

ON THE MAX-ALGEBRAIC CORE OF A NONNEGATIVE MATRIX*

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Abstract. The max-algebraic core of a nonnegative matrix is the intersection of column spans of all max-algebraic matrix powers. This paper investigates the action of a matrix on its core. Being closely related to ultimate periodicity of matrix powers, this study leads to new modifications and geometric characterizations of robust, orbit periodic and weakly stable matrices.

Key words. Max algebra, Ultimate periodicity, Perron-Frobenius, Eigenspace, Core.

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1. Introduction. The concept of matrix core was introduced by Pullman [26] in the case of nonnegative algebra. Defined as the intersection of nonnegative column spans of matrix powers

$$\text{core}(A) = \bigcap_{t=1}^{\infty} \text{span}(A^t),$$

it was used to derive an alternative geometric proof of the Perron-Frobenius theorem. Tam and Schneider [31] developed an extension of Pullman's results to cone-preserving maps. The core is also related to the limiting sets of Markov chains investigated by Sierksma [29], or the limiting sets of matrix groups described, e.g., by Seneta [27] and Hartfiel [19].

In a previous work [11], we investigated two cores of a nonnegative matrix: The “traditional” core defined by Pullman, and the “new” core defined in max algebra. Generalizing an argument of Pullman we gave a simultaneous proof that in both algebras, $\text{core}(A)$ equals the Minkowski sum of all eigencones (i.e., cones of nonnegative eigenvectors) of matrix powers. We also proved the periodicity of the sequence of eigencones of matrix powers and described extremals of the core, in both algebras. These results can be seen in the framework of unification and parallel development

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of nonnegative linear algebra and max algebra. To this end, let us also mention the previous works on generators, extremals and bases of max cones [8], and Z-matrix equations [10].

In the present work, we are interested in the action of a nonnegative matrix on its max-algebraic core. We observe that while this action is bijective in the usual (nonnegative) algebra, it is in general only surjective in the case of max algebra. Further, Butkovič et al. previously investigated certain classes of matrices related to the ultimate periodicity, in particular, robust matrices [7] and weakly stable matrices [9]. Following the same ideas, Sergeev and Schneider [28] considered orbit periodic matrices. Here we introduce the core restrictions of these notions. We show that this leads to a more geometric and in a sense, more transparent characterisation of robust and orbit periodic matrices. In the case of weakly stable matrices, the core restriction does not yield any alternative characterization, but it gives a necessary and sufficient condition for a matrix to be bijective on its core, in max algebra.

The paper is organized as follows. Section 2 is occupied with preliminaries on max-algebraic cyclicity, Frobenius normal form and reducible spectral theory. Here we also give a review of the results on access relations between classes of Frobenius normal form of matrix powers, formulated in [11]. In Section 3, we review the equality between the core and Minkowski sum of eigencones of matrix powers obtained in [11] for the case of max algebra, and introduce the classes of matrices mentioned above. We add here some new observations, on the core of integer matrices in max-plus algebra, and on the core in max-min algebra. Section 4 contains our main results on core restrictions of robustness, orbit periodicity and weak stability. Here we develop a geometric characterization of robust and orbit periodic matrices in terms of the core, and establish an equivalence between bijectivity and the core restriction of weak stability.

2. Max-algebraic cyclicity and spectral theory.

2.1. Max algebra: Main objects. By max algebra we understand the set of nonnegative numbers \mathbb{R}_+ where the role of addition is played by taking maximum of two numbers: $a \oplus b := \max(a, b)$, and the multiplication is as in the usual arithmetics. This is carried over to matrices and vectors like in the usual linear algebra so that for two matrices $A = (a_{ij})$ and $B = (b_{ij})$ of appropriate sizes, $(A \oplus B)_{ij} = a_{ij} \oplus b_{ij}$ and $(A \otimes B)_{ik} = \bigoplus_j a_{ij} b_{kj}$. Notation A^k will stand for the k th max-algebraic power.

With a matrix $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$ we associate a weighted (di)graph $\mathcal{G}(A)$ with the set of nodes $N = \{1, \dots, n\}$ and set of edges $E \subseteq N \times N$ containing a pair (i, j) if and only if $a_{ij} \neq 0$; the weight of an edge $(i, j) \in E$ is defined to be $w(i, j) := a_{ij}$. A graph with just one node and no edge will be called *trivial*.

A path P in $\mathcal{G}(A)$ is a sequence of nodes i_0, i_1, \dots, i_t such that each pair $(i_0, i_1), (i_1, i_2), \dots, (i_{t-1}, i_t)$ is an edge in $\mathcal{G}(A)$. It has *length* $l(P) := t$ and *weight* $w(P) := w(i_0, i_1) \cdot w(i_1, i_2) \cdots w(i_{t-1}, i_t)$, and is called an $i - j$ path if $i_0 = i$ and $i_t = j$. P is called a *cycle* if $i_0 = i_t$, and a cycle is called *elementary* if all nodes of the cycle are distinct.

$A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$ is irreducible if $\mathcal{G}(A)$ is trivial or for any $i, j \in \{1, \dots, n\}$ there is an $i - j$ path. Otherwise A is reducible.

A set $V \subseteq \mathbb{R}_+^n$ will be called a *max cone* if 1) $\alpha v \in V$ for all $v \in V$ and $\alpha \in \mathbb{R}_+$, 2) $u \oplus v \in V$ for $u, v \in V$. Max cones are also known as idempotent semimodules, see [22, 23]. A max cone V is said to be *generated* by $S \subseteq \mathbb{R}_+^n$ if each $v \in V$ can be represented as a max combination $v = \bigoplus_{x \in S} \alpha_x x$ where only finitely many (nonnegative) α_x are different from zero. When V is generated (we also say “spanned”) by S , this is denoted $V = \text{span}(S)$. When V is generated by the columns of a matrix A , this is denoted $V = \text{span}(A)$. Max cones are max-algebraic analogues of convex cones.

A vector z in a max cone $V \subseteq \mathbb{R}_+^n$ is called an *extremal* if $z = u \oplus v$ and $u, v \in V$ imply $z = u$ or $z = v$. Any finitely generated max cone is generated by its extremals, see Wagner [32] and [8, 17] for recent extensions.

The *maximum cycle geometric mean* (m.c.g.m.) of A is defined by

$$(2.1) \quad \rho(A) = \max\{w(C)^{1/l(C)}; C \text{ is a cycle in } \mathcal{G}(A)\}$$

The *critical graph* of A , denoted by $\mathcal{C}(A)$, consists of all nodes and edges belonging to the cycles which attain the maximum in (2.1). The set of such nodes will be called *critical* and denoted N_c ; the set of such edges will be called *critical* and denoted E_c . Observe that the critical graph, defined as above, consists of several strongly connected subgraphs of $\mathcal{G}(A)$. Maximal such subgraphs are the *strongly connected components* of $\mathcal{C}(A)$.

If for $A \in \mathbb{R}_+^{n \times n}$ we have $A \otimes x = \rho x$ with $\rho \in \mathbb{R}_+$ and a nonzero $x \in \mathbb{R}_+^n$, then ρ is a *max(-algebraic) eigenvalue* and x is a *max(-algebraic) eigenvector* associated with ρ . The set of max eigenvectors x associated with ρ , with the zero vector adjoined to it, is a max cone further denoted by $V(A, \rho)$. It is called the *eigencone* of A associated with ρ .

In general, a matrix $A \in \mathbb{R}_+^{n \times n}$ may have several max eigenvalues. The greatest max eigenvalue is equal to $\rho(A)$ (see [2, 6, 14, 21]), and it is called the *principal eigenvalue*. The corresponding eigencone is called the *principal eigencone*. It is also known that if A is irreducible, then $\rho(A)$ is the only eigenvalue, which we call the *max(-algebraic) Perron root* of A .

