SPARSE SPECTRALLY ARBITRARY PATTERNS*

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Abstract. We explore combinatorial matrix patterns of order n for which some matrix entries are necessarily nonzero, some entries are zero, and some are arbitrary. In particular, we are interested in when the pattern allows any monic characteristic polynomial with real coefficients, that is, when the pattern is spectrally arbitrary. We describe some order n patterns that are spectrally arbitrary. We show that each superpattern of a sparse companion matrix pattern is spectrally arbitrary. We determine all the minimal spectrally arbitrary patterns of order 2 and 3. Finally, we demonstrate that there exist spectrally arbitrary patterns for which the nilpotent-Jacobian method fails.

Key words. Companion matrix pattern, Spectrally arbitrary pattern, Nilpotent-Jacobian method, Nonzero pattern.

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1. Introduction. In this paper, a *pattern* of order n is an n-by-n matrix with entries in $\{0, *, \circledast\}$. The *qualitative class* of a pattern $\mathcal{A} = [\mathcal{A}_{i,j}]$, denoted by $Q(\mathcal{A})$, is the set of all real matrices $A = [A_{i,j}]$ such that $A_{i,j} \neq 0$ if $\mathcal{A}_{i,j} = *$, $A_{i,j} = 0$ if $\mathcal{A}_{i,j} = 0$, and $A_{i,j}$ is unrestricted if $\mathcal{A}_{i,j} = \circledast$. We say $\mathcal{A}_{,ij}$ is nonzero if $\mathcal{A}_{i,j} \in \{*, \circledast\}$. While the notation # has been used in the literature (see e.g. [1, 6]) instead of $\circledast$, we find the latter more visibly descriptive.

A pattern $\mathcal{A}$ *realizes* a polynomial $p(x)$ if there is a matrix $A \in Q(\mathcal{A})$ such that the characteristic polynomial of A is $p(x)$. A pattern $\mathcal{A}$ is *spectrally arbitrary* if $\mathcal{A}$ realizes every monic polynomial of degree n with real coefficients. For example the

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order n pattern

$$\mathcal{C}_n = \begin{bmatrix} \circledast & * & 0 & \cdots & 0 \\ \circledast & 0 & * & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & * \\ \circledast & 0 & \cdots & \cdots & 0 \end{bmatrix}$$

is spectrally arbitrary since it is the pattern of a companion matrix.

The concept of spectrally arbitrary pattern was introduced in [4] in the context of *sign* patterns (whose entries are in $\{0, +, -\}$) and have since been explored by various other authors. Spectrally arbitrary zero-nonzero patterns (whose entries are restricted to $\{0, *\}$) have also been explored; see for example [2]. More recently, Cavers and Fallat [1] have developed spectral results for more general classes of combinatorial patterns.

A pattern is *reducible* if it is permutationally equivalent to a block triangular matrix pattern and *irreducible* otherwise. Note that if $\mathcal{A}$ is irreducible, it is possible that there is a reducible matrix $A \in Q(\mathcal{A})$. Cavers and Fallat [1] noted that an irreducible spectrally arbitrary pattern of order n has at least $2n - 1$ nonzero entries. Throughout this paper, we call an order n spectrally arbitrary pattern *sparse* if it has exactly $2n - 1$ nonzero entries.

In Section 3, we characterize all the order 2 and 3 patterns that are spectrally arbitrary. In Section 3, we also introduce a spectrally arbitrary pattern for which a commonly-used method, the nilpotent-Jacobian method, fails. In Section 4, we answer a question raised by [1, Question 4.4]. In particular, we show that there exist sparse irreducible spectrally arbitrary patterns that have at least two nonzero entries on the main diagonal.

A pattern $\mathcal{B}$ is a *superpattern* of pattern $\mathcal{A}$ if $\mathcal{B}$ can be obtained from $\mathcal{A}$ by replacing some (or none) of the zero entries of $\mathcal{A}$ with $*$. A pattern $\mathcal{B}$ is a *relaxation* of $\mathcal{A}$ if $\mathcal{B}$ can be obtained from $\mathcal{A}$ by changing some of the $*$ entries of $\mathcal{A}$ to $\circledast$ and/or some of the zero entries of $\mathcal{A}$ to $\circledast$. Note that in [1], the definition of superpattern is slightly different; it includes some relaxations of the superpatterns. In particular, an entry of the superpattern $\mathcal{B}$ could be $\circledast$ if $\mathcal{A}$ is 0 in [1]. If $\mathcal{A}$ is spectrally arbitrary, then every relaxation of $\mathcal{A}$ is also spectrally arbitrary (see [1, Lemma 2.1]). It was shown in [1] that every superpattern of the companion pattern $\mathcal{C}_n$ is spectrally arbitrary. Recently [5], the class of all sparse companion matrix patterns were classified, and it was noted that some superpatterns of these patterns are spectrally arbitrary. In Section 2, we demonstrate that all superpatterns of the sparse companion patterns are spectrally arbitrary.

2. Techniques for spectrally arbitrary patterns. A common technique for showing a pattern is spectrally arbitrary, introduced in [4] for sign patterns, is the nilpotent-Jacobian method. This method was extended in [1, Theorem 2.11] to apply to other classes of patterns such as those discussed in this paper.

The nilpotent-Jacobian method. *Let $\mathcal{N}$ be a pattern of order n with $m \geq n$ nonzero entries and N be a nilpotent matrix in $Q(\mathcal{N})$. Among the m nonzero entries, choose n nonzero entries, say $N_{i_1,j_1}, \ldots, N_{i_n,j_n}$ and let A be the matrix obtained from N by replacing the entries $N_{i_1,j_1}, \ldots, N_{i_n,j_n}$ by real variables $a_1, \ldots, a_n$. If the characteristic polynomial of A is $p_A(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$, let $\mathrm{Jac}(N)$ be the Jacobian matrix of order n with (i,j) entry equal to $\frac{\partial p_i}{\partial a_j}$ evaluated at $(a_1, \ldots, a_n) = (N_{i_1,j_1}, \ldots, N_{i_n,j_n})$. If $\mathrm{Jac}(N)$ has rank n, then every superpattern of $\mathcal{N}$, including $\mathcal{N}$ itself, is spectrally arbitrary.*

To simplify the wording, we say that $\mathcal{N}$ *allows a nonzero Jacobian* if $\mathrm{Jac}(N)$ has rank n for some nilpotent matrix $N \in Q(\mathcal{N})$ and choice of variables. Thus the nilpotent-Jacobian method can be summarized as: if $\mathcal{N}$ allows a nonzero Jacobian, then every superpattern of $\mathcal{N}$ is spectrally arbitrary. It is not known whether every zero-nonzero spectrally arbitrary pattern allows a nonzero Jacobian over $\mathbb{R}$, but it was shown in [8] that there exist spectrally arbitrary zero-nonzero patterns that do not allow a nonzero Jacobian over $\mathbb{C}$. In Section 3, and again in Section 4, we demonstrate that not every spectrally arbitrary $\{0, *, \circledast\}$-pattern allows a nonzero Jacobian over $\mathbb{R}$.

