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EIGENVALUE LOCALIZATION FOR COMPLEX MATRICES*

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Abstract. Let A be an $n \times n$ complex matrix with $n \geq 3$. It is shown that at least $n-2$ of the eigenvalues of A lie in the disk

$$\left|z - \frac{\operatorname{tr} A}{n}\right| \leq \sqrt{\frac{n-1}{n}\left(\sqrt{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n}\right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}} - \frac{\operatorname{spd}^2(A)}{2}\right)},$$

where $\|A\|_2$, $\operatorname{tr} A$, and $\operatorname{spd}(A)$ denote the Frobenius norm, the trace, and the spread of A, respectively. In particular, if $A = [a_{ij}]$ is normal, then at least $n-2$ of the eigenvalues of A lie in the disk

$$\left|z - \frac{\operatorname{tr} A}{n}\right| \leq \sqrt{\frac{n-1}{n}\left(\frac{\|A\|_2^2}{2} - \frac{|\operatorname{tr} A|^2}{n} - \frac{3}{2}\max_{i,j=1,\ldots,n}\left(\sum_{\substack{k=1\\k\neq i}}^{n}|a_{ki}|^2 + \sum_{\substack{k=1\\k\neq j}}^{n}|a_{kj}|^2 + \frac{|a_{ii}-a_{jj}|^2}{2}\right)\right)}.$$

Moreover, the constant $\frac{3}{2}$ can be replaced by 4 if the matrix A is Hermitian.

Key words. Eigenvalue, Localization, Frobenius norm, Trace, Spread, Normal matrix.

AMS subject classifications. 15A18, 15A42.

1. Introduction. Let $\mathbb{M}_n(\mathbb{C})$ be the set of all $n \times n$ complex matrices. For a matrix $A \in \mathbb{M}_n(\mathbb{C})$, let $\lambda_j(A)$, $j = 1, \ldots, n$, be the eigenvalues of A repeated according to multiplicity, and let the symbols $\|A\|_2$ and $\operatorname{tr} A$ denote the Frobenius norm and the trace of A, respectively. We have to keep in mind that the Frobenius norm is unitarily invariant, that is, $\|UAV\|_2 = \|A\|_2$ for all unitary matrices U, V in $\mathbb{M}_n(\mathbb{C})$ and $\operatorname{tr} A = \sum_{j=1}^{n} \lambda_j(A)$.

An estimate for the eigenvalues of matrices [14] says that if A is an $n \times n$ real

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symmetric matrix, then

$$\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}\right| \le \sqrt{\frac{n-1}{n}\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n}\right)}. \tag{1.1}$$

for $\ell = 1, \ldots, n$. Moreover, a generalization of (1.1) for arbitrary matrices $A \in \mathbb{M}_n(\mathbb{C})$ has been obtained in [11].

One of the interesting estimates that presents a refinement of (1.1) for nonnormal matrices has been established in [12]. This refinement asserts that if $A \in \mathbb{M}_n(\mathbb{C})$, then

$$\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}\right| \le \sqrt{\frac{n-1}{n}\left(\|A\|_2^2 - \frac{\|A^*A - AA^*\|_2^2}{6\,\|A\|_2^2} - \frac{|\operatorname{tr} A|^2}{n}\right)} \tag{1.2}$$

for $\ell = 1, \ldots, n$. An improvement of the bound for $\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}$ given in (1.2) has been established in [13], which asserts that if $A \in \mathbb{M}_n(\mathbb{C})$, then

$$\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}\right| \le \sqrt{\frac{n-1}{n}}\sqrt[4]{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n}\right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}} \tag{1.3}$$

for $\ell = 1, \ldots, n$.

In this paper, we obtain bounds and localization results for the eigenvalues of matrices. Our results, which involve the traces and the spreads of matrices, are better than some known bounds and localization results. In particular, refinements of (1.2) and (1.3) will be given.

2. Eigenvalue localization for nonnormal matrices. In this section, we present refinements of (1.2) and (1.3) for nonnormal matrices. Throughout the paper, we let the symbol S_ℓ denote the set $\{1, \ldots, n\}\backslash\{\ell\}$ for $\ell = 1, \ldots, n$.

We start with the following result for scalars.