There is an explicit description of $V(A, \rho(A))$, see Theorem 2.5 below. It uses the *Kleene star*

$$(2.2) \quad A^* = I \oplus A \oplus A^2 \oplus A^3 \oplus \cdots,$$

where I denotes the identity matrix. Series (2.2) converges if and only if $\rho(A) \leq 1$, in which case $A^* = I \oplus A \oplus \cdots \oplus A^{n-1}$. Note that if $\rho(A) \neq 0$, then $\rho(A/\rho(A)) = 1$, and so $(A/\rho(A))^*$ always converges.

The *path interpretation* of max-algebraic matrix powers $A^{\otimes l}$ is that each entry $a_{ij}^{\otimes l}$ is equal to the greatest weight of $i - j$ paths with length l . Consequently, for $i \neq j$, the entry a_{ij}^* of A^* is equal to the greatest weight of $i - j$ paths (with no length restrictions).

For a strongly connected graph \mathcal{G} define its *cyclicity* σ as the gcd of the lengths of all elementary cycles and the cyclicity of a trivial graph to be 1. For a (general) graph containing several maximal strongly connected components (such as the critical graph $\mathcal{C}(A)$), cyclicity is defined as the lcm of the cyclicities of the strongly connected components. A graph with cyclicity 1 is called *primitive*.

The following result demonstrates importance of cyclicity of critical graph in max algebra.

THEOREM 2.1 (Cyclicity Theorem, Cohen et al. [13]). *Let $A \in \mathbb{R}_+^{n \times n}$ be irreducible and let σ be the cyclicity of $\mathcal{C}(A)$. Then σ is the smallest p such that there exists $T(A)$ with $A^{t+p} = \rho^p(A)A^t$ for all $t \geq T(A)$.*

This result is closely related to the theory of graph exponents as presented, for instance, in Bruualdi-Ryser [4]. In particular, it exploits the following number-theoretic lemma due to Schur and Frobenius:

LEMMA 2.2 ([4], Lemma 3.4.2). *Let n_1, \dots, n_m be integers and let $k = \gcd(n_1, \dots, n_m)$. Then there exists a number T such that for all integers l with $kl \geq T$, we have $kl = t_1 n_1 + \cdots + t_m n_m$ for some $t_1, \dots, t_m \geq 0$.*

2.2. Ultimate periodicity and immediate periodicity. Theorem 2.1 also shows that the periodicity of sequences is crucial in max algebra. We will need the following formal definitions.

A sequence of scalars/vectors/matrices $\{X_t\}_{t \geq 1}$ over \mathbb{R}_+ is called *ultimately periodic with the period τ growth rate $r \in \mathbb{R}_+$* if τ is the smallest integer $p \geq 1$ such that there exists T with $X_{t+p} = r^p X_t$ for all $t \geq T$.

A sequence of max cones $\{V_t\}_{t \geq 1}$ in \mathbb{R}_+^n is called *ultimately periodic with the period σ* if τ is the smallest integer $p \geq 1$ such that there exists T with $V_{t+p} = V_t$ for all $t \geq T$.

An ultimately periodic sequence of scalars/vectors/matrices $\{X_t\}_{t \geq 1}$ over \mathbb{R}_+ or max cones $\{V_t\}_{t \geq 1}$ in \mathbb{R}_+^n is called *immediately periodic* or just *periodic*, if T can be set to 1 in the definitions above.

In terms of the ultimate periodicity, Theorem 2.1 can be formulated as follows: For any irreducible nonnegative matrix $A \in \mathbb{R}_+^{n \times n}$, the sequence of matrix powers $\{A^t\}_{t \geq 1}$ is ultimately periodic with the period equal to the cyclicity of critical graph, and with the growth rate equal to the maximal cycle geometric mean of A .

Strictly speaking, the ultimate periodicity is also characterized by the smallest possible parameter T in the above definitions, which is called the *defect* of sequence or its *periodicity threshold*. The reader is referred to, e.g., Akian, Gaubert, Walsh [1] and Hartman, Arguelles [20] for various bounds and algorithms estimating the value of $T(A)$ in Theorem 2.1.

We also note that for a general reducible matrix $A \in \mathbb{R}_+^{n \times n}$, not all the sequences $\{a_{ij}^{(t)}\}_{t \geq 1}$ for $i, j \in \{1, \dots, n\}$, are ultimately periodic in the sense of the definition given above. Such sequences can be decomposed into ultimately periodic subsequences with different growth rates, and the reader is referred to De Schutter [15], Gavalec [18] and Molnárová [24] for more details. However, we do not need such general results and details in the present paper.

2.3. Other idempotent algebras. Max-plus algebra \mathbb{R}_{\max} is defined over $\mathbb{R} \cup \{-\infty\}$, the set of real numbers completed with the “least” element $-\infty$. It is equipped with operations $a \oplus b := \max(a, b)$ and $a \otimes b := a + b$, where it is assumed that $a \otimes -\infty = -\infty \otimes a = -\infty$ and $a \oplus -\infty = -\infty \oplus a = a$. When applying the exponential mapping $x \mapsto e^x$, the max-plus algebra is seen to be isomorphic to the max(-times) algebra.

In the present paper, we still choose to work with the max-times version for its theoretical convenience. The development of max-plus linear algebra is very similar to that of max(-times) linear algebra.

However, we will touch upon \mathbb{Z}_{\max} : Extended set of integers $\mathbb{Z} \cup \{-\infty\}$ equipped with the same max-plus arithmetics. This is a subsemiring of \mathbb{R}_{\max} , whose “dual” \mathbb{Z}_{\min} was treated in a work of Simon [30], with motivations in computer science and automata theory.

We will also touch upon the *max-min algebra*: The interval $[0, 1]$ equipped with “addition” $a \oplus b := \max(a, b)$ and “multiplication” $a \otimes b := \min(a, b)$. For more information on the linear algebra over this semiring, also known as *fuzzy linear algebra*, the reader is referred to, e.g., Gavalec [18]. The only observation that we will later use, is that unlike in max algebra, the “multiplication” defined as the minimum of

two numbers a, b does not result in any other numbers than a or b . In particular, it implies that the sequence $\{A^t\}_{t \geq 1}$ is ultimately periodic for each matrix A with the growth rate 1.

2.4. Frobenius normal form. Every matrix $A = (a_{ij}) \in \mathbb{R}_+^{n \times n}$ can be transformed by simultaneous permutations of the rows and columns to a *Frobenius Normal Form* (FNF) [3, 4]

$$(2.3) \quad \begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{pmatrix},$$

where A_{11}, \dots, A_{rr} are irreducible square submatrices of A , corresponding to the partition $N_1 \cup N_2 \cup \dots \cup N_r = N$. Generally, A_{KL} denotes the submatrix of A extracted from rows with indices in $K \subseteq N$ and columns with indices in $L \subseteq N$, and $A_{\mu\nu}$ is a shortcut for $A_{N_\mu N_\nu}$.

Consider the graph associated with $A_{\mu\mu}$. It is the same as the graph induced by N_μ : $\mathcal{G}(A_{\mu\mu}) = (N_\mu, E \cap (N_\mu \times N_\mu))$. It follows that each of the graphs $\mathcal{G}(A_{\mu\mu})$ ($\mu = 1, \dots, r$) is strongly connected and an arc from N_μ to N_ν in $\mathcal{G}(A)$ may exist only if $\mu \geq \nu$.

If A is in the Frobenius Normal Form (2.3) then the *reduced graph*, denoted $R(A)$, is the (di)graph whose nodes correspond to N_μ , for $\mu = 1, \dots, r$, and the set of arcs is $\{(\mu, \nu); (\exists k \in N_\mu)(\exists \ell \in N_\nu) a_{k\ell} > 0\}$.