The nilpotent-Jacobian method is an analytic method for determining if a pattern is spectrally arbitrary. In [7], an algebraic method, called the nilpotent-centralizer method, was introduced for sign patterns. Taking appropriate account of the $\circledast$ positions, this method can also be extended to $\{0, *, \circledast\}$-patterns as described below. We use $A \circ B$ to represent the Hadamard, or the entry-wise product, of matrices A and B of the same size. If A is a matrix and $\mathcal{B}$ is a pattern, then an entry of $A \circ \mathcal{B}$ is nonzero if and only if the corresponding entries in both A and $\mathcal{B}$ are nonzero. Note that the *index* of a nilpotent matrix N is the smallest positive integer k such that $N^k = O$. Based on Theorem 3.5 and Lemma 3.6 in [7], we have the following lemma which implies the validity of the subsequent nilpotent-centralizer method.

Lemma 2.1. *Let $N \in Q(\mathcal{N})$ be a nilpotent matrix of order n. Then $\mathrm{Jac}\,(N)$ has rank n if and only if N has index n and the only matrix B in the centralizer of N satisfying $B^T \circ \mathcal{N} = O$ is the zero matrix.*

The nilpotent-centralizer method. *Let $N \in Q(\mathcal{N})$ be an order n nilpotent matrix of index n. If the only matrix B in the centralizer of N satisfying $B^T \circ \mathcal{N} = O$ is the zero matrix, then the pattern $\mathcal{N}$ and each of its superpatterns is spectrally arbitrary pattern.*

Two patterns $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if $\mathcal{B}$ can be obtained from $\mathcal{A}$ by transposition and/or permutation similarity. In [1, Example 2.12], using a pattern equivalent to $\mathcal{C}_n$, the nilpotent-Jacobian method is used to demonstrate that every superpattern of $\mathcal{C}_n$ is spectrally arbitrary. Below we use the nilpotent-centralizer method to get the same result for a larger class of matrices that includes $\mathcal{C}_n$.

For $1 \leq k \leq n-1$, we say the k^{th} *subdiagonal* of a pattern is the set of positions $\{(i, i-k) : k+1 \leq i \leq n\}$. A family of patterns $\mathscr{C}_n$ was introduced in [5]: a pattern $\mathcal{A}$ is in $\mathscr{C}_n$ if $\mathcal{A}$ has an entry $*$ in each superdiagonal position, exactly one $\circledast$ along the main diagonal, exactly one $\circledast$ on each subdiagonal, and zeros elsewhere. For example, the pattern $\mathcal{C}_n$ is in $\mathscr{C}_n$. In [5], it was observed that the patterns in $\mathscr{C}_n$ characterize those patterns that uniquely realize each characteristic polynomial up to diagonal similarity. We now show that each superpattern of a pattern in $\mathscr{C}_n$ is spectrally arbitrary.

THEOREM 2.2. *Let $\mathcal{A}$ be a pattern in $\mathscr{C}_n$. Then every superpattern of $\mathcal{A}$ is spectrally arbitrary.*

Proof. Let $\mathcal{N}$ be in $\mathscr{C}_n$. Let $N \in Q(\mathcal{N})$ be the matrix with ones on the superdiagonal and zeros elsewhere. Note that N is nilpotent of index n. Let B be a matrix in the centralizer of N such that $B^T \circ \mathcal{N} = O$. Note that since B is in the centralizer of N, $B = q(N)$ for some polynomial $q(x)$ of degree at most $n-1$. Let $k \in \{1, \ldots, n-1\}$. Observe that the transpose of N^{n-k} has exactly k nonzero entries: it has k ones on the $(n-k)^{\text{th}}$ subdiagonal, and zeros elsewhere. But $\mathcal{N}$ has a $\circledast$ entry on each subdiagonal, and hence the coefficient of N^{n-k} must be zero in $q(N)$ since $B^T \circ \mathcal{N} = O$. It follows that $q(x)$ is a constant. But $\mathcal{N}$ has a nonzero entry on the main diagonal, hence $q(x)$ must be the zero polynomial. Thus every superpattern of $\mathcal{N}$ is spectrally arbitrary by the nilpotent-centralizer method. $\square$

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are order n patterns, each with only one nonzero element in the main diagonal, $\mathcal{A}_{k,k} \neq 0$ and $\mathcal{B}_{l,l} \neq 0$ for some $l \neq k$. Then $\mathcal{B}$ is said to be a *diagonal slide* of $\mathcal{A}$ if $\mathcal{B}_{i,j} = \mathcal{A}_{i,j}$ for all $i \neq j$. In the next argument, we use the fact that when N is an order n nilpotent matrix of index n, then the matrix B is in the centralizer of N if and only if $B = q(N)$ for some polynomial $q(x)$.

THEOREM 2.3. *Suppose $\mathcal{A}$ is a spectrally arbitrary pattern with only one nonzero entry on the main diagonal and $\mathcal{A}$ allows a nonzero Jacobian. If $\mathcal{B}$ is a diagonal slide of $\mathcal{A}$, then any superpattern of $\mathcal{B}$ is spectrally arbitrary.*

Proof. Let $\mathcal{A}$ be an order n pattern, and suppose that $\mathcal{B}$ is a diagonal slide of $\mathcal{A}$ and $N \in Q(\mathcal{A})$ is a nilpotent matrix such that $\mathrm{Jac}(N)$ has rank n. Then by Lemma 2.1, N has index n and if $p(x)$ is a polynomial of degree at most n such that $p(N) \circ \mathcal{A}^T = O$, then $p(x)$ is the zero polynomial.

