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SQUARE ROOTS OF DOUBLY REGULAR TOURNAMENT MATRICES*

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Abstract. Fletcher asked whether there is a $(0, 1)$-matrix of order greater than 3 whose square is a regular tournament matrix. We give a negative answer for a special class of regular tournament matrices: There is no $(0, 1)$-matrix of order greater than 3 whose square is a doubly regular tournament matrix.

Key words. Tournament matrix, Matrix square root, Doubly regular tournament

AMS subject classifications. 05C50, 05C20, 15B36.

1. Introduction. With a $(0, 1)$-matrix $A = [a_{hj}]$ of order v we associate a directed graph G_A in the usual manner. The vertex set is $\{1, 2, \ldots, v\}$, and there is an arc from vertex h to vertex j exactly when $a_{hj} = 1$. We say the $(0, 1)$-matrix A is *regular* of degree d provided the number of 1's in each row and column is d, that is, provided

$$AJ = JA = dJ,$$

where J is the all 1's matrix of order v. In the associated directed graph G_A each vertex has exactly d arcs exiting it and d arcs entering it.

A $(0, 1)$-matrix $B = [b_{hj}]$ of order v is a *tournament matrix* provided

$$B + B^T = J - I,$$

where J is the all 1's matrix of order v. A tournament matrix records the outcomes of the matches in a round robin tournament among the players $\{1, 2, \ldots, v\}$ with no tied matches. We have $b_{hj} = 1$ when player h defeats player j, and $b_{hj} = 0$ otherwise. The corresponding directed graph G_B is a *tournament*. In a regular tournament matrix

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ELA

B of order v the number of 1's in each row and column is $(v-1)/2$, and we have

$$B\mathbf{u} = \left(\frac{v-1}{2}\right)\mathbf{u},$$

where $\mathbf{u} = (1, 1, \ldots, 1)^T$ is a column vector of 1's.

The matrix A is a *square root* of the tournament matrix B provided $A^2 = B$. We restrict ourselves to square roots that are $(0, 1)$-matrices, so that the graph-theoretic interpretations given below are valid. Suppose that the $(0, 1)$-matrix A is a square root of a tournament matrix B. In the directed graph G_A there are no loops and no closed walks of length 2. Also, each pair of distinct vertices h and j is joined by a unique walk of length 2 either from h to j or from j to h, but not both.

Fletcher [2] gave several general constructions for square roots of some families of tournament matrices. Square roots of tournament matrices of order at most 3 are readily analyzed.

Example 1.1. (a) *The tournament matrix* $[0]$ *of order* 1 *is its own square root.*
(b) *The two tournament matrices of order* 2 *are*

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

neither of which has a square root.
(c) *The tournament matrices*

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

of order 3 *are square roots of one another. The six other tournament matrices of order* 3 *do not have square roots.*

The tournament matrices of orders 1 and 3 in Example 1.1 are regular and have square roots. Fletcher [2] asked whether there is any $(0, 1)$-matrix of order greater than 3 whose square is a regular tournament matrix. We provide a negative answer for a special class of regular tournament matrices.

A tournament matrix B of order v is *doubly regular* with *subdegree* λ provided

$$BB^T = (\lambda + 1)I + \lambda J.$$

This matrix equation implies that a doubly regular tournament matrix is regular of degree $2\lambda + 1$, and it follows that $v = 4\lambda + 3$. The matrix equation also requires that for each pair of distinct vertices h and j in the tournament G_B there are exactly λ

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vertices g such that the arcs $h \to g$ and $j \to g$ both occur. The tournament matrices of order 3 in Example 1.1(c) are doubly regular with subdegree $\lambda = 0$. A doubly regular tournament matrix is the incidence matrix of a skew Hadamard symmetric block design with parameters $(v, k, \lambda) = (4\lambda + 3, 2\lambda + 1, \lambda)$, and is thus equivalent [6] to a skew Hadamard matrix of order $v + 1$.

Our main result asserts that no doubly regular tournament matrix of order greater than 3 has a square root.

