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NORMALIZED RATIONAL SEMIREGULAR GRAPHS*

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Dedicated to the memory of Prof. David Gregory

Abstract. Let G be a graph and let A and D be the adjacency matrix of G and diagonal matrix of vertex degrees of G respectively. If each vertex degree is positive, then the *normalized adjacency matrix* of G is $\hat{A} = D^{-1/2}AD^{-1/2}$. A classification is given of those graphs for which the all eigenvalues of the normalized adjacency matrix are integral. The problem of determining those graphs G for which $\lambda \in \mathbb{Q}$ for each eigenvalue λ of $\hat{A}(G)$ is considered. These graphs are called *normalized rational*. It will be shown that a semiregular bipartite graph G with vertex degrees r and s is normalized rational if and only if every eigenvalue of A is a rational multiple of $\sqrt{rs}$. This result will be used to classify the values of n for which the semiregular graph (with vertex degrees 2 and $n-1$) obtained from subdividing each edge of K_n is normalized rational. Necessary conditions for the k-uniform complete hypergraph on n vertices to be normalized rational are also given. Finally, conditions for the incidence graphs of Steiner triple and quadruple systems to be normalized rational are given.

Key words. graphs, hypergraphs, semiregular graphs, normalized Laplacian matrix

AMS subject classifications. 05C50, 05C65

1. Introduction. Let G be a graph, $A = A(G)$, $D = D(G)$, and $L = L(G) = D - A$ its adjacency matrix, diagonal matrix of vertex degrees, and Laplacian matrix respectively. The *normalized adjacency matrix* of G is $\hat{A} = \hat{A}(G) = D^{-1/2}AD^{-1/2}$ and the *normalized Laplacian matrix* is $\hat{L} = \hat{L}(G) = D^{-1/2}LD^{-1/2}$. If v is a vertex of degree 0 in G, then $D_{vv} = 0$. In that case, D^{-1} and $D^{-1/2}$ are not defined. Therefore, we assume throughout that G has no vertices of degree 0 or, equivalently, that G contains no isolated vertices.

G is called *adjacency integral* if every eigenvalue of A is an integer and *Laplacian integral* if every eigenvalue of L is an integer. The problems of determining which graphs are adjacency integral graphs began with Harary and Schwenk [8] in 1974. Later, Grone and Merris considered Laplacian integral graphs [7]. Balińska et al [1] provide a survey on graphs with adjacency and Laplacian integral eigenvalues. More recent results on these topics include [2], [9], and [10]. Since these problems

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continue to generate interest and $\hat{A}$ and $\hat{L}$ have received more attention in recent years (For example, [5] and [4]), it may be worthwhile to consider the analogues for these matrices to the adjacency and Laplacian integral problems.

The natural analogues are to determine those graphs for which every eigenvalue of either $\hat{A}$ or $\hat{L}$ is an integer. Since $\hat{L} = I - A$, $1 - \lambda$ is an eigenvalue of $\hat{A}$ if and only if λ is an eigenvalue of $\hat{A}$. Therefore, the problems are equivalent. Since $\hat{A}$ is similar to the stochastic matrix AD^{-1}, we know that the eigenvalues of $\hat{A}$ are in the interval $[-1, 1]$, so the only possibly integer eigenvalues of $\hat{A}$ are $-1, 0$, and 1. This restriction makes it easy to classify the graphs for which the normalized matrices have only integer eigenvalues. They are graphs whose connected components are complete bipartite graphs. We prove this in Section 2.

We consider instead the problem of determining those graphs for which every eigenvalue of $\hat{A}$ is a rational number. As in the case of integer eigenvalues, the corresponding problem for $\hat{L}$ is equivalent. Therefore we say that G is *normalized rational* if every eigenvalue of $\hat{A}$ is a rational number.