Since N is nilpotent and $\mathcal{A}$ has only one nonzero entry on the main diagonal, N has all zeros on the main diagonal. Since $\mathcal{B}$ is a diagonal slide of $\mathcal{A}$ and N has all zeros on its diagonal, N is a nilpotent realization of $\mathcal{B}$. Suppose $q(x)$ is a polynomial of degree at most n such that $q(N) \circ \mathcal{B}^T = O$. Then for some constant c, we have $(q(N) - cI) \circ \mathcal{A}^T = O$, and thus $p(x) = q(x) - c$ is the zero polynomial. But then $q(x) = c$, and $q(N) \circ \mathcal{B}^T = O$ implies that $c = 0$. Thus $q(x) = 0$ and, by the nilpotent-centralizer method, any superpattern of $\mathcal{B}$ is spectrally arbitrary. □

The pattern $\mathcal{C}_n$ is an example of a pattern that preserves the property of being spectrally arbitrary under a diagonal slide, since each pattern in $\mathscr{C}_n$ is spectrally arbitrary as illustrated in Theorem 2.2. In fact, Theorem 2.2 implies that $\mathcal{C}_n$ preserves the property under any subdiagonal slide as well (where a *subdiagonal slide* involves moving a lone $\circledast$ entry on the k^{th} subdiagonal to another position on the same subdiagonal, for some $k \in \{1, \ldots, n-1\}$). In Section 4, we have an example of a spectrally arbitrary pattern, that does not allow a nonzero Jacobian, for which a subdiagonal slide restricts the possible characteristic polynomials. In particular, moving entry $(4, 1)$ of the spectrally arbitrary pattern $\mathcal{Y}_6(6, 2)$ to position $(5, 2)$ produces a pattern that requires singularity (see Section 4 for the definition of $\mathcal{Y}_n(n, k)$).

We do not know if the conclusion of Theorem 2.3 will still hold if we drop the condition of allowing a nonzero Jacobian. We do have examples of spectrally arbitrary patterns that do not allow a nonzero Jacobian that preserve the property of being spectrally arbitrary under a diagonal slide: $\mathcal{F}$ in Section 3 (see Theorem 3.8) and $\mathcal{Y}_n(s, k)$ in Section 4 (see Theorem 4.4 and 4.7).

3. Spectrally arbitrary patterns of order 2 and 3. In this section we characterize all the patterns of order 2 and 3 that are spectrally arbitrary up to equivalence. A tool that is used to classify permutationally equivalent patterns is the digraph of the pattern. Given an order n matrix pattern $\mathcal{A}$, the *digraph* $D(\mathcal{A})$ has vertex set $\{v_1, v_2, \ldots, v_n\}$ with arc set $\{(v_i, v_j) : \mathcal{A}_{i,j} \neq 0\}$. Two patterns are permutationally equivalent if and only if their labeled digraphs are isomorphic (labeling arcs with $*$ and $\circledast$ as appropriate). Further, the digraph of the transpose of $\mathcal{A}$ is obtained from $D(\mathcal{A})$ by reversing all the arcs. For $1 \leq k \leq n$, a cycle of length k, or *k-cycle*, in a digraph D is a sequence of k arcs $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_k}, v_{i_1})$ with k distinct vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$. A 1-cycle is often called a *loop* of D.

Lemma 3.1. [1, Corollary 2.8] *If $\mathcal{A}$ is a spectrally arbitrary pattern, then the digraph $D(\mathcal{A})$ has at least one loop and at least one 2-cycle.*

If $D(\mathcal{A})$ has cycle $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_k}, v_{i_1})$ and $A = [A_{i,j}] \in \mathcal{A}$ then the associated *cycle product* is $(-1)^{k-1} A_{v_{i_1}, v_{i_2}} A_{v_{i_2}, v_{i_3}} \cdots A_{v_{i_k}, v_{i_1}}$. A *composite* cycle of length k is a set of vertex disjoint cycles with lengths summing to k (with an associated cycle product being the product of the cycle products of the individual cycles). If a

k-cycle is not composite, then we say the k-cycle is a proper k-cycle. We will use the following fact (see for example [6]):

LEMMA 3.2. *Given $A \in Q(\mathcal{A})$, if f_k is the sum of all the composite cycle products of length k in $D(\mathcal{A})$, then the characteristic polynomial of A is*

$$p_A(x) = x^n - f_1 x^{n-1} + f_2 x^{n-2} + \cdots + (-1)^n f_n.$$

By Lemma 3.2, if the digraph of a pattern $\mathcal{A}$ has exactly one k-cycle for some k, $1 \leq k \leq n$, and $\mathcal{A}$ has only $*$ entries in the positions corresponding to the arcs of the k-cycle, then that pattern does not realize a polynomial with $f_k = 0$. Thus we have the following lemma:

LEMMA 3.3. *Suppose $k \in \{1, 2, \ldots, n\}$. If $\mathcal{A}$ is a spectrally arbitrary pattern of order n, and $D(\mathcal{A})$ has exactly one composite cycle of length k, then that cycle must have an arc labeled $\circledast$.*

The next result is a $\{0, *, \circledast\}$-pattern version of a corresponding sign pattern result [1, Theorem 3.2].

THEOREM 3.4. *If $\mathcal{A}$ is a spectrally arbitrary pattern of order 2 then $\mathcal{A}$ is, up to equivalence, a relaxation of either*

$$\mathcal{C}_2 = \begin{bmatrix} \circledast & * \\ \circledast & 0 \end{bmatrix} \qquad \text{or} \qquad \mathcal{T}_2 = \begin{bmatrix} * & * \\ * & * \end{bmatrix}.$$

The following result was stated for sign patterns in [3, Proposition 2.1], but the proof did not rely on the signs.

THEOREM 3.5. *The direct sum of patterns of which at least two are of odd order is not spectrally arbitrary. Furthermore if the direct sum of spectrally arbitrary patterns has at most one odd order summand, then the direct sum is spectrally arbitrary.*

A pattern $\mathcal{B}$ is a *minimal spectrally arbitrary pattern* if $\mathcal{A}$ is not spectrally arbitrary for any $\mathcal{A} \neq \mathcal{B}$ such that $\mathcal{B}$ is a superpattern of $\mathcal{A}$ or $\mathcal{B}$ is a relaxation of $\mathcal{A}$. The following theorem follows from Theorems 3.4 and 3.5 and demonstrates that, up to equivalence, the reducible, minimal spectrally arbitrary patterns of order 3 are $\mathcal{T}_2 \oplus [\circledast]$ and $\mathcal{C}_2 \oplus [\circledast]$.

THEOREM 3.6. *If $\mathcal{R}$ is a reducible spectrally arbitrary pattern of order 3 then $\mathcal{R}$ is equivalent to either a superpattern or a relaxation of $\mathcal{T}_2 \oplus [\circledast]$ or $\mathcal{C}_2 \oplus [\circledast]$.*

Next, we next explore irreducible patterns.

Lemma 3.7. [2, Theorem 1.1] *If $\mathcal{A}$ is an irreducible, minimal spectrally arbitrary zero-nonzero pattern of order 3, then $\mathcal{A}$ is equivalent to*

$$\mathcal{V}_3 = \begin{bmatrix} * & * & 0 \\ * & 0 & * \\ * & 0 & * \end{bmatrix}, \qquad \text{or} \quad \mathcal{T}_3 = \begin{bmatrix} * & * & 0 \\ * & 0 & * \\ 0 & * & * \end{bmatrix}.$$

Further, every superpattern of $\mathcal{V}_3$ and $\mathcal{T}_3$ is spectrally arbitrary.