Theorem 1.2. *If the square of a $(0,1)$-matrix A of order v is a doubly regular tournament matrix, then $v = 3$ and A is one of the two matrices in Example 1.1(c).*

Our proof appears in Sections 2 and 3. A similar argument occurs in [5].

2. Square Roots of Regular Tournament Matrices are Regular. We begin with a lemma that asserts that the square root of a regular tournament matrix must itself be regular. Our proof uses the well known Perron-Frobenius theorem for irreducible nonnegative matrices.

Lemma 2.1. *If A is a $(0,1)$-matrix of order v whose square is a regular tournament matrix, then the number of 1's in every row and column of A is $\sqrt{(v-1)/2}$.*

Proof. The result is trivial for $v = 1$. Suppose that $v \geq 2$. The regular tournament matrix A^2 of order v is irreducible (see [4, p. 2]), and it follows that A itself is irreducible. By the Perron-Frobenius theorem [3] there is a positive vector $\mathbf{w}$ and positive eigenvalue ρ such that $A\mathbf{w} = \rho\mathbf{w}$. Thus

$$A^2\mathbf{w} = \rho^2\mathbf{w},$$

and the positive vector $\mathbf{w}$ is an eigenvector of the regular tournament matrix A^2 with eigenvalue ρ^2. We know that up to a scalar multiple A^2 has the unique positive eigenvector $(1, 1, \ldots, 1)^T$ with corresponding eigenvalue $(v-1)/2$. Hence $\rho = \sqrt{(v-1)/2}$, and we can take $\mathbf{w} = (1, 1, \ldots, 1)^T$. Therefore the number of 1's in every row of A is $\sqrt{(v-1)/2}$. Apply the same argument to A^T to see that the number of 1's in every column of A is $\sqrt{(v-1)/2}$. ☐

3. Proof of Theorem 1.2. The result is clear for $v = 1$. Suppose that $v \geq 2$ and let A be a $(0,1)$-matrix of order v such that A^2 is a doubly regular tournament matrix. By Lemma 2.1 A is regular of degree $d = \sqrt{(v-1)/2}$. It is known from [7] (also see the article by Dom de Caen, David Gregory et al. [1]) that a doubly regular tournament matrix of order v has 3 distinct eigenvalues:

- $(v-1)/2$ of multiplicity 1;

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- r and $\overline{r}$, each of multiplicity $(v-1)/2$, where
$$r = -\frac{1}{2} + \left(\frac{\sqrt{v}}{2}\right) i.$$

The eigenvalues of the real matrix A must be square roots of eigenvalues of A^2 given above and must occur in complex conjugate pairs. Let us write $(x+yi)^2 = r$ with
$$x = \frac{\sqrt{\sqrt{v+1}-1}}{2} \qquad \text{and} \qquad y = \frac{\sqrt{\sqrt{v+1}+1}}{2},$$
It follows that for some nonnegative integer m the eigenvalues of A are

- $\sqrt{(v-1)/2}$ of multiplicity 1;
- $x \pm yi$, each of multiplicity m;
- $-x \pm yi$, each of multiplicity $(v-1-2m)/2$.

Because each diagonal entry of the tournament matrix A^2 is 0, each diagonal entry of A is 0. Thus the sum of the eigenvalues of A is 0, and we obtain the condition
$$0 = \sqrt{\frac{v-1}{2}} + m(2x) + (v-1-2m)(-x) = \sqrt{\frac{v-1}{2}} + x(4m-v+1).$$
It follows that
$$(4m-v+1)^2 = \frac{v-1}{2x^2} = \frac{2(v-1)}{\sqrt{v+1}-1} = \frac{2(v-1)\left(\sqrt{v+1}+1\right)}{v}$$
is (the square of) an integer. Thus $\sqrt{v+1}$ is a rational square root of an integer. It follows that $\sqrt{v+1}$ is an integer. Because v and $2(v-1)$ are relatively prime, v must be a divisor of $\sqrt{v+1}+1$. In particular, $v \leq \sqrt{v+1}+1$. Because $v \geq 3$, the only possibility is $v = 3$, and A must be one of the tournament matrices in Example 1.1(c).

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