If $G = K_{m,n}$, then the vertex degree is constant in each vertex part. More generally, if G is bipartite with vertex parts R and S, each vertex in R has degree r, and each vertex in S has degree s, then we say that G is *(r, s)-semiregular*. In section 3, we will study normalized rational (r, s)-semiregular graphs that are derived from regular graphs by subdividing edges and from uniform regular hypergraphs.

2. Normalized Integral and Normalized Rational Graphs. In this section, we classify those graphs for which every eigenvalue of the adjacency matrix is an integer. In both of the of the following lemmas, we use Sylvester's Law of Inertia [12, sec. IV.17], which states that if matrices A and B are congruent, then A and B have the same numbers of positive, negative, and zero eigenvalues.

LEMMA 2.1. *Suppose G has integral normalized adjacency eigenvalues. Then the eigenvalues of $\hat{A}$ are $1, 0$ and -1. The multiplicity of the eigenvalue 1 is the number of connected components of G.*

Proof. The first part of the lemma follows from the fact that $\hat{A}$ is similar to AD^{-1}, a stochastic matrix whose eigenvalues must therefore be in the interval $[-1, 1]$.

Because $\hat{L}$ and L are congruent, the multiplicity of 0 as an eigenvalue of $\hat{L}$ is same as that of L, and because $\hat{L} = I - \hat{A}$, this is the multiplicity of 1 as an eigenvalue of $\hat{A}$. But the multiplicity of the eigenvalue 0 of L is just the number of connected components of G [6, sec. 13.1]. The second assertion follows. ∎

LEMMA 2.2. *$\hat{A}(G)$ has only integral eigenvalues if and only if the connected components of G are complete bipartite.*

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Proof. If A is the adjacency matrix of $K_{m,n}$, then the normalized adjacency matrix of $K_{m,n}$ is $\hat{A} = \frac{1}{\sqrt{mn}}A$. The eigenvalues of A are $\pm\sqrt{mn}$ with multiplicity 1 and 0 with multiplicity $n-2$. It follows that the eigenvalues of $\hat{A}$ are ± 1 and 0 with the respective multiplicities. Thus if the components of G are complete bipartite, then $\hat{A}(G)$ has only integral eigenvalues.

Now suppose that $\hat{A}(G)$ has only integral eigenvalues. By Lemma 2.1, the eigenvalues of $\hat{A}$ are $-1, 0$, or 1 and if H is a connected component of G, then the multiplicity of the eigenvalue 1 is 1. The trace of $\hat{A}(H)$ is 0, so the sum of the eigenvalues is 0 and H has at least 1 negative eigenvalue. Since all eigenvalues are integral that eigenvalue must be -1, and because the sum of eigenvalues is 0, the multiplicity must be 1. The remaining eigenvalue is 0 with multiplicity $n-2$.

It is well known that a graph with exactly 1 positive adjacency eigenvalue and exactly 1 negative adjacency eigenvalue is complete bipartite. Since A is congruent to $\hat{A}$ for any graph, it is equivalent to say that a graph with exactly 1 positive and exactly 1 negative normalized adjacency eigenvalue must be complete bipartite. In particular, the components of G are complete bipartite. $\square$

So, in contrast to adjacency or Laplacian integral graphs, the problem of determining which graphs have integral normalized adjacency eigenvalues is easy.

Now suppose M is a square matrix. Then the coefficient of the highest degree term in the characteristic polynomial $\phi_M(\lambda) = (\lambda I - M)$ of M is 1. Furthermore, if M is integral, then ϕ_M will have only integral coefficients. It follows from the Rational Roots Theorem that if M has only integer entries, then any rational eigenvalues of M will be also be integers. In particular, all rational eigenvalues of the adjacency and Laplacian matrices are integral. However, the matrix AD^{-1} can have nonintegral rational entries (and usually does), so there can be nonintegral rational eigenvalues. Thus, we consider the problem of determining those graphs for which every eigenvalue of normalized adjacency matrix is rational. Since $(1-\lambda)$ is an eigenvalue of $\hat{L}$ if and only if λ is an eigenvalue of $\hat{A}$, G has only rational normalized Laplacian eigenvalues if and only if it has only normalized rational eigenvalues. Therefore, call a graph G *normalized rational* if all of the eigenvalues of $\hat{A}(G)$ are rational.