We will see in Theorem 3.9 that the following $\{0, *, \circledast\}$-patterns are irreducible, minimal spectrally arbitrary patterns:

$$\mathcal{C}_3 = \begin{bmatrix} \circledast & * & 0 \\ \circledast & 0 & * \\ \circledast & 0 & 0 \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} \circledast & * & 0 \\ * & 0 & * \\ \circledast & * & 0 \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} \circledast & * & 0 \\ * & \circledast & * \\ 0 & * & 0 \end{bmatrix},$$

$$\mathcal{G} = \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & \circledast & 0 \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} * & * & 0 \\ * & * & * \\ \circledast & 0 & 0 \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} \circledast & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix},$$

$$\text{and} \qquad \mathcal{F} = \begin{bmatrix} 0 & \circledast & 0 \\ * & \circledast & \circledast \\ * & * & 0 \end{bmatrix}.$$

We first observe that $\mathcal{F}$ is a spectrally arbitrary pattern for which the nilpotent-Jacobian method fails.

Theorem 3.8. *If $\mathcal{A}$ is equivalent to $\mathcal{F}$ or a diagonal slide of $\mathcal{F}$, then $\mathcal{A}$ is spectrally arbitrary, but $\mathcal{A}$ does not allow a nonzero Jacobian.*

Proof. Suppose $\mathcal{A}$ is a diagonal slide of $\mathcal{F}$ and $N \in Q(\mathcal{A})$ is nilpotent. It follows from Lemma 3.2 that the $\circledast$ diagonal position of $\mathcal{A}$ will be zero in N. Also, since $\det(N) = 0$, at least one of $N_{1,2}$ and $N_{2,3}$ is zero. Considering the 2-cycles of $D(\mathcal{A})$, it follows from Lemma 3.2 that $N_{1,2} = N_{2,3} = 0$. Thus N has zero entries in each $\circledast$ position of $\mathcal{A}$. Let B be a matrix of order 3 having only zero entries, except $B_{3,1} = 1$. Then $B^T \circ \mathcal{A} = O$ and B is in the centralizer of N for any nilpotent matrix in $Q(\mathcal{A})$. Thus, by Lemma 2.1, $\mathcal{A}$ does not allow a nonzero Jacobian.

Suppose $r_1, r_2, r_3 \in \mathbb{R}$, and $p(x) = x^3 + r_1x^2 + r_2x + r_3$. Let

$$F = \begin{bmatrix} 0 & a & 0 \\ 1 & b & c \\ d & 1 & 0 \end{bmatrix} \in Q(\mathcal{F}).$$

If $r_3 \neq 0$, let $t \in \mathbb{R}\backslash\{0, -r_2\}$ and $(a, b, c, d) = \left(-r_2 - t, -r_1, t, \frac{r_3}{(r_2+t)t}\right)$. If $r_3 = 0$, let $(a, b, c, d) = (-r_2, -r_1, 0, 1)$. In each case, d is nonzero as required for $\mathcal{F}$, and the characteristic polynomial of F is $p(x)$. Thus $\mathcal{F}$ is spectrally arbitrary.

Next, we will show that any diagonal slide of $\mathcal{F}$ is also a spectrally arbitrary. Let $\mathcal{F}^*$ be a diagonal slide of $\mathcal{F}$. Then either $\mathcal{F}^*_{1,1} = \circledast$ or $\mathcal{F}^*_{3,3} = \circledast$. By permutation equivalence, we can assume $\mathcal{F}^*_{1,1} = \circledast$. Consider the following matrix

$$F^* = \begin{bmatrix} b & a & 0 \\ 1 & 0 & c \\ d & 1 & 0 \end{bmatrix} \in Q(\mathcal{F}^*).$$

Suppose $r_1, r_2, r_3 \in \mathbb{R}$, and $p(x) = x^3 + r_1x^2 + r_2x + r_3$. If $r_1 \neq 0$, let $t \in \mathbb{R} \setminus \left\{0, -r_2, -\frac{r_3}{r_1}\right\}$ and $(a, b, c, d) = \left(-r_2 - t, -r_1, t, \frac{tr_1+r_3}{(r_2+t)t}\right)$. If $r_1 = 0$ and $r_3 \neq 0$, let $t \in \mathbb{R} \setminus \{0, -r_2\}$ and $(a, b, c, d) = \left(-r_2 - t, 0, t, \frac{r_3}{(r_2+t)t}\right)$. If $r_1 = 0$ and $r_3 = 0$, let $(a, b, c, d) = (0, 0, -r_2, 1)$. In each case d is nonzero and the characteristic polynomial of F^* is $p(x)$. Therefore every diagonal slide of $\mathcal{F}$ is spectrally arbitrary. □

The next theorem determines the minimal spectrally arbitrary $\{0, *, \circledast\}$-patterns of order 3.

Theorem 3.9. *If $\mathcal{A}$ is an irreducible pattern of order 3, then $\mathcal{A}$ is spectrally arbitrary if and only if, up to equivalence, $\mathcal{A}$ is a relaxation of a superpattern of one of the patterns $\mathcal{E}, \mathcal{G}, \mathcal{H}, \mathcal{K}, \mathcal{V}_3$ or $\mathcal{T}_3$, or of a diagonal slide of $\mathcal{C}_3, \mathcal{D}$, or $\mathcal{F}$.*

Proof. By Lemma 3.7, both $\mathcal{V}_3$ and $\mathcal{T}_3$ are spectrally arbitrary. By Theorem 2.2, every superpattern of a diagonal slide of $\mathcal{C}_3$ is spectrally arbitrary and it follows from Lemma 3.3 that $\mathcal{C}_3$ is not the relaxation of another spectrally arbitrary pattern. Hence any diagonal slide of $\mathcal{C}_3$ is a minimal spectrally arbitrary pattern. Consider the matrices

$$\begin{bmatrix} a_1 & 1 & 0 \\ a_2 & 0 & 1 \\ a_3 & 1 & 0 \end{bmatrix}, \begin{bmatrix} a_1 & 1 & 0 \\ a_2 & a_3 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ a_1 & a_2 & 1 \\ 0 & a_3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ a_1 & a_2 & 1 \\ a_3 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} a_1 & 1 & -1 \\ a_2 & 0 & 1 \\ a_3 & 2 & 0 \end{bmatrix}.$$

Setting the variables (a_1, a_2, a_3) as $(0, -1, 0), (0, -1, 0), (-1, -1, 0), (-1, -1, 0)$, and $(0, 2, 4)$ respectively, for each pattern $\mathcal{D}, \mathcal{E}, \mathcal{G}, \mathcal{H}$, and $\mathcal{K}$, we obtain a nilpotent matrix for which the corresponding Jacobian has full rank. Thus the superpatterns of $\mathcal{D}, \mathcal{E}, \mathcal{G}, \mathcal{H}$, and $\mathcal{K}$ are spectrally arbitrary by the nilpotent-Jacobian method and, by Theorem 2.3, any superpattern of a diagonal slide of $\mathcal{D}$ is spectrally arbitrary.

