SQUARE ROOTS OF DOUBLY REGULAR TOURNAMENT MATRICES

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Abstract. Fletcher asked whether there is a (0, 1)-matrix of order greater than 3 whose square is a regular tournament matrix. We give a negative answer for a special class of regular tournament matrices: There is no (0, 1)-matrix of order greater than 3 whose square is a doubly regular tournament matrix.

Key words. Tournament matrix, Matrix square root, Doubly regular tournament

AMS subject classifications. 05C50, 05C20, 15B36.

1. Introduction. With a (0, 1)-matrix $A = [a_{hj}]$ of order $v$ we associate a directed graph $G_A$ in the usual manner. The vertex set is $\{1, 2, \ldots, v\}$, and there is an arc from vertex $h$ to vertex $j$ exactly when $a_{hj} = 1$. We say the (0, 1)-matrix $A$ is regular of degree $d$ provided the number of 1’s in each row and column is $d$, that is, provided

$$AJ = JA = dJ,$$

where $J$ is the all 1’s matrix of order $v$. In the associated directed graph $G_A$ each vertex has exactly $d$ arcs exiting it and $d$ arcs entering it.

A (0, 1)-matrix $B = [b_{hj}]$ of order $v$ is a tournament matrix provided

$$B + B^T = J - I,$$

where $J$ is the all 1’s matrix of order $v$. A tournament matrix records the outcomes of the matches in a round robin tournament among the players $\{1, 2, \ldots, v\}$ with no tied matches. We have $b_{hj} = 1$ when player $h$ defeats player $j$, and $b_{hj} = 0$ otherwise. The corresponding directed graph $G_B$ is a tournament. In a regular tournament matrix

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*Received by the editors on September 20, 2014. Accepted for publication on November 9, 2014. Handling Editor: Kevin vander Meulen.
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of order \( v \) the number of 1’s in each row and column is \((v - 1)/2\), and we have

\[
Bu = \left(\frac{v - 1}{2}\right) u,
\]

where \( u = (1, 1, \ldots, 1)^T \) is a column vector of 1’s.

The matrix \( A \) is a square root of the tournament matrix \( B \) provided \( A^2 = B \). We restrict ourselves to square roots that are \((0, 1)\)-matrices, so that the graph-theoretic interpretations given below are valid. Suppose that the \((0, 1)\)-matrix \( A \) is a square root of a tournament matrix \( B \). In the directed graph \( G_A \) there are no loops and no closed walks of length 2. Also, each pair of distinct vertices \( h \) and \( j \) is joined by a unique walk of length 2 either from \( h \) to \( j \) or from \( j \) to \( h \), but not both.

Fletcher [2] gave several general constructions for square roots of some families of tournament matrices. Square roots of tournament matrices of order at most 3 are readily analyzed.

**Example 1.1.** (a) The tournament matrix \([0]\) of order 1 is its own square root.
(b) The two tournament matrices of order 2 are

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix},
\]

neither of which has a square root.
(c) The tournament matrices

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

of order 3 are square roots of one another. The six other tournament matrices of order 3 do not have square roots.

The tournament matrices of orders 1 and 3 in Example 1.1 are regular and have square roots. Fletcher [2] asked whether there is any \((0, 1)\)-matrix of order greater than 3 whose square is a regular tournament matrix. We provide a negative answer for a special class of regular tournament matrices.

A tournament matrix \( B \) of order \( v \) is doubly regular with subdegree \( \lambda \) provided

\[
BB^T = (\lambda + 1)I + \lambda J.
\]

This matrix equation implies that a doubly regular tournament matrix is regular of degree \( 2\lambda + 1 \), and it follows that \( v = 4\lambda + 3 \). The matrix equation also requires that for each pair of distinct vertices \( h \) and \( j \) in the tournament \( G_B \) there are exactly \( \lambda \)
vertices $g$ such that the arcs $h \rightarrow g$ and $j \rightarrow g$ both occur. The tournament matrices of order 3 in Example 1.1(c) are doubly regular with subdegree $\lambda = 0$. A doubly regular tournament matrix is the incidence matrix of a skew Hadamard symmetric block design with parameters $(v, k, \lambda) = (4\lambda + 3, 2\lambda + 1, \lambda)$, and is thus equivalent [6] to a skew Hadamard matrix of order $v + 1$.

