each block G' of G on more than 3 vertices is either a K_4 (in which case $\mathrm{tf}(G') = \mathrm{cpr}(G') = 4$), or bipartite (in which case $\mathrm{tf}(G') = \mathrm{cpr}(G')$ by (a)), or a T_k (in which case $\mathrm{cpr}(T_k) = 2k-4$, by [3, Theorem 3.9], and $\mathrm{tf}(T_k) = 2k-4$, since its maximal size triangle free subgraph is $K_{2,k-2}$). Thus by Lemma 5.5 such G has $\mathrm{tf}(G) = \mathrm{cpr}(G)$.

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In case (c),

$$n = \mathrm{tf}(G) \le \mathrm{cpr}(G) = n.$$

(The left equality by Theorem 5.4 and the right one by Proposition 2.2.) □

Next we show that every connected outerplanar graph G that has at least one block on more than 3 vertices also satisfies $\mathrm{cpr}(G) = \mathrm{tf}(G)$. This generalizes (c) of the previous theorem. An *outerplanar graph* is a graph that can be drawn in the plane so that no two edges cross, and all the vertices lie on the boundary of the outer face. We use the well known facts that every subgraph of an outerplanar graph is outerplanar, has a vertex of degree at most 2, and if it is 2-connected, it has a unique Hamiltonian cycle.

THEOREM 5.7. *Every connected outerplanar graph G on n vertices with $\mathrm{tf}(G) \ge n$ satisfies $\mathrm{tf}(G) = \mathrm{cpr}(G)$.*

Proof. We prove the theorem by induction on n. By the assumption that $\mathrm{tf}(G) \ge n$, we have $n \ge 4$. If $n = 4$, then $4 \le \mathrm{tf}(G) \le \mathrm{cpr}(G) = 4$ implies the desired result. Let $n > 4$ and assume the claim holds for graphs with fewer than n vertices.

If G is not 2-connected, each of its blocks on more than 3 vertices satisfies the equality by the induction hypothesis, and there has to be at least one such block by Theorem 5.3. Hence by Lemma 5.5 G also satisfies $\mathrm{tf}(G) = \mathrm{cpr}(G)$. Thus it suffices to consider the case that G is 2-connected. In that case, G has a unique Hamiltonian cycle, and a vertex v of degree exactly 2.

Case 1: The two vertices adjacent to v are not adjacent to each other.

If $G - v$ has only blocks that are edges and triangles, then G itself is a subgraph of S_{2k} for some $k \ge 3$ and, by Theorem 5.4 and Proposition 2.2, $\mathrm{tf}(G) = \mathrm{cpr}(G) = n$. Otherwise, $G - v$ satisfies the induction hypothesis, and thus

$$\mathrm{tf}(G) \le \mathrm{cpr}(G) \le 2 + \mathrm{cpr}(G - v) = 2 + \mathrm{tf}(G - v) = \mathrm{tf}(G),$$

where the second inequality follows from Lemma 3.4, and the last equality from Lemma 5.1(b).

Case 2: The vertices u and w that are adjacent to v are also adjacent to each other.

Without loss of generality let $u = 1$, $v = 2$ and $w = 3$. Since the graph is Hamiltonian, there exists an additional vertex which is adjacent to 3 other than 1 and 2, label this vertex 4. (The vertex 4 may or may not be adjacent to 1, and there is at least one more vertex on the Hamiltonian cycle.)

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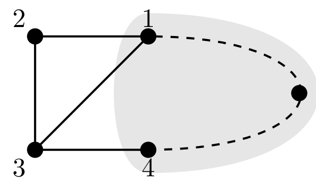

Figure 3: Case 2

Let $\alpha_1 = \{1,\dots,4\}$, $\alpha_2 = \{1,4,\dots,n\}$ and $G_i = G[\alpha_i]$, $i = 1,2$. Let A be a completely positive matrix such that $G(A) = G$ and $\mathrm{cpr}(A) = \mathrm{cpr}(G)$, and let $A = A_1+A_2$, where $G(A_1[\alpha_1]) \subseteq G_1$ and $G(A_2[\alpha_2]) \subseteq G_2$, and $\mathrm{cpr}(A) = \mathrm{cpr}(A_1)+\mathrm{cpr}(A_2)$.