By Theorem 3.8 we know every diagonal slide of $\mathcal{F}$ is spectrally arbitrary. We next show that all of their superpatterns are also spectrally arbitrary.

Let $\mathcal{B}$ be a superpattern of $\mathcal{F}$. If $\mathcal{B}_{1,3} \neq 0$ then $\mathcal{B}$ is equivalent to a relaxation of $\mathcal{K}$ via the permutation (12). If $\mathcal{B}_{1,1} \neq 0$ then $\mathcal{B}^T$ is a relaxation of a superpattern of $\mathcal{G}$. If $\mathcal{B}_{3,3} \neq 0$ then $\mathcal{B}$ is equivalent, via transpose and the permutation (13), to a relaxation of a superpattern of $\mathcal{V}_3$. Thus any superpattern of $\mathcal{F}$ is spectrally arbitrary.

Now let $\mathcal{B}$ be a superpattern of a diagonal slide of $\mathcal{F}$. By transpose and the permutation (13), we can assume $\mathcal{B}_{1,1} = \circledast$. If $\mathcal{B}_{1,3} \neq 0$ then $\mathcal{B}$ is a relaxation of $\mathcal{K}$. If $\mathcal{B}_{2,2} \neq 0$ then $\mathcal{B}^T$ is a relaxation of a superpattern of $\mathcal{G}$. If $\mathcal{B}_{3,3} \neq 0$ then $\mathcal{B}$ is a relaxation of a superpattern of $\mathcal{V}_3$. Thus any superpattern of $\mathcal{B}$ is also spectrally arbitrary. Therefore any superpattern of a diagonal slide of $\mathcal{F}$ is spectrally arbitrary.

Suppose $\mathcal{A} = [a_{i,j}]$ is an irreducible, minimal spectrally arbitrary pattern of order 3. As noted in the introduction, a spectrally arbitrary pattern of order n must have at least $2n-1$ nonzero entries. Thus $\mathcal{A}$ must have at least five nonzero entries. If $\mathcal{A}$ has no $\circledast$ entries, then by Lemma 3.7, $\mathcal{A}$ is equivalent to $\mathcal{V}_3$ or $\mathcal{T}_3$. So suppose $\mathcal{A}$ has at least one $\circledast$ entry. By Lemma 3.1 and the fact that $\mathcal{A}$ is irreducible, $\mathcal{A}$ must be equivalent to a relaxation of a superpattern of

$$\mathscr{V}_1 = \begin{bmatrix} 0 & * & 0 \\ * & 0 & * \\ * & 0 & 0 \end{bmatrix}, \quad \text{or} \quad \mathscr{V}_2 = \begin{bmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & * & 0 \end{bmatrix}.$$

We consider the following cases:

1. Suppose $\mathcal{A}$ has exactly five nonzero entries. By Lemmas 3.1 and 3.3, one of the nonzero entries must be a $\circledast$ on the main diagonal. If $\mathcal{A}$ is a relaxation of $\mathscr{V}_2$ and not $\mathscr{V}_1$ then $\mathcal{A}$ does not realize the polynomial x^3+1, regardless of the position of the diagonal entry. Thus $\mathcal{A}$ is a relaxation of $\mathscr{V}_1$. By Lemma 3.3, either $a_{1,2} = \circledast$ or $a_{2,1} = \circledast$. If $a_{1,2} = \circledast$ and $a_{2,1} = *$, then $\mathcal{A}$ does not realize the polynomial x^3+1. Thus $a_{2,1} = \circledast$. If $a_{2,3} = a_{3,1} = *$, then $\mathcal{A}$ does not realize the polynomial x^3+x. Thus $a_{3,1} = \circledast$ or $a_{2,3} = \circledast$. If $a_{3,1} = \circledast$, then $\mathcal{A}$ is a diagonal slide of a relaxation of $\mathcal{C}_3$. If $a_{2,3} = \circledast$, then $\mathcal{A}^T$ is equivalent to a diagonal slide of a relaxation of $\mathcal{C}_3$, via permutation (12).
2. Suppose $\mathcal{A}$ has exactly six nonzero entries and $\mathcal{A}$ is a relaxation of a superpattern of $\mathscr{V}_1$ with only one entry on the main diagonal. Thus exactly one of $a_{3,2}$ and $a_{1,3}$ is not zero. But the case with $a_{1,3} \neq 0$ is equivalent to the former via permutation (123). Thus assume $a_{3,2} \neq 0$. In order to realize the polynomial x^3, either $a_{3,1} = \circledast, a_{1,2} = \circledast$, or $a_{2,3} = \circledast$.
   (a) Suppose $a_{3,1} = \circledast$. Then $\mathcal{A}$ is a relaxation of a diagonal slide of $\mathcal{D}$.
   (b) Suppose $a_{1,2} = \circledast$ and $a_{3,1} = *$. If $a_{2,3} = a_{3,2} = *$ then $\mathcal{A}$ does realize characteristic polynomial x^3. Thus either $a_{2,3} = \circledast$ or $a_{3,2} = \circledast$. If $a_{3,2} = \circledast$, then $\mathcal{A}$ would be equivalent to a superpattern of a diagonal slide of $\mathcal{C}_3$ via permutation (132). Thus $a_{2,3} = \circledast$, and $\mathcal{A}$ is a diagonal slide of $\mathcal{F}$.
   (c) Suppose $a_{2,3} = \circledast$ and $a_{3,1} = a_{1,2} = *$. In order for $\mathcal{A}$ to realize a characteristic polynomial x^3, it is necessary that $a_{2,1} = \circledast$. Therefore $\mathcal{A}$ is equivalent to a superpattern of a diagonal slide of $\mathcal{C}_3$ via transposition and permutation (12).