Lemma 2.1. *Let $z_1, \ldots, z_n$ be complex numbers such that $\sum_{j=1}^{n} z_j = 0$. Then*

$$|z_\ell|^2 + \frac{1}{2n}\sum_{j,k \in S_\ell} |z_j - z_k|^2 = \frac{n-1}{n}\sum_{j=1}^{n} |z_j|^2$$

for $\ell = 1, \ldots, n$.

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Proof. Let $\ell \in \{1, \ldots, n\}$. Then

$$\begin{aligned}
|z_\ell|^2 + \frac{1}{2} \sum_{j,k \in S_\ell} |z_j - z_k|^2 &= \left| - \sum_{j \in S_\ell} z_j \right|^2 + \frac{1}{2} \sum_{j,k \in S_\ell} |z_j - z_k|^2 \\
&= \left| \sum_{j \in S_\ell} z_j \right|^2 + \frac{1}{2} \sum_{j,k \in S_\ell} |z_j - z_k|^2 \\
&= \left(\overline{\sum_{j \in S_\ell} z_j} \right) \left(\sum_{j \in S_\ell} z_j \right) + \frac{1}{2} \sum_{j,k \in S_\ell} |z_j - z_k|^2 \\
&= \sum_{j \in S_\ell} |z_j|^2 + \frac{1}{2} \sum_{\substack{j,k \in S_\ell \\ j \neq k}} (\bar{z}_j z_k + \bar{z}_k z_j) + \frac{1}{2} \sum_{\substack{j,k \in S_\ell \\ j \neq k}} |z_j - z_k|^2 \\
&= \sum_{j \in S_\ell} |z_j|^2 + \frac{1}{2} \sum_{\substack{j,k \in S_\ell \\ j \neq k}} \left(\bar{z}_j z_k + \bar{z}_k z_j + |z_j - z_k|^2 \right) \\
&= \sum_{j \in S_\ell} |z_j|^2 + \frac{1}{2} \sum_{\substack{j,k \in S_\ell \\ j \neq k}} \left(|z_j|^2 + |z_k|^2 \right) \\
&= (n-1) \sum_{j \in S_\ell} |z_j|^2 \\
&= (n-1) \sum_{j=1}^{n} |z_j|^2 - (n-1) |z_\ell|^2 . \qquad (2.1)
\end{aligned}$$

It follows from the identity (2.1) that

$$n |z_\ell|^2 + \frac{1}{2} \sum_{j,k \in S_\ell} |z_j - z_k|^2 = (n-1) \sum_{j=1}^{n} |z_j|^2 ,$$

and so

$$|z_\ell|^2 + \frac{1}{2n} \sum_{j,k \in S_\ell} |z_j - z_k|^2 = \frac{n-1}{n} \sum_{j=1}^{n} |z_j|^2 ,$$

as required. ∎

Another result for scalars that we need is the following. Its proof is similar to that of Lemma 2.1 and is left to the reader.

LEMMA 2.2. *Let $z_1, \ldots, z_n$ be complex numbers. Then*

$$\sum_{j=1}^{n} |z_j|^2 = \frac{1}{n} \left| \sum_{j=1}^{n} z_j \right|^2 + \frac{1}{n} \sum_{1 \le j < k \le n} |z_j - z_k|^2 .$$

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Our first result is the following identity.

THEOREM 2.3. *Let* $A \in \mathbb{M}_n(\mathbb{C})$ *with* $n \geq 3$. *Then*

$$\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}\right|^2 + \frac{1}{2n}\sum_{j,k \in S_\ell} |\lambda_j(A) - \lambda_k(A)|^2 = \frac{n-1}{n}\left(\sum_{j=1}^n |\lambda_j(A)|^2 - \frac{|\operatorname{tr} A|^2}{n}\right)$$

for $\ell = 1, \ldots, n$.