3. Normalized rational semiregular graphs.

A bipartite graph G with vertex parts X and Y is called (r,s)-*semiregular* if each vertex in X has degree r and each vertex in Y has degree s. If G is (r,s)-semiregular, then $\pm\sqrt{rs}$ are eigenvalues of A and they are the largest in absolute value.

LEMMA 3.1. *Suppose that G is an (r,s)-semiregular graph. Then G is normalized rational if and only if every eigenvalue of A is a rational multiple of $\sqrt{rs}$.*

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Proof. By permuting vertices if necessary, we can write

$$\hat{A}(G) = \begin{bmatrix} O & \frac{1}{\sqrt{rs}}B \\ \frac{1}{\sqrt{rs}}B^T & O \end{bmatrix} = \frac{1}{\sqrt{rs}}A$$

so $\lambda/\sqrt{rs}$ is an eigenvalue of $\hat{A}$ if and only if λ is an eigenvalue of A. Therefore, $\lambda/\sqrt{rs}$ is rational if and only if λ is a rational multiple of $\sqrt{rs}$. ∎

Corollary 3.2. *If G is adjacency integral and rs is a square in $\mathbb{Z}$, then G is normalized rational.*

The complete bipartite graph $K_{m,n}$ is (m,n)-semiregular with adjacency eigenvalues $\pm\sqrt{mn}$ and 0. Thus, by the preceding theorem, $K_{m,n}$ is normalized rational. One infinite family of graphs satisfying the conditions of the preceding corollary will be given in the next section. Another sporadic example is given by the bipartite subdivision $G = S(T(11))$ (defined below) of the triangular graph $T(11)$ (triangular graphs are defined in [6, Ch. 10]). G is a $(2,18)$-semiregular graph with eigenvalues $\pm 6, \pm 5$, and ± 4. For this graph, $rs = 36$, which is a square in $\mathbb{Z}$. It follows from the corollary that G is normalized rational. A search of the literature on adjacency integral graphs did not reveal any other examples.

3.1. Bipartite subdivisions of complete graphs. The *bipartite subdivision* of a graph G is the graph $S(G)$ obtained by replacing each edge of G by a path of length 2. If x is a vertex in $S(G)$ that is not in G, then x is adjacent to exactly two vertices, so the degree of x in $S(G)$ is 2. If x is in both G and $S(G)$, then the degree of x is the same in both. In particular, if G is a k-regular graph, then every vertex in both G and $S(G)$ has degree k, so $S(G)$ is a $(2,k)$-semiregular graph.

Lemma 3.3. *Let G be a k-regular graph. Then $\pm\sqrt{\lambda + k}$ is an eigenvalue of $A(S(G))$ whenever λ is an eigenvalue of $A(G)$. The remaining eigenvalues of $A(S(G))$ (if any) are 0.*

Proof. If M is the incidence matrix of G, then the adjacency matrix of $S(G)$ is

$$A(S(G)) = \begin{bmatrix} O & M \\ M^T & O \end{bmatrix}$$

The n singular values of M and their negatives are eigenvalues of $A(S(G))$. These are the square roots of the eigenvalues of $MM^T = A + kI$ and their negatives. The remaining eigenvalues must be 0. ∎

Suppose that $n \geq 2$. Then $G = K_n$ is $(n-1)$-regular and G has at least one edge, so $S(G)$ is $(2, n-1)$-semiregular. The nonzero eigenvalues of G are n and -1, so the eigenvalues of $S(G)$ are $\pm\sqrt{2n-2}$, $\pm\sqrt{n-2}$, and 0. Thus, G is normalized rational

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if and only if

$$\sqrt{n-2} = \frac{a}{b}\sqrt{2n-2}$$

for some integers a and b. We assume that a/b is in lowest terms, so $\gcd(a,b) = 1$. Solving for n gives

$$n = \frac{2(b^2 - a^2)}{b^2 - 2a^2} \tag{3.1}$$

Thus, $\sqrt{n-2}$ is a rational multiple of $\sqrt{2n-2}$ if and only if $(b^2 - 2a^2)$ divides $2(b^2 - a^2)$.