3. Suppose $\mathcal{A}$ has exactly six nonzero entries with two nonzero entries on the main diagonal and $\mathcal{A}$ is a relaxation of a superpattern of $\mathscr{V}_1$. If $a_{1,1} \neq 0$ and $a_{3,3} \neq 0$ then $\mathcal{A}$ is a relaxation of $\mathcal{V}_3$. If $a_{2,2} \neq 0$ and $a_{3,3} \neq 0$ then $\mathcal{A}$ is equivalent to a relaxation of $\mathcal{V}_3$ via transposition and permutation (12). Suppose that $a_{1,1} \neq 0$ and $a_{2,2} \neq 0$. By Lemma 3.3 either $a_{3,1} = \circledast, a_{1,2} = \circledast$, or $a_{2,3} = \circledast$. If $a_{1,2} = \circledast$ and $a_{3,1} = a_{2,3} = *$, then $\mathcal{A}$ does not realize characteristic polynomial $x^3 + x$. Thus $a_{3,1} = \circledast$ or $a_{2,3} = \circledast$. If $a_{3,1} = \circledast$, then $\mathcal{A}$ is a relaxation of $\mathcal{H}$. If $a_{2,3} = \circledast$, then $\mathcal{A}$ is equivalent to a relaxation of $\mathcal{H}$ via transpose and permutation (12).
4. Suppose $\mathcal{A}$ is a relaxation of a superpattern of $\mathscr{V}_2$ but not of $\mathscr{V}_1$, and $\mathcal{A}$ has six nonzero entries. Then $\mathcal{A}$ has two nonzero entries on the main diagonal. If $a_{1,1} \neq 0$ and $a_{3,3} \neq 0$, then $\mathcal{A}$ is a relaxation of $\mathcal{T}_3$. The case with $a_{1,1} \neq 0$ and $a_{2,2} \neq 0$ is equivalent, via permutation (13), to the case with $a_{2,2} \neq 0$ and $a_{3,3} \neq 0$. Suppose $a_{1,1}$ and $a_{2,2}$ are nonzero. By Lemma 3.3, $a_{3,2} = \circledast, a_{2,3} = \circledast$ or $a_{1,1} = \circledast$.
   (a) Suppose $a_{3,2} = \circledast$. Then $\mathcal{A}$ is a relaxation of $\mathcal{G}$.
   (b) Suppose $a_{2,3} = \circledast$. Then $\mathcal{A}^T$ is a relaxation of $\mathcal{G}$.
   (c) Suppose $a_{1,1} = \circledast$ and $a_{3,2} = a_{2,3} = *$. Then $a_{2,2} = \circledast$, otherwise $\mathcal{A}$ does not realize characteristic polynomial x^3. Thus $\mathcal{A}$ is a relaxation of $\mathcal{E}$.
5. Suppose $\mathcal{A}$ has seven nonzero entries. Since $\mathcal{A}$ is irreducible, up to equivalence, $\mathcal{A}$ is a relaxation of one of the five patterns:

$$\begin{bmatrix} * & * & * \\ * & 0 & * \\ * & 0 & * \end{bmatrix}, \begin{bmatrix} * & * & 0 \\ * & * & * \\ * & 0 & * \end{bmatrix}, \begin{bmatrix} * & * & 0 \\ * & 0 & * \\ * & * & * \end{bmatrix}, \begin{bmatrix} * & * & 0 \\ * & * & * \\ 0 & * & * \end{bmatrix} \text{ and } \begin{bmatrix} * & * & * \\ * & 0 & * \\ * & * & 0 \end{bmatrix}.$$

   The first and second patterns are superpatterns of $\mathcal{V}_3$. The third and fourth patterns are superpatterns of $\mathcal{T}_3$. Suppose that $\mathcal{A}$ is a relaxation of the fifth pattern. By Lemma 3.3 the diagonal entry must be a $\circledast$ and thus $\mathcal{A}$ is a relaxation of $\mathcal{K}$.

Note that any pattern with more than 7 nonzero entries is equivalent to a relaxation of a superpattern of $\mathcal{V}_3$. □

Corollary 3.10. *Each named pattern in Theorem* 3.9 *is a minimal spectrally arbitrary pattern.*

We conclude this section by noting a consequence for $\{0, \circledast\}$-patterns that follows from the fact that every $\{0, \circledast\}$-pattern is a $\{0, *, \circledast\}$-pattern.

Corollary 3.11. *If $\mathcal{A}$ is an irreducible spectrally arbitrary $\{0, \circledast\}$-pattern of* *order* 3*, then* $\mathcal{A}$ *is equivalent to a relaxation of a diagonal slide of*

$$\begin{bmatrix} \circledast & \circledast & 0 \\ \circledast & 0 & \circledast \\ \circledast & 0 & 0 \end{bmatrix},$$

or a relaxation of one of

$$\begin{bmatrix} \circledast & \circledast & 0 \\ \circledast & 0 & \circledast \\ 0 & \circledast & \circledast \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \circledast & \circledast & 0 \\ \circledast & \circledast & \circledast \\ 0 & \circledast & 0 \end{bmatrix}.$$

4. A class of spectrally arbitrary patterns that do not allow a nonzero Jacobian. It is unknown (see for example [9]) whether an irreducible zero-nonzero spectrally arbitrary pattern must allow a nonzero Jacobian. As we saw in the previous section, the pattern $\mathcal{F}$ is an example of an irreducible spectrally arbitrary $\{0, *, \circledast\}$-pattern that does not allow a nonzero Jacobian. In this section we demonstrate that $\mathcal{F}$ is not a solitary exception; we introduce a whole class of irreducible spectrally arbitrary $\{0, *, \circledast\}$-patterns that do not allow a nonzero Jacobian. We also answer Question 4.4 raised in [1] in the affirmative with Corollary 4.6 below.

Given integers k, s and n with $1 \leq k < s \leq n$, let $\mathcal{Y}_n(s, k)$ be the lower Hessenberg pattern of order n with a $*$ in each superdiagonal position, a $\circledast$ in position $(i, 1)$ for all $i \neq s$, $1 \leq i \leq n$ and a $\circledast$ in position $(s, s-k+1)$. Note there are two $\circledast$ entries on the $(k-1)^{\text{st}}$ subdiagonal in this pattern. We will see that this pattern does not allow a nonzero Jacobian. Nevertheless, it can be shown, by examining characteristic polynomials, that for certain choices of s, k, and n, the pattern is spectrally arbitrary. Any matrix in $Q(\mathcal{Y}_n(s, k))$ is diagonally similar to an order n unit Hessenberg matrix

$$Y_n(s,k) = \begin{bmatrix}
y_1 & 1 & 0 & \cdots & & & & & & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & & & & & & & \vdots \\
\vdots & \vdots & \ddots & & & & & & & & \\
y_k & \vdots & & & & & & & & & \\
\vdots & 0 & & & & & & & & & \\
\vdots & \vdots & \ddots & & & & & & & & \\
y_{s-1} & & & 0 & & & & & & & \\
0 & & & & t & & & & & \ddots & \vdots \\
y_{s+1} & & & & & 0 & & & & \ddots & 0 \\
\vdots & \vdots & & & & & \ddots & & & \ddots & 1 \\
y_n & 0 & \cdots & & & & \cdots & 0 & \cdots & \cdots & 0
\end{bmatrix}$$

with $y_i, t \in \mathbb{R}$. Note that the digraph of $\mathcal{Y}_n(s,k)$ has two k-cycles, no s-cycle, and exactly one l-cycle for each $l \notin \{s,k\}$, $1 \le l \le n$. The only composite l-cycles that are not proper l-cycles involve the entry t in position $(s, s-k+1)$. The following description of the characteristic polynomial of $Y_n(s,k)$ then follows from Lemma 3.2.