Proof. Let $z_j = \lambda_j(A) - \frac{\operatorname{tr} A}{n}$, $j = 1, \ldots, n$. Then $\sum_{j=1}^n z_j = 0$, and so

$$\begin{aligned}
&\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}\right|^2 + \frac{1}{2n}\sum_{j,k \in S_\ell} |\lambda_j(A) - \lambda_k(A)|^2 \\
&\quad = |z_\ell|^2 + \frac{1}{2n}\sum_{j,k \in S_\ell} |z_j - z_k|^2 \\
&\quad = \frac{n-1}{n}\sum_{j=1}^n |z_j|^2 \quad \text{(by Lemma 2.1)} \\
&\quad = \frac{n-1}{n}\left(\frac{1}{n}\left|\sum_{j=1}^n z_j\right|^2 + \frac{1}{n}\sum_{1 \leq j < k \leq n}^n |z_j - z_k|^2\right) \quad \text{(by Lemma 2.2)} \\
&\quad = \frac{n-1}{n}\left(\frac{1}{n}\sum_{1 \leq j < k \leq n}^n |z_j - z_k|^2\right) \\
&\quad = \frac{n-1}{n}\left(\frac{1}{n}\sum_{1 \leq j < k \leq n}^n |\lambda_j(A) - \lambda_k(A)|^2\right) \\
&\quad = \frac{n-1}{n}\left(\sum_{j=1}^n |\lambda_j(A)|^2 - \frac{1}{n}\left|\sum_{j=1}^n \lambda_j(A)\right|^2\right) \quad \text{(by Lemma 2.2)} \\
&\quad = \frac{n-1}{n}\left(\sum_{j=1}^n |\lambda_j(A)|^2 - \frac{|\operatorname{tr} A|^2}{n}\right)
\end{aligned}$$

for $\ell = 1, \ldots, n$. $\square$

Also, we need the following bound for the eigenvalues of a given matrix $A \in \mathbb{M}_n(\mathbb{C})$ [4].

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LEMMA 2.4. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\sum_{j=1}^{n} |\lambda_j(A)|^2 \leq \sqrt{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n} \right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}} + \frac{|\operatorname{tr} A|^2}{n}.$$

REMARK 2.5. The bound given in Lemma 2.4 is sharper than the bounds:

$$\sum_{j=1}^{n} |\lambda_j(A)|^2 \leq \|A\|_2^2 - \frac{\|A^*A - AA^*\|_2^2}{6\,\|A\|_2^2}$$

and

$$\sum_{j=1}^{n} |\lambda_j(A)|^2 \leq \sqrt{\|A\|_2^4 - \frac{\|A^*A - AA^*\|_2^2}{2}}$$

given earlier in [2] and [7], respectively. These bounds, in turn, are sharper than the classical Schur's inequality [15, p. 50].

Based on Theorem 2.3 and Lemma 2.4, we have the following localization result for the eigenvalues of matrices. This result includes a refinement of (1.3).

THEOREM 2.6. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $n \geq 3$. Then*

$$\begin{aligned} &\left| \lambda_\ell(A) - \frac{\operatorname{tr} A}{n} \right|^2 + \frac{1}{2n} \sum_{j,k \in S_\ell} |\lambda_j(A) - \lambda_k(A)|^2 \\ &\leq \frac{n-1}{n} \sqrt{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n} \right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}} \end{aligned} \tag{2.2}$$

for $\ell = 1, \ldots, n$. In particular,

$$\left| \lambda_\ell(A) - \frac{\operatorname{tr} A}{n} \right| \leq \sqrt{\frac{n-1}{n}} \sqrt[4]{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n} \right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}} \tag{2.3}$$

for $\ell = 1, \ldots, n$.

Proof. The result follows from Theorem 2.3 and Lemma 2.4. ∎

We remark here that (2.3) has been obtained earlier in [13].

Applications of Theorem 2.6 are given in the following result. This result includes a refinement of (1.2).

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COROLLARY 2.7. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $n \geq 3$. Then:*

(a)

$$\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}\right| \leq \sqrt{\frac{n-1}{n}\left(\sqrt{\|A\|_2^4 - \frac{\|A^*A - AA^*\|_2^2}{2}} - \frac{|\operatorname{tr} A|^2}{n}\right)} \tag{2.4}$$

for $\ell = 1, \ldots, n$.

(b)

$$\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}\right| \leq \sqrt{\frac{n-1}{n}\left(\|A\|_2^2 - \frac{\|A^*A - AA^*\|_2^2}{4\|A\|_2^2} - \frac{|\operatorname{tr} A|^2}{n}\right)} \tag{2.5}$$

for $\ell = 1, \ldots, n$.