The nodes of $R(A)$ are *marked* by the corresponding max-algebraic eigenvalues (Perron roots) denoted by $\rho_\mu := \rho(A_{\mu\mu})$. These nodes will be called *classes* of A . We naturally attribute to a class μ also the graph $\mathcal{G}(A_{\mu\mu})$ with set of nodes N_μ and cyclicity σ_μ .

We say that a class μ is *trivial* if $A_{\mu\mu}$ consists of a single diagonal zero entry, i.e., when $\mathcal{G}(A_{\mu\mu})$ is trivial. Class μ *accesses* class ν , denoted $\mu \rightarrow \nu$, if there exists a $\mu - \nu$ path in $R(A)$. A class is called *initial*, resp., *final*, if it is not accessed by, resp., if it does not access, any other class. Node i accesses class ν , denoted by $i \rightarrow \nu$, if i belongs to a class μ such that $\mu \rightarrow \nu$.

Simultaneous permutations of the rows and columns of A are equivalent to calculating $P^{-1}AP$, where P is a permutation matrix. Such transformations do not change the eigenvalues, and the eigenvectors before and after such a transformation only differ by the order of their components. Hence, we will assume without loss of generality that A is in FNF (2.3).

2.5. Reducible spectral theory. A class ν of A is called a *spectral class* of A associated with eigenvalue $\rho \neq 0$, or sometimes (A, ρ) -spectral class for short, if

$$\rho_\nu = \rho, \text{ and } \mu \rightarrow \nu \text{ implies } \rho_\mu \leq \rho_\nu.$$

Note that there may be several spectral classes associated with the same eigenvalue.

Denote by $\Lambda(A)$, the set of **nonzero** eigenvalues of $A \in \mathbb{R}_+^{n \times n}$. The following description is standard.

THEOREM 2.3 ([6] Th. 4.5.4, [16]). *Let $A \in \mathbb{R}_+^{n \times n}$. Then $\Lambda(A) = \{\rho_\nu \neq 0; \nu \text{ is spectral}\} = \{\rho_\nu \neq 0; \forall \mu, \mu \rightarrow \nu \text{ implies } \rho_\mu \leq \rho_\nu\}$.*

For each $\rho \in \Lambda(A)$ define

$$A_\rho := \rho^{-1} \begin{pmatrix} 0 & 0 \\ 0 & A_{M_\rho M_\rho} \end{pmatrix}, \text{ where} \\ M_\rho := \{i; i \rightarrow \nu, \nu \text{ is } (A, \rho)\text{-spectral}\}.$$

The next proposition allows us to reduce a general eigencone to the case of a principal eigencone. We will assume an appropriate ordering of indices (i.e., nodes of the corresponding graph). Such assumptions will sometimes be made also in the sequel, without special mention of them.

PROPOSITION 2.4 ([6, 16]). *For $A \in \mathbb{R}_+^{n \times n}$ and each $\rho \in \Lambda(A)$, we have $V(A, \rho) = V(A_\rho, 1)$, where 1 is the principal eigenvalue of A_ρ .*

Using Proposition 2.4, we define the *critical graph associated with $\rho \in \Lambda(A)$* as the critical graph of A_ρ . The strongly connected components of critical graphs associated with ρ , for all $\rho \in \Lambda(A)$, will be called the *critical components of A* . These components will be (similarly as the classes of FNF) denoted by $\tilde{\mu}$, with the node set $N_{\tilde{\mu}}$.

We now describe the principal eigencones in max algebra. By means of Proposition 2.4, this description can be obviously extended to the case of general eigencones. Recall that support of $x \in \mathbb{R}_+^n$, denoted by $\text{supp } x$, is the set of indices i with $x_i \neq 0$.

THEOREM 2.5 ([6] Th. 4.3.5, [11] Th. 2.11, [16]). *Let $A \in \mathbb{R}_+^{n \times n}$ have $\rho(A) = 1$. Then*

- (i) *Each component $\tilde{\mu}$ of $\mathcal{C}(A)$ corresponds to an eigenvector $x_{\tilde{\mu}}$ defined as one of the columns A_i^* with $i \in N_{\tilde{\mu}}$, all columns with $i \in N_{\tilde{\mu}}$ being multiples of each other.*
- (i') *Each component $\tilde{\mu}$ of $\mathcal{C}(A)$ is contained in a (spectral) class μ with $\rho_\mu = 1$, and the support of each $x_{\tilde{\mu}}$ of (i) consists of all indices in all classes that have access to μ .*

- (ii) $V(A, 1)$ is generated by $x_{\tilde{\mu}}$ of (i), for $\tilde{\mu}$ ranging over all components of $\mathcal{C}(A)$.
- (iii) $x_{\tilde{\mu}}$ of (i) are extremals in $V(A, 1)$. (Moreover, $x_{\tilde{\mu}}$ are strongly linearly independent in the sense of [5].)

2.6. Access relations in matrix powers. In [11], it was demonstrated that access relations and spectral classes of all matrix powers are essentially the same, and that the case of an arbitrary eigenvalue reduces to the case of the principal eigenvalue. We start from the following basic result.

LEMMA 2.6 ([11], Lemma 5.1). *Let A be irreducible with the (unique) eigenvalue ρ , let $\mathcal{G}(A)$ have cyclicity σ and t be a positive integer. Then, A^t is a direct sum of $\gcd(t, \sigma)$ irreducible blocks with eigenvalues ρ^t , and A^t does not have eigenvalues other than ρ^t . The cyclicity of each block is $\sigma/\gcd(t, \sigma)$. In particular, all blocks of A^σ are primitive.*

Recall that each class μ of A corresponds to an irreducible submatrix $A_{\mu\mu}$. It is easy to see that $(A^t)_{\mu\mu} = (A_{\mu\mu})^t$ for any positive integer t . Suppose that the cyclicity of $\mathcal{G}(A_{\mu\mu})$ is σ . Applying Lemma 2.6 to $A_{\mu\mu}$ we see that μ gives rise to $\gcd(t, \sigma)$ classes in A^t , which are said to be *derived* from their common *ancestor* μ . The classes of A^t and A^l derived from the common ancestor will be called *related*. Note that this is an equivalence relation on the set of classes of all powers of A .

It can be checked that the same notions can be defined for the components of critical graphs, see [11].

Let us recall the following results on the similarity of access relations in matrix powers.

LEMMA 2.7 ([11], Lemma 5.3). *For all $t, l \geq 1$ and $\rho > 0$, an index $i \in \{1, \dots, n\}$ accesses (resp., is accessed by) a class with Perron root ρ^t in A^t if and only if it accesses (resp., is accessed by) a related class with Perron root ρ^l in A^l .*

A similar result holds for the strongly connected components of the critical graphs of matrix powers, see [11], Theorem 3.3. We will only need the following observation, which we formulate for the *critical matrix* $A_C = (a_C)_{ij}$ defined by

$$(a_C)_{ij} = \begin{cases} 1, & \text{if } (i, j) \in E_c, \\ 0, & \text{otherwise.} \end{cases}$$

LEMMA 2.8. *Let $A \in \mathbb{R}_+^{n \times n}$. Then $(A_C)^t = (A^t)_C$.*

This observation will allow us to apply Lemma 2.6 to the critical graphs $\mathcal{C}(A^t)$ of matrix powers, that is, to the critical matrices $(A^t)_C$.

All eigenvalues and spectral classes of matrix powers are derived from those of A .