Lemma 3.4. *If* $\gcd(a^2, b^2) > 1$*, then* $\gcd(a, b) > 1$.

Proof. Let $g = \gcd(a^2, b^2) > 1$ and suppose that p is a prime divisor of g. Then $p|a^2$ and $p|b^2$. Since p is prime, $p|a$, $p|b$, and $p > 1$. Since p is a common divisor of a and b, $p|\gcd(a, b)$. $\square$

Theorem 3.5. *Suppose that* $a, b \in \mathbb{Z}$ *and* $\gcd(a, b) = 1$. *Then* $n = \frac{2(b^2-a^2)}{b^2-2a^2} \in \mathbb{Z}$ *if and only if* $|b^2 - 2a^2| < 3$.

Proof. If $|b^2 - 2a^2| < 3$, then the denominator in the formula for n is ± 1 or ± 2. Since each of these numbers divides 2 and 2 is a factor of the numerator, it follows that n is an integer.

Proceeding by contradiction for the converse, suppose that $b^2 - 2a^2 = k$ where $|k| \geq 3$. Then $b^2 - 2a^2 = k|2(b^2 - a^2)$, since $n \in \mathbb{Z}$. Therefore, $2(b^2 - a^2) = ck$ for some $c \in \mathbb{Z}$. Since $b^2 - 2a^2 = k$, $b^2 = 2a^2 + k$. Substituting this into $2(b^2 - a^2) = ck$ yields the equation $2a^2 = k(c - 2)$.

If k is odd, then $k|a^2$, and therefore, $a^2 = pk$ for some $p \in \mathbb{Z}$. It follows that $b^2 = 2(pk) + k = k(2p + 1)$, and so $k|b^2$. Thus, $k|\gcd(a^2, b^2)$. If k is even, then $k = 2l$ for some $l \in \mathbb{Z}$, $|l| \geq 2$. Then $2a^2 + 4l = 2cl$, so $a^2 = l(c - 2)$. Therefore, $l|a^2$ and $a^2 = pl$ for some $p \in \mathbb{Z}$. It follows that $b^2 = l(2p+2)$, and so $l|b^2$. Thus, $l|\gcd(a^2, b^2)$.

In either case, by Lemma 3.4, there is a number $q > 1$ such that $q|\gcd(a, b)$, a contradiction. Therefore $|b^2 - 2a^2| < 3$. $\square$

Therefore, in looking for n such that $\sqrt{n-2} = \frac{a}{b}\sqrt{2n-2}$ for some (a, b) we only need to consider (a, b) where $\gcd(a, b) = 1$ and $b^2 - 2a^2 = c$ for some $c \in \{-2, -1, 1, 2\}$. For any choice of c and for any integer solution (a, b) to $b^2 - 2a^2$, we can substitute $b^2 = 2a^2 + c$ into 3.1 to find the corresponding value of n. Since n is the number of vertices in K_n, n must be positive. If $c = -2$, then $n = -(a^2 - 2) \geq 2$ only when $a = 0$. But there is no integer solution $(0, b)$ to $b^2 - 2a^2 = -2$. If $c = -1$, then $n = -2(a^2 - 1) \geq 2$ only when $a = 0$. But there is no integer solution $(0, b)$ to

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$b^2 - 2a^2 = -1$. Therefore, there is no value of n for which bipartite subdivision of K_n is normalized rational when $b^2 - 2a^2 = -2$ or $b^2 - 2a^2 = -1$ in (3.1).