Proof. By direct computations, it can be seen that

$$\sqrt{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n}\right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}} \leq \sqrt{\|A\|_2^4 - \frac{\|A^*A - AA^*\|_2^2}{2}} - \frac{|\operatorname{tr} A|^2}{n} \tag{2.6}$$

and

$$\sqrt{\|A\|_2^4 - \frac{\|A^*A - AA^*\|_2^2}{2}} \leq \|A\|_2^2 - \frac{\|A^*A - AA^*\|_2^2}{4\|A\|_2^2}. \tag{2.7}$$

Now, part (a) of the corollary follows from (2.3) and (2.6), while part (b) follows from (2.4) and (2.7). $\square$

REMARK 2.8. It is clear from (2.6) and (2.7) that the bound for $\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}$ given in (2.3) is sharper than that given in (2.4), which, in turn, is sharper than that given in (2.5). Another bound for the eigenvalues of a given matrix $A \in \mathbb{M}_n(\mathbb{C})$ says that

$$\sum_{j=1}^{n} |\lambda_j(A)|^2 \leq \|A\|_2^2 - \frac{\|A^*A - AA^*\|_2^2}{4\|A\|_2^2}.$$

This bound can be inferred from Lemma 2.4, (2.6), and (2.7), and it can also be obtained from Theorem 2 in [3], concerning measures of nonnormality of matrices.

To give another application of Theorem 2.6, we need the following lemma [9].

LEMMA 2.9. *Let $z_1, \ldots, z_n$ be complex numbers. Then*

$$\frac{n}{2} \max_{j,k=1,\ldots,n} |z_j - z_k|^2 \leq \sum_{1 \leq j < k \leq n} |z_j - z_k|^2. \tag{2.8}$$

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Based on Theorem 2.6 and Lemma 2.9, we have the following result. This result presents another refinement of (1.3).

Corollary 2.10. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $n \geq 3$. Then*

$$\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}\right| \leq \sqrt{\frac{n-1}{n}\left(\sqrt{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n}\right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}} - \frac{s^2(A)}{2}\right)} \tag{2.9}$$

for $\ell = 1, \ldots, n$, where $s(A) = \min_{1\leq \ell \leq n} \max_{j,k\in S_\ell} |\lambda_j(A) - \lambda_k(A)|$.

Proof. Since S_ℓ contains $n-1$ numbers, we have

$$\begin{aligned}
\sum_{j,k\in S_\ell} |\lambda_j(A) - \lambda_k(A)|^2 &= 2 \sum_{\substack{j,k\in S_\ell \\ j<k}} |\lambda_j(A) - \lambda_k(A)|^2 \\
&\geq (n-1) \max_{j,k\in S_\ell} |\lambda_j(A) - \lambda_k(A)|^2 \ \text{(by Lemma 2.9)} \\
&\geq (n-1)\, s^2(A). \qquad (2.10)
\end{aligned}$$

Now, (2.9) follows from (2.2) and (2.10). ∎

Remark 2.11. A localization result for the eigenvalues of matrices has been given in [4]. This result asserts that if the eigenvalues of a matrix $A \in \mathbb{M}_n(\mathbb{C})$ are arranged as $|\lambda_1(A)| \geq \cdots \geq |\lambda_n(A)|$, then

$$\begin{aligned}
&\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{m}\right| \\
&\leq \sqrt{\frac{m-1}{2m}\left(\sqrt{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n}\right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}} + \left|\operatorname{tr} A^2 - \frac{(\operatorname{tr} A)^2}{m}\right|\right)}
\end{aligned}$$

for $\ell = 1, \ldots, m$, where m is any integer satisfying $\operatorname{rank} A \leq m \leq n$.

Another result analogous to (2.2) can be stated as follows.

Theorem 2.12. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $n \geq 3$. Then*

$$\begin{aligned}
&\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}\right|^2 + \frac{1}{n}\left|(n-1)\left(\operatorname{tr}\left(A^2\right) - \lambda_\ell^2(A)\right) - (\operatorname{tr} A - \lambda_\ell(A))^2\right| \\
&\leq \frac{n-1}{n}\sqrt{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n}\right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}} \qquad (2.11)
\end{aligned}$$

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for $\ell = 1, \ldots, n$.