THEOREM 2.9 ([11], Th. 5.4, Coro. 5.5). *Let $A \in \mathbb{R}_+^{n \times n}$ and $t \geq 1$.*

- (i) $\Lambda(A^t) = \{\rho^t; \rho \in \Lambda(A)\}$.
- (ii) *For each spectral class μ of A with cyclicity σ there are $\gcd(t, \sigma)$ spectral classes of A^t derived from it. Conversely, each spectral class of A^t is derived from a spectral class of A .*

As in the case of eigencones of a matrix, when working with $V(A^t, \rho^t)$ we can assume that $\rho = 1$ is the principal eigenvalue of A , and hence of all A^t .

THEOREM 2.10 ([11], Th. 5.7). *Let $A \in \mathbb{R}_+^{n \times n}$, $t \geq 1$ and $\rho \in \Lambda(A)$.*

- (i) $(A^t)_{M_\rho M_\rho} = (\rho^t (A_\rho)^t)_{M_\rho M_\rho}$.
- (ii) $V(A^t, \rho^t) = V((A_\rho)^t, 1)$.

3. Core and eigenvectors. In this section, we recall the main results of [11] on the matrix core, and its finite stabilization in special cases. We also consider the case of integer entries and, briefly, the core in max-min algebra and other algebras where every sequence of matrix powers $\{A^t\}_{t \geq 1}$ is ultimately periodic.

3.1. Core in general. The purpose of [11], Section 4 was to show, by adapting an argument of Pullman [26] to max algebra, that the max-algebraic core can be also represented as the (Minkowski) sum of the eigencones of matrix powers, that is,

$$(3.1) \quad \text{core}(A) = \bigoplus_{t \geq 1, \rho \in \Lambda(A)} V(A^t, \rho^t).$$

By definition, on the right hand side of (3.1) we have a max cone consisting of all combinations $\bigoplus_t \alpha_t y^{(t)}$ with $y^{(t)} \in V(A^t, \rho^t)$ and finite number of nonzero α_t .

It was also shown in [11], Section 7 that the sequence of eigencones of matrix powers is periodic. To describe this periodicity, the following notation was introduced:

1. σ_ρ : Cyclicity of the critical graph associated with an eigenvalue $\rho \in \Lambda(A)$,
2. σ_Λ : The lcm of all σ_ρ , over $\rho \in \Lambda(A)$,

and the Minkowski sums of all eigencones of matrix powers were considered:

$$V^\Sigma(A^t) := \bigoplus_{\rho \in \Lambda(A)} V(A^t, \rho^t).$$

The periodicity can be described as follows.

THEOREM 3.1 ([11], Main Theorem 2). *Let $A \in \mathbb{R}_+^{n \times n}$ and $\rho \in \Lambda(A)$. Then the sequence of max cones $\{V(A^t, \rho^t)\}_{t \geq 1}$ is immediately periodic with the period σ_ρ , and $V(A^t, \rho^t) \subseteq V(A^{\sigma_\rho}, \rho^{\sigma_\rho})$ for all t .*

THEOREM 3.2 ([11], Main Theorem 2). *Let $A \in \mathbb{R}_+^{n \times n}$. Then the sequence of max cones $\{V^\Sigma(A^t)\}_{t \geq 1}$ is immediately periodic with the period σ_Λ , and $V^\Sigma(A^t) \subseteq V^\Sigma(A^{\sigma_\Lambda})$ for all t .*

In particular, $V(A^t, \rho^t) \subseteq V(A^{\sigma_\Lambda}, \rho^{\sigma_\Lambda})$ for all $\rho \in \Lambda(A)$, and hence,

$$\bigoplus_{t \geq 1, \rho \in \Lambda(A)} V(A^t, \rho^t) = \bigoplus_{\rho \in \Lambda(A)} V(A^{\sigma_\Lambda}, \rho^{\sigma_\Lambda}) = V^\Sigma(A^{\sigma_\Lambda}).$$

The central result of [11] can be formulated as follows:

THEOREM 3.3 ([11], Main Theorem 1). *Let $A \in \mathbb{R}_+^{n \times n}$. Then $\text{core}(A) = \bigoplus_{\rho \in \Lambda(A)} V(A^{\sigma_\rho}, \rho^{\sigma_\rho}) = V^\Sigma(A^{\sigma_\Lambda})$.*

Note that the inclusion $\bigoplus_{\rho \in \Lambda(A)} V(A^{\sigma_\rho}, \rho^{\sigma_\rho}) \subseteq \text{core}(A)$ holds since each vector in $V(A^{\sigma_\rho}, \rho^{\sigma_\rho})$ is in $\text{span}(A^{t\sigma_\rho})$ for all $t \geq 1$, hence in $\text{core}(A)$.

The proof of the reverse inclusion relied on the facts collected in Lemma 3.4 below. Here, a vector $v \in \mathbb{R}_+^n$ is called *scaled* if $\|v\| := \max_{i=1}^n v_i = 1$, and A induces a mapping on the scaled vectors of $\text{core}(A)$ by $v \mapsto Av/\|Av\|$.

LEMMA 3.4. *Let $A \in \mathbb{R}_+^{n \times n}$, then*

- (i) *core(A) is generated by no more than n vectors,*
- (ii) *the mapping induced by A on core(A) is a surjection,*
- (iii) *the mapping induced by A on the scaled extremals of core(A) is a permutation (i.e., a bijection).*

Note that in **nonnegative linear algebra** A is always bijective on its core. This follows since the extreme generators of the nonnegative core are linearly independent in the usual linear algebra, so A just permutes and rescales the elements of basis of the linear-algebraic span of the nonnegative core.

In **max algebra**, A is in general non-bijective on $\text{core}(A)$, and we later describe when A is bijective. Moreover, *finite stabilization of the core* (when $\text{core}(A) = \text{span}(A^t)$ for all large enough t) is often observed. Note that in this case, Theorem 3.3 follows immediately.

3.2. Ultimate periodicity and finite stabilization. In max algebra, there are wide classes of matrices $A \in \mathbb{R}_+^{n \times n}$ where we have finite stabilization of the core. We list some of them.

- \mathcal{S}_1 : **Irreducible matrices.**
- \mathcal{S}_2 : **Ultimately periodic matrices.** This is when we have $A^{t+\sigma} = \rho^\sigma A^t$ for all sufficiently large t , with $\rho = \rho(A)$. As shown in [25], this happens if and only if the Perron roots of all nontrivial classes of A equal $\rho(A)$.
- \mathcal{S}_3 : **Robust matrices.** For any nonzero vector $x \in \mathbb{R}_+^n$, there exists $t \geq 1$ such that $A^t x$ is an eigenvector of A . This implies that the whole remaining part of the orbit of A , that is, $\{A^s x\}_{s \geq t}$, consists of multiples of that eigenvector. The notion of robustness was introduced and studied in [7], and it will be revisited below.
- \mathcal{S}_4 : **Orbit periodic matrices:** For any nonzero vector $x \in \mathbb{R}_+^n$, there exists $t \geq 1$ such that $A^t x$ is an eigenvector of A^σ , implying that the remaining part of the orbit of A is periodic (with some growth rate). See [28], Section 7 and below for characterization.
- \mathcal{S}_5 : **Column periodic matrices.** This is when for any $i = 1, \dots, n$ we have $(A^{t+\sigma})_{\cdot i} = \rho_i^\sigma A_{\cdot i}^t$ for all large enough t and some ρ_i .

Observe that $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{S}_4 \subseteq \mathcal{S}_5$ and $\mathcal{S}_3 \subseteq \mathcal{S}_4$ (see, e.g., [11], Section 4). To see that $\text{span}(A^t) = \text{core}(A)$ for all large enough t in all these cases, observe that in the column periodic case all sequences of columns end up with periodically repeating eigenvectors of A^σ , which implies that $\text{span}(A^t) \subseteq \text{core}(A)$ for all large enough t , and hence also $\text{span}(A^t) = \text{core}(A)$. Thus, finite stabilization of the core occurs in all these classes.