If $c = 1$, then $n = 2(a^2 + 1) \geq 2$ for each integer solution (a, b) to $b^2 - 2a^2 = 1$, and if $c = 2$, then $n = (a^2 + 2) \geq 2$ for each integer solution (a, b) to $b^2 - 2a^2 = 2$. Thus, we have the following theorem.

Theorem 3.6. *The subdivision of K_n is normalized rational if and only if $n = \frac{2(b^2-a^2)}{b^2-2a^2}$ where $b^2 - 2a^2 = 1$ or $b^2 - 2a^2 = 2$ for some $a, b \in \mathbb{Z}$. Furthermore,*

1. *if (a, b) is a solution to $b^2 - 2a^2 = 1$, then $n = 2(a^2 + 1)$,*
2. *if (a, b) is a solution to $b^2 - 2a^2 = 2$, then $n = (a^2 + 2)$*

It remains to determine the integer solutions to $b^2 - 2a^2 = 1$ (a Pell equation) and $b^2 - 2a^2 = 2$ (a generalized Pell equation). In each case, the general solution can be found, using the algorithm in [11, Ch. 6] for example. The solutions to $b^2 - 2a^2 = 1$ are

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\left(3 + 2\sqrt{2}\right)^m}{4} \begin{bmatrix} 4 + 3\sqrt{2} \\ 6 + 4\sqrt{2} \end{bmatrix} + \frac{\left(3 - 2\sqrt{2}\right)^m}{4} \begin{bmatrix} 4 - 3\sqrt{2} \\ 6 - 4\sqrt{2} \end{bmatrix},$$

where $m \in \mathbb{Z}$. The solutions to $b^2 - 2a^2 = 2$ are

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{\left(3 + 2\sqrt{2}\right)^m}{2} \begin{bmatrix} 7 + 5\sqrt{2} \\ 10 + 7\sqrt{2} \end{bmatrix} + \frac{\left(3 - 2\sqrt{2}\right)^m}{2} \begin{bmatrix} 7 - 5\sqrt{2} \\ 10 - 7\sqrt{2} \end{bmatrix},$$

where $m \in \mathbb{Z}$. Some resulting values of n, a, b, and n, as well as the corresponding value for c are given in Table 3.1

We can also use Corollary 3.2 to show that the graphs described in Part 2 of Theorem 3.6 are normalized rational. It follows from the spectrum of K_n given above that $S(K_n)$ is adjacency integral if and only if $2n - 2 = \sigma^2$ and $n - 2 = \tau^2$ for some integers σ and τ. This implies $n = \tau^2 + 2$ for some τ such that there is an integral solution (σ, τ) to $\sigma^2 - 2\tau^2 = 2$. If we let $(\sigma, \tau) = (b, a)$, this is the same equation as in Part 2 of the theorem, and the resulting value of n is also the same. Since $r = 2$ and $s = n - 2$, we have $rs = 2n - 2 = \sigma^2$, and so by Corollary 3.2, the graph is normalized rational.

3.2. Uniform Hypergraphs. An r-regular k-uniform hypergraph is a hypergraph in which each hyperedge contains exactly k vertices and each vertex is contained in exactly r hyperedges. The *incidence graph* of a hypergraph H is the bipartite graph $I(H)$ whose vertex parts are $V(H)$ and $E(H)$. A vertex $v \in V(H)$ and hyperedge $e \in E(H)$ are adjacent in $I(H)$ if v is a vertex in e. If H is an r-regular k-uniform hypergraph, then $I(H)$ is (k, r)-semiregular graph. Conversely, any (k, r)-semiregular graph is the incidence graph of an r-regular k-uniform hypergraph (possibly with

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TABLE 3.1

Some values of n for which $S(K_n)$ is normalized rational, along with the corresponding a, b, and c.

n	a	b	c
2	0	1	1
3	1	2	2
10	2	3	1
51	7	10	2
290	12	17	1
1683	41	58	2
9802	70	99	1
57123	239	338	2
332930	408	577	1
1940451	1393	1970	2
11309770	2378	3363	1

multiple hyperedges). In this language, a k-regular graph is a k-regular 2-uniform hypergraph. Since the incidence graph $I(H)$ contains isolated vertices if H does, and thus $\hat{A}(I(H))$ and $\hat{L}(I(H))$ are not defined, we assume that H contains no isolated vertices. This means that $n \geq k$.