Now set

$$t = \min_{1\le j\le 3} \frac{a_{1j}}{a_{2j}}.$$

Suppose t is attained at $j = m$. Since A is positive semidefinite and its first two rows are linearly independent by their sign pattern, m cannot be equal to 1, so $m = 2$ or $m = 3$. Let $S = I - tE_{12}$. Then SA_1S^T is doubly nonnegative and, compared to A_1, has one or two additional zero entries in the first row, in position $1,2$ or in position $1,3$. Because row 2 of A_2 is zero, $SA_2S^T = A_2$. Every 4×4 doubly nonnegative matrix is completely positive, hence SA_1S^T is completely positive, and so is $SAS^T = SA_1S^T + A_2$. The graph of SAS^T is a subgraph of G, and thus $\mathrm{cpr}(SAS^T) \le \mathrm{cpr}(G)$. But since $S^{-1} \ge 0$, we have that $\mathrm{cpr}(G) = \mathrm{cpr}(A) \le \mathrm{cpr}(SAS^T)$ (see [3, Proposition 3.3]). Thus $\mathrm{cpr}(SAS^T) = \mathrm{cpr}(G)$.

Denote the edge $\{1,2\}$ by e_1 and the edge $\{1,3\}$ by e_2. By Lemma 5.5 and the induction hypothesis $\mathrm{cpr}(G-e_1) = \mathrm{tf}(G-e_1)$, since the block of $G-e_1$ on vertices $\{1,3,4,\dots,n\}$ is 2-connected and has at least 4 vertices. Thus if $G(SAS^T) = G-e_1$ we have

$$\mathrm{tf}(G) \le \mathrm{cpr}(G) = \mathrm{cpr}(SAS^T) \le \mathrm{cpr}(G-e_1) = \mathrm{tf}(G-e_1) \le \mathrm{tf}(G),$$

and the equality $\mathrm{cpr}(G) = \mathrm{tf}(G)$ follows.

Since $\mathrm{cpr}(G-e_2) = \mathrm{tf}(G-e_2)$ by Case 1, a similar argument shows that if $G(SAS^T) = G-e_2$ then $\mathrm{cpr}(G) = \mathrm{tf}(G)$.

Finally, suppose $G(SAS^T) = G-\{e_1,e_2\}$. If $G-\{e_1,e_2\}$ has only K_2 and K_3 blocks, then G is a 2-connected subgraph of S_{2k} for some $k\ge 3$, and thus $\mathrm{cpr}(G) = n = \mathrm{tf}(G)$ by Proposition 2.2 and Theorem 5.6(c). Otherwise, $\mathrm{cpr}(G-\{e_1,e_2\}) = \mathrm{tf}(G-\{e_1,e_2\})$ by the induction hypothesis and Lemma 5.5. $\square$

A particular example of an outerplanr graph is the *fan* on n vertices, F_n: a graph on n vertices, consisting of a path on $n-1$ vertices and a vertex adjacent to all vertices

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A publication of the International Linear Algebra Society
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of the path (i.e., a suspended path).

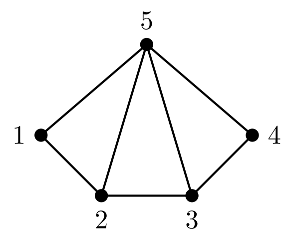

Figure 4: F_5

Corollary 5.8. *For $n \geq 4$,*

$$\operatorname{cpr}(F_n) = \begin{cases} \dfrac{3n-5}{2} & n \text{ is odd} \\[2ex] \dfrac{3n-4}{2} & n \text{ is even} \end{cases}.$$

Proof. Since $n \geq 4$, F_n is a 2-connected outerplanar graph on more than 3 vertices, so $\operatorname{cpr}(F_n) = \operatorname{tf}(F_n)$. It is easy to see that the triangle free subgraph consisting of the spanning n-cycle and every other chord is a maximum size triangle free subgraph of F_n. (More explicitly: Label the path vertices consecutively by $1, \ldots, n-1$ and the vertex of degree $n-1$ by n. The triangle free subgraph consists of the Hamiltonian cycle on all n vertices, and the chords jn, $j = 2k+1$, $k = 2, \ldots, \lfloor (n-3)/2 \rfloor$.) $\square$

Finally, let W_n be the *wheel*: a graph on n vertices, which consists of the cycle on $n-1$ vertices, and a vertex adjacent to all the cycle vertices (i.e., a suspended cycle). The wheel is a planar graph, but for $n \geq 4$ it is not an outerplanar one, and for $n \geq 5$ it does not belong to any of the graph classes mentioned in Theorem 5.6.