3.3. Integer max-plus case. Let us consider the semiring \mathbb{Z}_{\max} defined on the set $\mathbb{Z} \cup \{-\infty\}$ with $\oplus = \max$ and $\otimes = +$. This is a subsemiring of the max-plus algebra \mathbb{R}_{\max} , which is isomorphic to the max-(times) algebra that we consider. Essentially this is the case of integer matrices in $\mathbb{R}_{\max}^{n \times n}$. For such integer matrices Theorem 3.3 holds by isomorphism, with max-plus arithmetics. We will show that the generators of the core of such matrices are in \mathbb{Z}_{\max}^n , i.e., that they have integer components. This implies that Theorem 3.3 is also true in \mathbb{Z}_{\max} .

THEOREM 3.5. *Let $A \in \mathbb{R}_{\max}^{n \times n}$ have integer entries only. Then $\text{core}(A)$ is generated by integer vectors.*

Proof. We show that $V(A^\sigma, \rho^\sigma)$, for any $\rho \in \Lambda(A)$ and $\sigma = \sigma_\rho$, is generated by integer vectors. For this, we need to show that all $\rho \in \Lambda(A^\sigma)$, that is, the maximal cycle means in the spectral blocks of A , are integer. More precisely, each $\rho \in \Lambda(A^\sigma)$ can be expressed as

$$(3.2) \quad \rho = \frac{a_{i_1 i_2}^{(\sigma)} + \dots + a_{i_m i_1}^{(\sigma)}}{m},$$

where (i_1, \dots, i_m) is any elementary cycle in a strongly connected component of the critical graph corresponding to ρ . Eqn. (3.2) implies that $\rho = K/M$, where M is the

lcm of all lengths of elementary cycles (i_1, \dots, i_m) , and K is a multiple of M/m for any such cycle. All strongly connected components of the critical graph of A^σ are primitive, hence the gcd of all denominators in (3.2), taken over all elementary cycles (i_1, \dots, i_m) in a critical component, is 1.

Next we use the following observation from the elementary number theory: If the numbers m_i , for $i = 1, \dots, \ell$ are coprime (i.e., $\gcd(m_i) = 1$), and $M = \text{lcm}(m_i)$, then $\text{lcm}(M/m_i) = M$.

Since K is a multiple of M/m_i and $\text{lcm}(M/m_i) = M$, we obtain that K is a multiple of M so ρ in (3.2) is integer. Since all $\rho \in \Lambda(A^\sigma)$ are integer, and since extremals of the cone are columns of $(A_\rho^\sigma)^*$ for $\rho \in \Lambda(A)$ (by Theorem 3.3 and Theorem 2.5), the result follows. \square

This result can be also deduced from the Cyclicity Theorem 2.1. By that result, ρ^σ satisfies $A^{t+\sigma} = \rho^\sigma A^t$ for all large enough t , for A irreducible. If A has only integer entries then so do all the powers of A , and we deduce that ρ^σ is integer. In the case of reducible A , this argument is applied to the powers of each submatrix of A that corresponds to a spectral class (observe that the cyclicity of the critical graph in each spectral class divides σ , which is the lcm of all such cyclicities).

3.4. Core in max-min algebra. Max-min algebra gives an example where the ultimate periodicity of A^t takes place for all A , with the growth rate 1 [18]. Indeed, the operations in this semiring are such that all entries of A^t are among the entries of A , so they start to repeat after some time, with a period bounded by n^2 . See [18] for more information. Hence, in this case, we have an analogue of Theorem 3.3, where $\bigoplus_{\rho \in \Lambda(A)} V(A^t, \rho^t)$ must be replaced with $V(A^t, 1)$. So

$$\text{core}(A) = \bigoplus_{k \geq 1} V(A^k, 1)$$

in max-min algebra and in any other algebra where the sequence $\{A^t\}$ is ultimately periodic with growth rate 1 for all A . Note that in max-min algebra each number is an eigenvalue, however the corresponding eigenvectors do not belong to the core unless they can also be associated with the eigenvalue 1.

4. Action of a matrix on the core. This section contains the main results of the present paper. Here we study the mapping induced by a matrix on its core in max algebra.

4.1. Finite stabilization. We first extend Cyclicity Theorem (Theorem 2.1) to the case of a *spectral index*, i.e., an index that belongs to a spectral class with $\rho_\mu = \rho(A)$. Note that in the case when A is irreducible there is only one class, which is spectral, and therefore every index is spectral in this case.

THEOREM 4.1. *Let $A \in \mathbb{R}_+^{n \times n}$, and let j be an index in a spectral class. Then the sequence of columns $\{A_{\cdot j}^t\}_{t \geq 1}$ is ultimately periodic.*

Proof. We can assume without loss of generality that the greatest eigenvalue (i.e., the m.c.g.m.) of A is 1. For now we will also assume that the critical graph $\mathcal{C}(A)$ is primitive, and that the spectral class containing j is associated with the eigenvalue 1, i.e., with the greatest eigenvalue. We will show how to omit these two assumptions in the end of the proof.

Let i be an index with access to j . We show that $a_{ij}^{(t)}$ is constant at all large enough t . Note that when i does not have access to j , we have $a_{ij}^{(t)} = 0$ for all t .

Denote by Π_1 the set of paths that connect i to j via a critical index, with length bounded by $2(n-1)$, and denote by $w(\Pi_1)$ the largest weight of paths in Π_1 . Observe that since j is in a spectral class, there are paths connecting i to j via a critical index. Indeed, there are critical nodes in the strongly connected component of $\mathcal{G}(A)$ containing j . Any such critical node can be connected to j by a path, and back. The resulting cycle can be appended to the access path from i to j , forming a path that connects i to j via a critical index.

Denote by Π_2 the set of paths that connect i to j and do not traverse any critical index, with length bounded by n , and by $w(\Pi_2)$ the biggest weight of paths in Π_2 . Denote by μ the second largest cycle geometric mean in A , taken among the simple cycles only, then $\mu < 1$.

Let us first prove that

$$a_{ij}^{(t)} \leq w(\Pi_1) \oplus w(\Pi_2)\mu^{t-n}$$

for all t .

Case 1. Suppose that $a_{ij}^{(t)}$ is the weight of a path P that connects i to j via a critical index k . Then it can be decomposed as $P_1 \circ P_2$ where P_1 connects i to k and P_2 connects k to j . Delete a cycle (if any) from P_1 , then remove another one (if it still exists), and so on until no cycle exists as a subpath of P_1 . Then repeat the same process with P_2 . By this we obtain a simple path P'_1 connecting i to k and a simple path P'_2 connecting k to j . We conclude that $P'_1 \circ P'_2 \in \Pi_1$, and

$$a_{ij}^{(t)} = w(P) = w(P_1 \circ P_2) \leq w(P'_1 \circ P'_2) \leq w(\Pi_1).$$

Note that since there exist paths connecting i to j through a critical index, the cycle deletion argument above implies that Π_1 is non-empty and $w(\Pi_1) > 0$.

Case 2. Suppose that $a_{ij}^{(t)}$ is the weight of a path \tilde{P} that connects i to j not traversing any critical index. Repeatedly applying the cycle deletion to \tilde{P} we obtain a simple path or a cycle \tilde{P}_2 with length bounded by n and weight bounded by $w(\Pi_2)$. Since we deleted only non-critical cycles with cycle mean not exceeding μ , we obtain

$$a_{ij}^{(t)} = w(\tilde{P}) \leq w(\tilde{P}_2)\mu^{t-l(\tilde{P}_2)} \leq w(\Pi_2)\mu^{t-n}.$$

So in this case we also have $a_{ij}^{(t)} \leq w(\Pi_1) \oplus w(\Pi_2)\mu^{t-n}$ for all t .