LEMMA 4.1. *Let $y_s = 0$ and $y_0 = -1$. The characteristic polynomial of $Y = Y_n(s,k)$ is*

$$p_Y = x^n - \sum_{i=1}^{n} y_i x^{n-i} + \sum_{i=k}^{s} (t y_{i-k}) x^{n-i}.$$

LEMMA 4.2. *If s is not a multiple of k, then $\mathcal{Y} = \mathcal{Y}_n(s,k)$ is not spectrally arbitrary.*

Proof. Suppose s is not a multiple of k, and the characteristic polynomial of $Y \in Q(\mathcal{Y})$ is $x^n + x^{n-s}$. By Lemma 4.1, $y_i = 0$ for $1 \le i < k$. Note that there exists an integer c such that $1 \le s - ck < k$. Since $y_{s-ck} = 0$, we can show by induction on h that $y_{s-(c-h)k} = 0$ for $h \in \{1, 2, \ldots, c-1\}$. In particular, $y_{s-k} = 0$. But the coefficient of x^{n-s} in p_Y is $y_{s-k}t$. Thus $\mathcal{Y}$ cannot realize the characteristic polynomial $x^n + x^{n-s}$. Therefore $\mathcal{Y}$ is not spectrally arbitrary. $\square$

LEMMA 4.3. *Suppose s is a multiple of k. If Y is a nilpotent realization of $\mathcal{Y} = \mathcal{Y}_n(s,k)$, then $Y_{ij} = 0$ whenever $\mathcal{Y}_{ij} = \circledast$.*

Proof. Suppose $Y = Y_n(s,k)$ is nilpotent, and $t \neq 0$. Let p_Y be the characteristic polynomial of Y as in Lemma 4.1. Since Y is nilpotent, $y_i = 0$ for all $i \notin \{k, \ldots, s\}$, $1 \le i \le n$. Thus the coefficient of x^{n-s} in p_Y is $t y_{s-k}$. Hence $y_{s-k} = 0$. We can show, by induction on c, that $y_{s-ck} = 0$ for $c \in \{1, \ldots, \frac{s-k}{k}\}$. Therefore the coefficient of x^{n-k} is $-t \neq 0$. But this contradicts the fact that Y is nilpotent. Therefore $t = 0$. By Lemma 4.1, it follows that $y_i = 0$ for all i since Y is nilpotent. $\square$

THEOREM 4.4. *Suppose s is a multiple of k. The pattern $\mathcal{Y}_n(s,k)$ does not allow a nonzero Jacobian.*

Proof. Suppose $Y = Y_n(s,k)$ is nilpotent. By Lemma 4.3, Y is a unit Hessenberg matrix with all entries zero except those on the superdiagonal. Thus Y has index n. Let B be the matrix such that B^T has ones on the $(s-1)^{\text{th}}$ subdiagonal and zeros elsewhere. Then $B^T \circ \mathcal{Y} = O$ and $BY = YB$. Thus by Lemma 2.1, $\mathcal{Y}$ does not allow a nonzero Jacobian. $\square$

THEOREM 4.5. *For any $n > 2$, the pattern $\mathcal{Y}_n(s,k)$ is a (minimal) spectrally arbitrary pattern if and only if s is an odd multiple of k.*

Proof. Let $Y = Y_n(s,k)$. By Lemma 4.2, we may assume $s = ck$ for some positive integer c. Let $p(x) = x^n + r_1 x^{n-1} + r_2 x^{n-2} + \cdots + r_{n-1}x + r_n$ for some $r_1, \ldots, r_n \in \mathbb{R}$. It is enough to show that $p_Y(x) = p(x)$ for some $y_1, \ldots, y_{s-1}, t, y_{s+1}, \ldots, y_n \in \mathbb{R}$ if and only if c is odd. By Lemma 4.1, we seek a real solution to the system of equations:

$$\begin{aligned} r_1 &= -y_1 \\ &\vdots \\ r_{k-1} &= -y_{k-1} \\ r_k &= -t - y_k \\ r_{k+1} &= ty_1 - y_{k+1} \\ &\vdots \\ r_{s-1} &= ty_{s-k-1} - y_{s-1} \\ r_s &= ty_{s-k} \\ r_{s+1} &= -y_{s+1} \\ &\vdots \\ r_n &= -y_n. \end{aligned} \tag{4.1}$$

It follows from the above that $y_q = -r_q$ for all $q \in \{1, \ldots, k-1, s+1, \ldots, n\}$. Setting $r_0 = 1$ and using forward substitution, we can solve for y_m for each $m \in \{k, k+1, \ldots, s-1\}$, starting at $m = k$, to obtain

$$y_m = -\sum_{i=0}^{\lfloor \frac{m}{k} \rfloor} r_{m-ki} t^i.$$

After making these substitutions, the s^{th} equation in (4.1) yields

$$r_s = -\sum_{i=1}^{c} r_{s-ki} t^i. \tag{4.2}$$

Suppose c is even. Then the characteristic polynomial of Y can not be $x^n + x^{n-s}$ since if $r_1 = \cdots = r_{s-1} = 0$, then (4.2) would imply $r_s = -t^c < 0$. Thus, if c is even, $\mathcal{Y}_n(s,k)$ is not spectrally arbitrary.

Suppose c is odd. Then we let t be a real root of the monic polynomial $\sum_{i=0}^{c} r_{s-ki} t^i$ of degree c. It follows that $\mathcal{Y} = \mathcal{Y}_n(s,k)$ can achieve any characteristic polynomial. Since $\mathcal{Y}$ is sparse, there is no spectrally arbitrary pattern $\mathcal{B}$ such that $\mathcal{Y}$ is a proper superpattern of $\mathcal{B}$. By Lemma 4.3, there exists no spectrally arbitrary $\mathcal{B}$ such that $\mathcal{Y}$ is a relaxation of $\mathcal{B}$. Therefore $\mathcal{Y}_n(s,k)$ is a minimal spectrally arbitrary pattern. □

When $k = 1$, then pattern $\mathcal{Y}_n(s,k)$ has two nonzero entries on the main diagonal.