3.2.1. Complete hypergraphs. The complete k-uniform hypergraph $K_n^{(k)}$ is the hypergraph on n vertices whose hyperedge set consists of all subsets of size k from $V(H)$. The complete graph K_n is the complete 2-uniform hypergraph $K_n^{(2)}$, and the bipartite subdivision of K_n is the incidence graph of K_n. The incidence graph of $K_n^{(k)}$ is an (r, k)-semiregular with $r = \binom{n-1}{k-1}$. The *incidence matrix* of the hypergraph H is the matrix whose rows are indexed by the vertices of H and whose columns are indexed by the hyperedges of H. The (v, e)-entry of the matrix is 1 if the vertex v is contained in the hyperedge e, and 0 otherwise. If M is the incidence matrix of H, then $I(H)$ has adjacency matrix

$$A(I(H)) = \begin{bmatrix} O & M^T \\ M & O \end{bmatrix}$$

As in the proof of Lemma 3.3, the square roots of the eigenvalues of MM^T and their negatives are eigenvalues of $A(I(H))$, and the remaining eigenvalues are 0. The (i, j)-entry of MM^T is the number of hyperedges containing vertices i and j. In the case where $i = j$, this is the number of hyperedges containing the vertex i. There are $\binom{n-1}{k-1}$ hyperedges containing a given vertex in H and $\binom{n-2}{k-2}$ hyperedges containing a given pair of vertices. Thus $MM^T = \binom{n-1}{k-1}I + \binom{n-2}{k-2}(J - I)$, which has nonzero eigenvalues $\binom{n-1}{k-1} - \binom{n-2}{k-2}$ and $\binom{n-1}{k-1} - \binom{n-2}{k-2} + n\binom{n-2}{k-2}$. After some manipulation, applying the

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identity $\binom{n-1}{k-1} = \frac{n-1}{k-1}\binom{n-2}{k-2}$, we find that the eigenvalues of $I(H)$ are

$$\pm\sqrt{\frac{n-k}{k-1}\binom{n-2}{k-2}} \text{ and } \pm\sqrt{rk} = \sqrt{\frac{(n-1)k}{k-1}\binom{n-2}{k-2}}$$

Therefore, the eigenvalues of $A(I(H))$ are rational multiples of $\sqrt{rk}$ if and only if

$$\sqrt{\frac{n-k}{k-1}\binom{n-2}{k-2}} = \frac{a}{b}\sqrt{\frac{(n-1)k}{k-1}\binom{n-2}{k-2}}$$

for some integers a and b for which we may assume that $\gcd(a,b) = 1$. Isolating n in this equation gives

$$n = \frac{k(b^2 - a^2)}{b^2 - ka^2} \tag{3.2}$$

Thus, in order for $I(H)$ to be normalized rational, the expression on the right must be a positive integer. The following lemma restricts the possible values of $b^2 - ka^2$ in the denominator on the right hand side of (3.2).