We need to show that $a_{ij}^{(t)} = w(\Pi_1)$ for all large enough t . The path attaining $w(\Pi_1)$ (composed of two simple paths) goes through a critical index k , which lies in a primitive critical component. Then by Lemma 2.2 for all large enough t there exist critical cycles of length t passing through k , hence for all large enough t there exist paths connecting i to j via k with weight $w(\Pi_1)$. This shows that $a_{ij}^{(t)} = w(\Pi_1)$ for all large enough t .

As i and j were chosen only with the restriction that j is in a spectral class with eigenvalue 1, and i has access to j , it follows that all columns of A^t with indices in a spectral class with m.c.g.m. 1 are ultimately constant (that is, ultimately periodic with period 1 and growth rate 1), when the critical graph of A is primitive and $\rho(A) = 1$.

Consider the general case, that is, the case of an index j in a general spectral class with general m.c.g.m. ρ and with general cyclicity $\sigma = \sigma_\rho$ of $\mathcal{C}(A_\rho)$. We first use the reduction to the powers of A_ρ , see Theorem 2.10(i), where 1 is the greatest eigenvalue. Raising A to the power σ and using Theorem 2.9 (ii), which shows that the set of indices in spectral classes corresponding to a given eigenvalue does not depend on power, we reduce to the main case considered above. In particular, by Lemmas 2.6 and 2.8 the critical graph $\mathcal{C}((A_\rho)^\sigma)$ is primitive.

We obtain that the sequence of the j th columns of the powers of A^σ is ultimately periodic with period 1 and growth rate ρ^σ , which implies that the sequence of the j th columns of the powers of A is ultimately periodic with growth rate ρ and period at most σ . Indeed, multiplying the equality $A_{\cdot j}^{(l+1)\sigma} = A_{\cdot j}^{l\sigma}$ by A^s for any $s \geq 1$ from the left, we get $A_{\cdot j}^{(l+1)\sigma+s} = A_{\cdot j}^{l\sigma+s}$ for all $s \geq 1$. The proof is complete. \square

We will need the following observation on support of the vectors belonging to the core.

PROPOSITION 4.2. *Let $z \in \text{core}(A)$, then*

- (i) *For each class μ of A^σ , either $N_\mu \subseteq \text{supp } z$, or $N_\mu \cap \text{supp } z = \emptyset$.*
- (ii) *Consider the set of classes μ of A^σ such that $N_\mu \subseteq \text{supp } z$, and let R_z be the subgraph of the marked reduced graph $R(A^\sigma)$ consisting only of such classes and edges between them. Then in R_z , all final classes are spectral.*

Proof. Part (i) holds for each eigenvector of A^σ , and in particular, for every fundamental eigenvector $(A_\rho)^*_i$ where $\rho \in \Lambda(A)$ and i belongs to the corresponding critical graph. Part (ii) also holds for every fundamental eigenvector, since in R_z there is a unique final class μ with $N_\mu \subseteq \text{supp } z$ containing i , and it is spectral.

To deduce both claims for general $z \in \text{core}(A)$, we notice that by Theorem 3.3, $\text{core}(A)$ is generated by such fundamental eigenvectors, and that taking max-linear combinations corresponds to taking the union of supports. \square

PROPOSITION 4.3.

- (i) *If i belongs to a spectral class of A then $A^t_i \in \text{core}(A)$ for all large enough t ;*
- (ii) *If i belongs to a non-trivial non-spectral class of A then $A^t_i \notin \text{core}(A)$ for any t .*

Proof. (i) Follows by Theorem 4.1.

(ii) We use Proposition 4.2. For any t , either the alternative “ $N_\mu \subseteq \text{supp}(A^t_i)$ or $N_\mu \cap \text{supp}(A^t_i) = \emptyset$ ” does not hold for some μ , or it holds but if for $z = A^t_i$ we take the subgraph R_z described in Proposition 4.2 part (ii), then the only final class of this subgraph is the one containing i , and it is not spectral. Thus, we have $A^t_i \notin \text{core}(A)$ for any t . \square

THEOREM 4.4. *Finite stabilization of $\text{core}(A)$ occurs if and only if all nontrivial classes of A are spectral.*

Proof. The “only if” part: If there is a nontrivial non-spectral class then the columns A^t_i with i in that class do not belong to $\text{core}(A)$ by Proposition 4.3.

The “if” part: Assume there are only spectral classes and trivial classes. By Theorem 4.1, there is an integer $T(A)$ such that all sequences $\{A^t_i\}_{t \geq T(A)}$, where i is an index in a spectral class, are periodic. Now take $t \geq T(A) + n$ and let i be an index of a trivial class. Let $\{P_k, k = 1, \dots, r\}$ be all paths connecting certain indices j_k in spectral classes to i with a property that in these paths j_k is not repeated and no index other than j_k can belong to a non-trivial class. Such paths will be called *direct access paths*. Their length is bounded by $n - 1$, for otherwise they contain a cycle, and any cycle belongs to a non-trivial class. So indeed, there is only a finite number of such paths. Let $l(P_k)$, $w(P_k)$ denote the length, resp. the weight, of P_k ,

then for all s for which there are no direct access paths with length t connecting s to i , we can express

$$a_{si}^{(t)} = \bigoplus_{k=1}^r a_{sj_k}^{(t-l(P_k))} w(P_k).$$

Theorem 4.1 implies that $\{a_{sj_k}^{(t-l(P_k))}\}_{t \geq l(P_k)}$ is periodic at $t \geq T(A) + l(P_k)$, so every such sequence is periodic at $t \geq T(A) + n$. There are no direct access paths of length $t \geq n$, and hence of length $t \geq T(A) + n$. Since for $t \geq T(A) + n$, all columns $A_{j_k}^{t-l(P_k)}$ are in $\text{core}(A)$. Using (4.1) we see that also $A_i^t \in \text{core}(A)$ for $t \geq T(A) + n$. \square

4.2. Robustness. We characterize the classes of robust and orbit periodic matrices (\mathcal{S}_3 and \mathcal{S}_4 of Subsection 3.2) by means of the restriction of these properties to the core.

Matrix A is called *core robust* if for any $x \in \text{core}(A) \setminus \{0\}$ (instead of any $x \in \mathbb{R}_+^n \setminus \{0\}$) the orbit $\{A^t x, t \geq 1\}$ hits an eigenvector of A in finite number of steps.

THEOREM 4.5. $A \in \mathbb{R}_+^{n \times n}$ with no zero columns is robust if and only if it is core robust and the core finitely stabilizes.

Proof. Evidently, core robustness is necessary. For the finite stabilization observe that the sequence of columns $\{A_i^t, t \geq 0\}$ is the orbit of the i th unit vector, and hence, if A is robust then this orbit converges to an eigenvector of A . So $A_i^t \in \text{core}(A)$ for all i and sufficiently large t .

The properties are also sufficient. Indeed, any orbit $\{A^t x, t \geq 0\}$, for $x \neq 0$, first gets to a vector of the core by the finite stabilization property, and this vector is nonzero since A does not have zero columns. Then the orbit hits an eigenvector of A in a finite number of steps, by the core robustness. \square

THEOREM 4.6. A is core robust if and only if

- (i) the mapping induced by A on the scaled extremals of its core is identity; i.e., $\text{core}(A)$ is generated by the eigenvectors of A , and
- (ii) $x \in V(A, \rho_1) \setminus \{0\}$, $y \in V(A, \rho_2) \setminus \{0\}$ and $\rho_1 < \rho_2$ imply that $\text{supp}(x) \subseteq \text{supp}(y)$.