Theorem 5.9. *For $n \geq 4$,*

$$\operatorname{cpr}(W_n) = \operatorname{tf}(W_n) = \begin{cases} \dfrac{3n-3}{2} & n \text{ is odd} \\[2ex] \dfrac{3n-4}{2} & n \text{ is even} \end{cases}$$

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A publication of the International Linear Algebra Society
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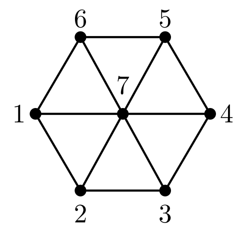

Figure 5: W_7

Proof. Since $\mathrm{cpr}(G) = 4$ for every graph G on 4 vertices, we have $\mathrm{cpr}(W_4) = 4$ as claimed. For $n \geq 5$, label the vertices of the $(n-1)$-cycle consecutively by $1, \dots, n-1$, and the vertex of degree $n-1$ by n. It is easy to see that the subgraph of W_n consisting of the cycle on vertices $1, \dots, n-1$ and every other edge adjacent to n (more explicitly: the edges jn, $j = 2k-1$, $k = 1, \dots, \lfloor (n-1)/2 \rfloor$) is a triangle free subgraph of W_n of maximal size, so that

$$\mathrm{tf}(W_n) = \begin{cases} \dfrac{3n-3}{2} & n \text{ is odd} \\[2ex] \dfrac{3n-4}{2} & n \text{ is even} \end{cases}. \tag{5.1}$$

Let A be a cp-realization of W_n attaining maximum cp-rank. Let $\alpha_i = \{i, i+1, n\}$, $i = 1, \dots, n-1$ (where $i+1 \in \{1, \dots, n-1\}$ is the sum modulo $n-1$). Then (by repeated use of Lemma 3.1) $A = \sum_{i=1}^{n-1} A_i$, where A_i is completely positive, and is zero except for $A_i[\alpha_i]$, and $\mathrm{cpr}(A) = \sum_{i=1}^{n-1} \mathrm{cpr}(A_i)$. By Lemma 3.8(a), for each i there exists a cp factorization $A_i = B_i B_i^T$, in which the support of exactly one column of B_i contains $\{i, i+1\}$. Suppose the column of B_1 whose support contains $\{1, n-1\}$ is $\mathbf{b}$. Then $A = \mathbf{b}\mathbf{b}^T + Q$, where $G(Q) \subseteq F_n$. Thus

$$\mathrm{cpr}(W_n) \leq 1 + \mathrm{cpr}(F_n)$$

For odd n combine (5.1) and Corollary 5.8 to get

$$\frac{3n-3}{2} = \mathrm{tf}(W_n) \leq \mathrm{cpr}(W_n) \leq \mathrm{cpr}(F_n) + 1 = \frac{3n-3}{2}.$$

In particular, $\mathrm{cpr}(W_n) = \mathrm{tf}(W_n)$.

For even n, the same calculation yields

$$\frac{3n-4}{2} = \mathrm{cpr}(F_n) \leq \mathrm{cpr}(W_n) \leq \mathrm{cpr}(F_n) + 1 = \frac{3n-2}{2}. \tag{5.2}$$

We now show that the upper bound on the right hand side of (5.2) can be reduced by 1, yielding the desired equality. Let A_i and B_i be as above. Suppose each $1 \leq i \leq n-1$ is in the support of at least 3 different columns of $B_1, \dots, B_{n-1}$. Then there are at

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
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least $2(n-1)$ different columns containing the vertices of the $n-1$ cycle: $n-1$ columns whose supports contain the cycle edges, and at least one more column for each $i = 1, \ldots, n-1$. Thus $\mathrm{cpr}(A) \geq 2(n-1) > \frac{3n-2}{2}$. But this contradicts (5.2). Thus there is at least one vertex i which belongs to at most two of the columns of $B_1, \ldots, B_{n-1}$. Suppose without loss of generality that $i = 1$, and the two columns are $\mathbf{b}_1$ and $\mathbf{b}_2$. Then $A = \mathbf{b}_1\mathbf{b}_1^T + \mathbf{b}_2\mathbf{b}_2^T + Q$, where $Q = 0 \oplus Q_1$ is a completely positive matrix and $G(Q_1)$ is a subgraph of F_{n-1}. Thus

$$\frac{3n-4}{2} \leq \mathrm{cpr}(W_n) = \mathrm{cpr}(A) \leq \mathrm{cpr}(F_{n-1}) + 2 = \frac{3(n-1)-5}{2} + 2 = \frac{3n-4}{2}. \qquad \square$$

Theorems 5.7 and 5.9, and some other results of this paper, may be used to partially prove the DJL conjecture for $n = 6$, see [12].

Acknowledgement. I would like to thank the referees for helpful suggestions.

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