Lemma 3.7. *If $n = \frac{k(b^2-a^2)}{b^2-ka^2}$ is an integer and $\gcd(a,b) = 1$, then $b^2 - ka^2$ is a divisor of $k(k-1)$.*

Proof. If $n = \frac{k(b^2-a^2)}{b^2-ka^2}$ is an integer, then $b^2 - ka^2$ must be a divisor of $k(b^2 - a^2)$. If $b^2 - ka^2 = \pm 1$, then it follows immediately that it is a divisor $k(k-1)$. So suppose that the denominator is some integer d such that $|d| > 1$. Then $b^2 = ka^2 + d$ and

$$n = \frac{k((k-1)a^2}{d} + k \tag{3.3}$$

Since n is an integer, $d|k(k-1)a^2$. Since $b^2 = ka^2 + d$, if $\gcd(d, a^2) > 1$, then $\gcd(a^2, b^2) > 1$ and so $\gcd(a,b) > 1$, a contradiction. Since d and a^2 are coprime, d must divide $k(k-1)$. ∎

The number n must be at least k, so by Equation 3.3, $\frac{k(k-1)a^2}{d}$ must be non-negative. If d is positive, then this is true for any feasible values of k and a. If d is negative, then this is true only when $a = 0$, but there are no solutions $(a,b) = (0,b)$ to $b^2 - ka^2 = d$ when d is negative. This gives the following.

Theorem 3.8. *If the incidence graph of $K_n^{(k)}$, $n \geq k$, is normalized rational, then $n = \frac{k(b^2-a^2)}{b^2-ka^2}$ for some integers a and b such that $\gcd(a,b) = 1$ and $b^2 - ka^2 = d$ for some positive divisor d of $k(k-1)$.*

Thus, to determine when the incidence graph of $K_n^{(k)}$ is normalized rational for a given k, we only need to find the integer solutions (a,b), if they exist, to the generalized

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Pell equation $b^2 - ka^2 = d$ for each positive divisor d of $k(k-1)$. If solutions exist, then the corresponding values of n are integral. For example, when $k = 3$, $k(k-1) = 6$ so $d \in \{1, 2, 3, 6\}$. The equation $a^2 - 3b^2 = d$ has integer solutions when $d = 1$ or $d = 6$, but not when $d = 2$ or $d = 3$.

If $d = 1$ or $d = k(k-1)$, then $(a, b) = (0, 1)$ and $(a, b) = (1, k)$ are solutions, respectively, to $b^2 - ka^2 = d$. Unfortunately, there does not appear to be a simple set of criteria for determining whether or not the equation has integer solutions for a particular k and d where $d \notin \{1, k(k-1)\}$. However, the algorithm given in [11] can be used to determine whether or not solutions to the equation exist and find all integer solutions in the case that they do.

Note that if k is a perfect square, say $k = \sigma^2$, then the equation becomes $b^2 - ka^2 = (b - \sigma a)(b + \sigma a) = d$. Thus, in this case, it is only necessary to solve the system

$$\begin{aligned} b - \sigma a &= d_1 \\ b + \sigma a &= d_2 \end{aligned}$$

for each factorization $d = d_1 d_2$ into integers. Therefore, there are only finitely many integer solutions when k is a square. It is shown in [11] that if k is not a square and there is at least 1 solution (a, b), then infinitely many solutions can be found from (a, b) by iterative application of a linear transformation. Otherwise, there are 0 solutions.

3.2.2. Incidence graphs of Steiner systems. Another class of k-uniform r-regular hypergraphs comes from t-(n, k, λ) designs (see, for example, [3, Ch. 1]). Such a design is equivalent to a hypergraph whose vertex set and hyperedge set are the point set and block set, respectively, of the design. A *Steiner system* $S(n, k, t)$ is a t-$(n, k, 1)$ design. We will consider *Steiner triple systems* $S(n, 3, 2)$ and *Steiner quadruple systems* $S(n, 4, 3)$.