Proof. Both (i) and (ii) are necessary. Indeed, the mapping induced by A on the scaled extremals of the core is a bijection, so the orbits of all extreme rays are periodic. If (i) does not hold then there are extreme rays with period greater than 1, and they never stabilize. If (ii) does not hold, then no orbit of any combinations $\alpha x \oplus \beta y$ with $\alpha > 0$ and $\beta > 0$ and x, y as in (ii) ever converges to an eigenvector.

Indeed, we have

$$A^t(\alpha x \oplus \beta y) = \alpha A^t x \oplus \beta A^t y = \alpha \rho_1^t x \oplus \beta \rho_2^t y.$$

For $i \in \text{supp } x \setminus \text{supp } y$, we have $A^t(\alpha x \oplus \beta y)_i = \alpha \rho_1^t x_i$ and for $i \in \text{supp } y$ we have $A^t(\alpha x \oplus \beta y)_i = \alpha \rho_1^t x_i \oplus \beta \rho_2^t y_i = \beta \rho_2^t y_i$ for all large enough t . Hence, an eigenvector will never be reached, a contradiction.

Let (i) and (ii) hold, then any $y \in \text{core}(A)$ can be written as $y = \bigoplus_{\mu} \alpha_{\mu} x_{\mu}$, where x_{μ} are eigenvectors corresponding to the eigenvalues

$$\rho_1 \leq \rho_2 \leq \cdots \leq \rho_l = \cdots = \rho_m.$$

It follows that the union of all supports corresponding to the vectors with the largest eigenvalue contains all other supports. Hence,

$$A^t y = \bigoplus_{\mu=1}^m \alpha_{\mu} \rho_{\mu}^t x_{\mu} = \rho_m^t \bigoplus_{\mu=l}^m \alpha_{\mu} x_{\mu}$$

at all large enough t , hence A is core robust. \square

Now we deduce the characterization of robust matrices obtained by Butkovič, Cuninghame-Green, Gaubert [7].

THEOREM 4.7 (Butkovič et al. [7]). $A \in \mathbb{R}_+^{n \times n}$ with no zero columns is robust if and only if the following conditions hold:

- (i) All nontrivial classes of Frobenius normal form are spectral;
- (ii) The critical graphs associated with all $\rho \in \Lambda(A)$ are primitive;
- (iii) For any two classes μ and ν , if both $\mu \not\rightarrow \nu$ and $\nu \not\rightarrow \mu$ then $\rho_{\mu} = \rho_{\nu}$.

Proof. By Theorem 4.4, part (i) is equivalent to finite stabilization of $\text{core}(A)$. In view of Theorems 4.5 and Theorem 4.6, it is sufficient to show that part (ii) is equivalent with Theorem 4.6(i), and part (iii) is equivalent with Theorem 4.6(ii).

For the first equivalence, we observe that $V(A^{\sigma_{\Lambda}}) = V(A)$ if and only if $\sigma_{\Lambda} = 1$. Indeed, the “if” part is trivial, and for the “only if” part recall that $V(A) \subseteq V(A^t) \subseteq V(A^{\sigma_{\Lambda}})$ for all $t \geq 1$. So that if $V(A^{\sigma_{\Lambda}}) = V(A)$ then the period of the sequence $\{V(A^t)\}_{t \geq 1}$ is 1, which equals σ_{Λ} by Theorem 3.2.

For the second equivalence, observe using the description of spectral classes of Theorem 2.3, that part (iii) of the present theorem is equivalent to the following condition:

- (iii') For any two classes μ and ν , $\rho_{\mu} < \rho_{\nu}$ implies $\mu \rightarrow \nu$.

It suffices to show that this condition is equivalent to the one of Theorem 4.6(ii).

For this, applying Theorem 2.5(i'), observe that (iii') holds if and only if for any pair of (spectral) classes μ and ν with $\rho_\mu < \rho_\nu$ and for any pair of critical components $\tilde{\mu}$ in μ and $\tilde{\nu}$ in ν , the associated fundamental eigenvectors satisfy $\text{supp}(x_{\tilde{\mu}}) \subseteq \text{supp}(y_{\tilde{\nu}})$.

This already shows that if (iii') is violated then Theorem 4.6(ii) does not hold. For the converse, we argue that if $x \in V(A, \rho_1)$ and $y \in V(A, \rho_2)$, where $x, y \neq 0$ and $\rho_1 < \rho_2$, then x , resp., y are combinations of fundamental eigenvectors $x_{\tilde{\mu}}$, resp., $y_{\tilde{\nu}}$ satisfying $\text{supp}(x_{\tilde{\mu}}) \subseteq \text{supp}(y_{\tilde{\nu}})$. As $\text{supp}(x)$, resp., $\text{supp}(y)$, is the union of supports of such $x_{\tilde{\mu}}$, resp., $y_{\tilde{\nu}}$, the inclusion $\text{supp}(x) \subseteq \text{supp}(y)$ follows. \square

Recall the notion of orbit periodicity defined in Subsection 3.2: A is called orbit periodic if the orbit of any vector $x \in \mathbb{R}_+^n \setminus \{0\}$ is ultimately periodic, which happens if and only if it hits an eigenvector of A^{σ^Λ} . The core restriction of orbit periodicity is defined as follows: A is called *core periodic* if the orbit of any vector $x \in \text{core}(A) \setminus \{0\}$ is ultimately periodic, i.e., it hits an eigenvector of A^{σ^Λ} .

Observe that A is orbit periodic if and only if A^σ is robust, and this also holds for the core restrictions (more generally, for any restrictions) of both notions. The next observations are analogous to Theorems 4.5 and 4.6.

THEOREM 4.8. $A \in \mathbb{R}_+^{n \times n}$ with no zero columns is orbit periodic if and only if it is core periodic and the core finitely stabilizes.

Proof. Follows the lines of the proof of Theorem 4.5. \square

THEOREM 4.9. $A \in \mathbb{R}_+^{n \times n}$ is core periodic if and only if $x \in V(A^{\sigma^\Lambda}, \rho_1^{\sigma^\Lambda}) \setminus \{0\}$, $y \in V(A^{\sigma^\Lambda}, \rho_2^{\sigma^\Lambda}) \setminus \{0\}$ with $\rho_1, \rho_2 \in \Lambda(A)$ and $\rho_1 < \rho_2$ imply that $\text{supp}(x) \subseteq \text{supp}(y)$.

Proof. $A \in \mathbb{R}_+^{n \times n}$ is core periodic if and only if A^σ is core robust. Observe that A^σ satisfies condition (i) in Theorem 4.6 since $\text{core}(A) = \text{core}(A^\sigma)$, and this is generated by the eigenvectors of A^σ . Hence, core robustness of A^σ is equivalent to condition (ii) in Theorem 4.6. As $\Lambda(A^\sigma) = \{\rho^\sigma \mid \rho \in \Lambda(A)\}$, this condition is identical to the one of the present proposition. \square

4.3. Bijection and weak stability. The concept of a *weakly stable matrix* was introduced by Butkovič et al. [9]. It requires that the orbit of any vector x **does not** hit any eigenvector of A , unless that vector x already is an eigenvector. Equivalently we can write

$$(4.1) \quad Ay = z, \quad z \in V(A, \rho) \Rightarrow y = \rho^{-1}z.$$

Weakly stable matrices can be characterized as follows [9]: Each spectral class is initial and each critical graph is a Hamiltonian cycle.