Our main result asserts that no doubly regular tournament matrix of order greater than 3 has a square root.

**Theorem 1.2.** If the square of a $(0,1)$-matrix $A$ of order $v$ is a doubly regular tournament matrix, then $v = 3$ and $A$ is one of the two matrices in Example 1.1(c).

Our proof appears in Sections 2 and 3. A similar argument occurs in [5].

2. Square Roots of Regular Tournament Matrices are Regular. We begin with a lemma that asserts that the square root of a regular tournament matrix must itself be regular. Our proof uses the well known Perron-Frobenius theorem for irreducible nonnegative matrices.

**Lemma 2.1.** If $A$ is a $(0,1)$-matrix of order $v$ whose square is a regular tournament matrix, then the number of 1’s in every row and column of $A$ is $\sqrt{(v-1)/2}$.

**Proof.** The result is trivial for $v = 1$. Suppose that $v \geq 2$. The regular tournament matrix $A^2$ of order $v$ is irreducible (see [4, p. 2]), and it follows that $A$ itself is irreducible. By the Perron-Frobenius theorem [3] there is a positive vector $w$ and positive eigenvalue $\rho$ such that $Aw = \rho w$. Thus

$$A^2w = \rho^2w,$$

and the positive vector $w$ is an eigenvector of the regular tournament matrix $A^2$ with eigenvalue $\rho^2$. We know that up to a scalar multiple $A^2$ has the unique positive eigenvector $(1, 1, \ldots, 1)^T$ with corresponding eigenvalue $(v-1)/2$. Hence $\rho = \sqrt{(v-1)/2}$, and we can take $w = (1, 1, \ldots, 1)^T$. Therefore the number of 1’s in every row of $A$ is $\sqrt{(v-1)/2}$. Apply the same argument to $A^T$ to see that the number of 1’s in every column of $A$ is $\sqrt{(v-1)/2}$. \[\square\]

3. **Proof of Theorem 1.2.** The result is clear for $v = 1$. Suppose that $v \geq 2$ and let $A$ be a $(0,1)$-matrix of order $v$ such that $A^2$ is a doubly regular tournament matrix. By Lemma 2.1 $A$ is regular of degree $d = \sqrt{(v-1)/2}$. It is known from [7] (also see the article by Dom de Caen, David Gregory et al. [1]) that a doubly regular tournament matrix of order $v$ has 3 distinct eigenvalues:

- $(v - 1)/2$ of multiplicity 1;
• $r$ and $\tau$, each of multiplicity $(v-1)/2$, where

$$r = \frac{1}{2} + \left( \sqrt{\frac{v}{2}} \right) i.$$

The eigenvalues of the real matrix $A$ must be square roots of eigenvalues of $A^2$ given above and must occur in complex conjugate pairs. Let us write $(x + yi)^2 = r$ with

$$x = \frac{\sqrt{v+1}-1}{2} \quad \text{and} \quad y = \frac{\sqrt{v+1}+1}{2},$$

It follows that for some nonnegative integer $m$ the eigenvalues of $A$ are

• $\sqrt{(v-1)/2}$ of multiplicity 1;
• $x \pm yi$, each of multiplicity $m$;
• $-x \pm yi$, each of multiplicity $(v-1-2m)/2$.

Because each diagonal entry of the tournament matrix $A^2$ is 0, each diagonal entry of $A$ is 0. Thus the sum of the eigenvalues of $A$ is 0, and we obtain the condition

$$0 = \sqrt{\frac{v-1}{2}} + m(2x) + (v-1-2m)(-x) = \sqrt{\frac{v-1}{2}} + x(4m - v + 1).$$

It follows that

$$(4m - v + 1)^2 = \frac{v-1}{2x^2} = \frac{2(v-1)}{\sqrt{v+1} - 1} = \frac{2(v-1) (\sqrt{v+1} + 1)}{v}$$

is (the square of) an integer. Thus $\sqrt{v+1}$ is a rational square root of an integer. It follows that $\sqrt{v+1}$ is an integer. Because $v$ and $2(v-1)$ are relatively prime, $v$ must be a divisor of $\sqrt{v+1} + 1$. In particular, $v \leq \sqrt{v+1} + 1$. Because $v \geq 3$, the only possibility is $v = 3$, and $A$ must be one of the tournament matrices in Example 1.1(c).

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