A Steiner triple system $S(n, 3, 2)$ is equivalent to a 3-uniform r-regular hypergraph H where $r = (n-1)/2$. Furthermore, any two points are contained in a unique hyperedge. Thus, if M is the incidence matrix of the hypergraph, then $MM^T = \frac{n-1}{2} I + (J - I) = \frac{n-3}{2} I + J$. This matrix has eigenvalues $\frac{n-3}{2}$ and $rk = \frac{3n-3}{2}$. Therefore, the eigenvalues of $A(I(H))$ are

$$\pm\sqrt{\frac{n-3}{2}} \text{ and } \pm\sqrt{rk} = \pm\sqrt{\frac{3n-3}{2}}$$

Therefore, $I(H)$ is normalized rational if and only if

$$\sqrt{\frac{n-3}{2}} = \frac{a}{b}\sqrt{\frac{3n-3}{2}}$$

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for some $a, b \in \mathbb{Z}$ for which we may assume $\gcd(a, b) = 1$. Solving for v in terms of a and b gives

$$n = \frac{3(b^2 - a^2)}{b^2 - 3a^2} \tag{3.4}$$

Thus, $I(H)$ is normalized rational if and only if the expression on the right hand side of 3.4 is an integer. This gives the following.

Lemma 3.9. *Let H be the hypergraph obtained from an $S(n, 3, 2)$. Then the incidence graph of $I(H)$ is normalized rational if and only if*

$$n = \frac{3(b^2 - a^2)}{b^2 - 3a^2}$$

for some integers a and b such that $\gcd(a, b) = 1$.

The right hand side of (3.4) is the same as the formula for n in Theorem 3.8 with $k = 3$. Using the same argument, we can show that $b^2 - 3a^2$ must be equal to some positive divisor of 6, and if it is, then there is a corresponding n. Note that an $S(n, 3, 2)$ exists if and only if $n \equiv 1$ or $3 \mod 6$, so n must satisfy this further condition for a corresponding normalized rational graph to exist. If $n = 9$ or $n = 99$, for example, then the incidence graph of an $S(n, 3, 2)$ is normalized rational.

A Steiner quadruple system is an $S(n, 4, 3)$. It is equivalent to a 4-uniform r-regular hypergraph where $r = \frac{(n-1)(n-2)}{6}$. Two distinct points are contained in $\frac{n-2}{2}$ blocks. Therefore, if M is is the incidence matrix of the hypergraph, then $MM^T = \frac{(n-1)(n-2)}{6} I + \frac{n-2}{2}(J - I)$. It follows that the eigenvalues of $A(I(H))$ are

$$\pm\sqrt{\frac{(n-4)(n-2)}{6}}, \pm\sqrt{rk} = \pm\sqrt{\frac{4(n-4)(n-2)}{6}}$$

Thus, $I(H)$ is normalized rational if and only if

$$\sqrt{\frac{(n-4)(n-2)}{6}} = \frac{a}{b}\sqrt{\frac{4(n-4)(n-2)}{6}}$$

for some integers a and b, for which we may assume that $\gcd(a, b) = 1$. Solving this equation for n in terms of a and b gives

$$n = \frac{4(b^2 - a^2)}{b^2 - 4a^2}$$

The right hand side of this equation is the same as in Theorem 3.8 with $k = 4$. As in the theorem, $b^2 - 4a^2$ must be a positive divisor of $k(k-1) = 12$. Since 4 is a square, there are only finitely many solutions, at most 1 for each factorization of a divisor

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of 12 into two positive integers. Therefore, if $I(H)$ is normalized rational for some integers a and b, then there must be integer solutions to

$$a - 2b = d_1$$
$$a + 2b = d_2$$

where $d_1 d_2 | 12$ for some $d_1, d_2 \in \mathbb{N}$. An exhaustive search over all possibilities yields integer solutions $(1, 4)$ when $d_1 = 2$ and $d_2 = 6$, $(-1, 4)$ when $d_1 = 6$ and $d_2 = 2$, $(0, 2)$ when $d_1 = d_2 = 2$, and $(0, 1)$ when $d_1 = d_2 = 1$. The corresponding values for n are 5, 5, 4, and 4. But a Steiner system exists only when $n \equiv 2$ or $4 \mod 6$. So the only $S(n, 4, 3)$ whose incidence graph is normalized rational is the trivial case of $n = 4$, consisting of one block (or hyperedge) containing all 4 vertices.

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