We now study the *core weakly stable matrices*: Restricting vector x in the above definition to $\text{core}(A)$. Thus, A is core weakly stable if and only if

$$Ay = z, z \in V(A, \rho) \text{ and } y \in \text{core}(A) \Rightarrow y = \rho^{-1}z.$$

As we will shortly see, the core weak stability is equivalent to the matrix being bijective on its core. We will need the following corollary of Lemma 2.7 and Theorem 2.9

COROLLARY 4.10. *Let μ and ν be classes of A with cyclicities σ_μ and σ_ν , and let $\mu^{(k)}$ for $k = 1, \dots, \gcd(t, \sigma_\mu)$ and $\nu^{(l)}$ for $l = 1, \dots, \gcd(t, \sigma_\nu)$ be the classes of A^t derived from μ and ν respectively. Then the following are equivalent:*

- (i) μ and ν are spectral and $\mu \rightarrow \nu$ (in A);
- (ii) All derived classes $\mu^{(k)}$ and $\nu^{(l)}$ are spectral, and for each k there is an l and for each l there is a k such that $\mu^{(k)} \rightarrow \nu^{(l)}$ (in A^t);
- (iii) There exists a pair of derived classes $\mu^{(k)}$ and $\nu^{(l)}$ that are spectral, such that $\mu^{(k)} \rightarrow \nu^{(l)}$ (in A^t).

Let us also recall the following well-known observation, which we briefly prove for the reader's convenience.

LEMMA 4.11 ([12] Lemma 1.4 part 4). *Let $x \in V(A, \rho)$, and let x_C be the subvector of x extracted from the node set of the critical graph associated with ρ . Then x is uniquely determined by x_C .*

Proof. Using Proposition 2.4, assume without loss of generality that $\rho = 1$ is the greatest eigenvalue of A (in other words, the maximum cycle geometric mean). The set of the critical nodes of A induces the following block decomposition:

$$A = \begin{pmatrix} A_{CC} & A_{CN} \\ A_{NC} & A_{NN} \end{pmatrix}$$

with A_{CC} , resp., A_{NN} being the principal submatrices extracted from the set of critical nodes, resp., its complement (the noncritical nodes). For the subvector x_N extracted from the noncritical nodes, we have

$$x_N = A_{NC}x_C \oplus A_{NN}x_N.$$

For a given x_C , this is a max-algebraic Z-matrix (or Bellman) equation on x_N as treated, for instance, in [10]. As A_{NN} has $\rho(A_{NN}) < 1$, we obtain that $(A_{NN})^* A_{NC}x_C$ is the only solution, hence x_N is uniquely determined by x_C . \square

THEOREM 4.12. *Let $A \in \mathbb{R}_+^{n \times n}$. The following are equivalent:*

- (i) *The mapping induced by A on its core is a bijection;*

- (ii) A is core weakly stable;
- (iii) If μ and ν are spectral classes of A and $\mu \rightarrow \nu$, then $\rho_\mu = \rho_\nu$. In other words, spectral classes with different Perron roots do not access each other.

Moreover, if A satisfies (i), (ii), or (iii) then A^t satisfies (i), (ii), or (iii), respectively, for every $t \geq 1$. Also if A^t satisfies (i), (ii), or (iii) for some $t \geq 1$ then the same properties hold for A .

Proof. We show first that each of claims (i) and (iii) holds for A if and only if it holds for A^t with an arbitrary t . Then, the implications (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) will be shown for arbitrary $t \geq 1$, and the implication (iii) \Rightarrow (i) will be shown for $t = \sigma_\Lambda$.

First observe that $\text{core}(A^t) = \text{core}(A)$ for all t . Then for (i), the equivalence for all t is clear.

To show the same equivalence for (iii) we use Corollary 4.10 and argue by contradiction. By that Corollary, if μ and ν are spectral in A with $\mu \rightarrow \nu$ but $\rho_\mu \neq \rho_\nu$, then for each A^t there exist derived spectral classes μ' and ν' in A^t such that $\mu' \rightarrow \nu'$ in A^t and $\rho_{\mu'} \neq \rho_{\nu'}$. So if (iii) does not hold for A then it does not hold for A^t . The converse implication follows similarly.

Also note that by Lemma 3.4 (ii), we know that A and hence all its powers are surjective on the core.

(i) \Rightarrow (ii) for every A^t : Assume by contradiction that the core weak stability (4.1), formulated for A^t as

$$A^t y = z, \quad z \in V(A^t, \rho^t) \text{ and } y \in \text{core}(A^t) \Rightarrow y = \rho^{-t} z,$$

is violated. Since $y = \rho^{-t} z$ satisfies $A^t y = z$, this means that there is $z \in \text{core}(A^t) = \text{core}(A)$ and $y' \in \text{core}(A)$ other than $\rho^{-t} z$ such that $A^t y' = z$. This means that A^t and hence A are not bijective on the core.

(ii) \Rightarrow (iii) for all A^t , assuming without loss of generality $t = 1$: If (iii) does not hold, then there exist μ, ν spectral with $\mu \rightarrow \nu$ such that $\rho_\mu < \rho_\nu$. Then by Theorem 2.5(i') $\text{core}(A)$ contains two eigenvectors x_μ and x_ν with $\text{supp}(x_\mu) \subset \text{supp}(x_\nu)$ corresponding to the Perron roots $\rho_\mu < \rho_\nu$. Taking $\beta \neq 0$ and α sufficiently large, we make a combination $\alpha x_\mu \oplus \beta x_\nu$, different from both αx_μ and βx_ν . However,

$$A^t(\alpha x_\mu \oplus \beta x_\nu) = \alpha \rho_\mu^t x_\mu \oplus \beta \rho_\nu^t x_\nu = \beta \rho_\nu^t x_\nu,$$

starting from some t , but not for all t . So x_ν has a preimage in the core which is not an eigenvector, and A is not core weakly stable.

(iii) \Rightarrow (i) for $s = \sigma_\Lambda$. Let $\sigma := \sigma_\Lambda$, for brevity. We need to show that if that if $A^\sigma z = A^\sigma z'$ for $z, z' \in \text{core}(A)$ then $z = z'$. We can write $z = \bigoplus_\mu x_\mu$ and $z' = \bigoplus_\mu y_\mu$

where x_μ, y_μ are eigenvectors of A^σ corresponding to ρ_μ^σ or zero vectors, with μ running over all eigenvalues of A . Then we have

$$A^\sigma \left(\bigoplus_\mu x_\mu \right) = \bigoplus_\mu \rho_\mu^\sigma x_\mu, \quad A^\sigma \left(\bigoplus_\mu y_\mu \right) = \bigoplus_\mu \rho_\mu^\sigma y_\mu,$$

and our goal is to show that

$$(4.2) \quad \bigoplus_\mu \rho_\mu^\sigma x_\mu = \bigoplus_\mu \rho_\mu^\sigma y_\mu \Rightarrow x_\mu = y_\mu \quad \forall \mu,$$

which implies $z = z'$. For (4.2), observe that by Theorem 2.5(i') and the assumed condition (iii), the support of any vector in $V(A^\sigma, \rho^\sigma)$ contains only indices in spectral classes with ρ^σ or indices in non-spectral and trivial classes. Then the *spectral part of support*, i.e., the part consisting of all indices in spectral classes, is disjoint from the spectral part of support of any vector in $V(A^\sigma, \tilde{\rho}^\sigma)$ when $\tilde{\rho} \neq \rho$. Therefore in any max-algebraic sum of $x_\mu \in V(A^\sigma, \rho_\mu^\sigma)$ with different ρ_μ , as in (4.2), the subvectors of x_μ in spectral classes are determined uniquely. This shows that the components of x_μ and y_μ in spectral classes associated with ρ_μ are the same.

By Lemma 4.11 any eigenvector is uniquely determined by its critical components, and hence, x_μ and y_μ in (4.2) are uniquely determined by their components in spectral classes, implying that $x_\mu = y_\mu$ for all μ , which is (4.2). Then $z = z'$, and the proof is complete. \square

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