For example,

$$\mathcal{Y}_4(3,1) = \begin{bmatrix} \circledast & * & 0 & 0 \\ \circledast & 0 & * & 0 \\ 0 & 0 & \circledast & * \\ \circledast & 0 & 0 & 0 \end{bmatrix}.$$

Further, each pattern $\mathcal{Y}_n(s,k)$ with $s < n$ is irreducible. Thus by Theorem 4.5, for odd $s < n$, $\mathcal{Y}_n(s,1)$ is a sparse spectrally arbitrary pattern that has the characteristics sought in [1, Question 4.4]:

Corollary 4.6. *For each $n \geq 4$, there exists a sparse irreducible spectrally arbitrary pattern that has two nonzero entries on the main diagonal.*

We finish this section by observing that other spectrally arbitrary patterns can be obtained from $\mathcal{Y} = \mathcal{Y}_n(s,k)$, for example, by using a diagonal or subdiagonal slide. In the proof of Theorem 4.5, for any particular polynomial $p(x)$, we determined entries y_i in $Y = Y_n(s,k)$ by forward substitutions, while solving for t depended on the polynomial equation in (4.2), so that $p_Y(x) = p(x)$. We will show that a diagonal slide of $\mathcal{Y}$ will change the coefficients in the characteristic polynomial of Y in such a way that one can still solve for each y_i via forward substitution and the corresponding equation to (4.2) will still be a monic polynomial in t of odd degree $c = \frac{s}{k}$.

Theorem 4.7. *Let s be an odd multiple of $k > 1$. Then any diagonal slide of $\mathcal{Y}_n(s,k)$ is a minimal spectrally arbitrary pattern.*

Proof. Let $k > 1$. Suppose $\mathcal{B}$ is a diagonal slide of $\mathcal{Y}_n(s,k)$ such that $\mathcal{B}_{l,l} = \circledast$ for some $1 \leq l \leq n$. Consider $Y = Y_n(s,k)$ and let $B \in Q(\mathcal{B})$ be a matrix such $B_{l,l} = y_1$ and $B_{i,j} = Y_{i,j}$ for all $i \neq j$. Note the digraph of B has two proper k-cycles (since $s > 2k$), no proper s-cycle, and exactly one proper m-cycle for each $m \notin \{s,k\}$, $1 \leq m \leq n$. The composite m-cycles that are not proper m-cycles involve either the entry t in position $(s, s-k+1)$ or the entry y_1 in position (l,l). If we consider the system that arises when equating the characteristic polynomial of B to $p(x) = x^n + r_1x^{n-1} + \cdots + r_{n-1}x + r_n$, a system of equations arises much like the system in (4.1) with the following adjustments:

1. for each m with $2 < m \leq l$, $m \neq s+1$, add summand $y_{m-1}y_1$ to the coefficient of x^{n-m},
2. if $l \leq s-k$, then for each m with $2 < m \leq l$, add summand $-y_1ty_{m-1}$ to the coefficient of x^{n-m-k},
3. if $l \geq s+1$, then for each m with $2 < m \leq s-k+1$, add summand $-y_1ty_{m-1}$ to the coefficient of x^{n-m-k}, and
4. if $s-k+1 \leq l \leq s$, then remove summand y_1t from the coefficient of x^{n-k-1}.

Note that the variable t is not removed from any coefficient of x^{n-m} with m a multiple of k. Thus, the corresponding equation to (4.2) still provides a monic polynomial in t of odd degree when s is an odd multiple of k. Therefore, as in the proof of Theorem 4.5, the system is solvable by forward substitution if s is an odd multiple of $k > 1$. □

In Section 2 we saw that any subdiagonal slide of the pattern $\mathcal{C}_n$ is spectrally arbitrary and noted that this is not true, in general, for the $\mathcal{Y}_n(s,k)$ patterns. In particular, moving entry $(4,1)$ of the spectrally arbitrary pattern $\mathcal{Y}_6(6,2)$ to position $(5,2)$ produces a pattern that requires singularity. We now present a subdiagonal slide that preserves the spectrally arbitrary property of $\mathcal{Y}_n(s,k)$.

Theorem 4.8. *If s is an odd multiple of $k > 1$, then for $s \leq m < n$, any m^{th}-subdiagonal slide of $\mathcal{Y}_n(s,k)$ is spectrally arbitrary.*

Proof. Suppose s is an odd multiple of $k > 1$, and $s \leq m < n$. Suppose $\mathcal{B}$ is an m^{th}-subdiagonal slide of $\mathcal{Y}_n(s,k)$ such that $\mathcal{B}_{l,l-m+1} = \circledast$ for some l, $m < l \leq n$. Let $Y = Y_n(s,k)$ and let $B \in Q(\mathcal{B})$ be a matrix such $B_{i,j} = Y_{i,j}$ for all i,j except $B_{l,l-m+1} = y_m$ and $B_{m,1} = 0$. Note the digraph of B has two k-cycles, no s-cycle, and exactly one q-cycle for $q \notin \{s,k\}$, $1 \leq q \leq n$. The composite q-cycles that are not proper q -cycles involve either the entry t in position $(s, s-k+1)$ or the entry y_m in position $(l, l-m+1)$. When equating the characteristic polynomial of B to $p(x) = x^n + r_1x^{n-1} + \cdots + r_{n-1}x + r_n$, a system of equations arises much like the system in (4.1) except the q^{th} equation may contain some extra terms for all $s < q \leq n$. In order to solve this system we make the same substitutions for the first s equations as we did in the proof for Theorem 4.5 so that $t, y_1, \ldots, y_{s-1}$ are fixed. The remaining $n-s$ equations are solvable via forward substitution. □

5. Concluding comments. Of the $\{0,*,\circledast\}$-patterns we observed, including those in this paper and those found via a computer search of small order patterns, the digraph of each irreducible spectrally arbitrary pattern with $2n-1$ nonzero entries contains a directed Hamilton path (i.e., a directed path on n vertices). In other words, any such pattern is equivalent to a superpattern of a proper Hessenberg pattern (i.e., a pattern with a nonzero entry in each position $(i,i+1)$, $1 \leq i < n$). We wonder if there exist examples that do not have a directed Hamilton path or if a Hamilton path is a characteristic of the digraph of every irreducible sparse spectrally arbitrary $\{0,*,\circledast\}$-pattern.

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