Proof. Observe that

$$\begin{aligned}
&\sum_{j,k \in S_\ell} |\lambda_j(A) - \lambda_k(A)|^2 \\
&\qquad = \sum_{j,k \in S_\ell} \left| (\lambda_j(A) - \lambda_k(A))^2 \right| \\
&\qquad \geq \left| \sum_{j,k \in S_\ell} (\lambda_j(A) - \lambda_k(A))^2 \right| \\
&\qquad = \left| \sum_{j,k \in S_\ell} \left(\lambda_j^2(A) + \lambda_k^2(A) - 2\lambda_j(A)\lambda_k(A) \right) \right| \\
&\qquad = \left| \sum_{j,k \in S_\ell} \lambda_j^2(A) + \sum_{j,k \in S_\ell} \lambda_k^2(A) - 2 \sum_{j,k \in S_\ell} \lambda_j(A)\lambda_k(A) \right| \\
&\qquad = \left| 2(n-1) \sum_{j \in S_\ell} \lambda_j^2(A) - 2 \left(\sum_{j \in S_\ell} \lambda_j(A) \right)^2 \right| \\
&\qquad = \left| 2(n-1) \left(\sum_{j=1}^{n} \lambda_j^2(A) - \lambda_\ell^2(A) \right) - 2 \left(\sum_{j=1}^{n} \lambda_k(A) - \lambda_\ell(A) \right)^2 \right| \\
&\qquad = \left| 2(n-1) \left(\sum_{j=1}^{n} \lambda_j(A^2) - \lambda_\ell^2(A) \right) - 2 (\operatorname{tr} A - \lambda_\ell(A))^2 \right| \\
&\qquad = \left| 2(n-1) \left(\operatorname{tr}\left(A^2\right) - \lambda_\ell^2(A) \right) - 2 (\operatorname{tr} A - \lambda_\ell(A))^2 \right|. \qquad (2.12)
\end{aligned}$$

Now, the result follows from Theorem 2.6 and (2.12). ☐

The spread $\operatorname{spd}(A)$ of a matrix $A \in \mathbb{M}_n(\mathbb{C})$ is defined to be the maximum distance between any two eigenvalues of A, that is,

$$\operatorname{spd}(A) = \max_{j,k=1,\ldots,n} |\lambda_j(A) - \lambda_k(A)|.$$

In the following result, we describe a disk that contains at least $n-2$ of the eigenvalues of a given matrix $A \in \mathbb{M}_n(\mathbb{C})$. A localization result for at least $n-2$ of the eigenvalues of $A \in \mathbb{M}_n(\mathbb{C})$ that improves (1.3) can be concluded from this result.

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THEOREM 2.13. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $n \geq 3$. Then at least $n-2$ of the eigenvalues of A lie in the disk*

$$\begin{aligned}&\left|z - \frac{\operatorname{tr} A}{n}\right| \\ &\leq \sqrt{\frac{n-1}{n}\left(\sqrt{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n}\right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}} - \frac{\operatorname{spd}^2(A)}{2}\right)}. \end{aligned} \tag{2.13}$$

Proof. Let $s, t \in \{1, \ldots, n\}$ be such that $\operatorname{spd}(A) = |\lambda_s(A) - \lambda_t(A)|$ and let $S = \{1, \ldots, n\} \backslash \{s, t\}$. Then $s, t \in S_\ell$ for all $\ell \in S$, and so

$$\operatorname{spd}(A) = \max_{j,k \in S_\ell} |\lambda_j(A) - \lambda_k(A)| \tag{2.14}$$

for all $\ell \in S$. It follows that

$$\begin{aligned}\sum_{j,k \in S_\ell} |\lambda_j(A) - \lambda_k(A)|^2 &= 2 \sum_{\substack{j,k \in S_\ell \\ j<k}} |\lambda_j(A) - \lambda_k(A)|^2 \\ &\geq (n-1) \max_{j,k \in S_\ell} |\lambda_j(A) - \lambda_k(A)|^2 \quad \text{(by Lemma 2.9)} \\ &= (n-1) \operatorname{spd}^2(A) \quad \text{(by the identity (2.14))} \end{aligned} \tag{2.15}$$

for all $\ell \in S$. Now, (2.15) and Theorem 2.6 imply that

$$\begin{aligned}&\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}\right|^2 + \frac{n-1}{2n} \operatorname{spd}^2(A) \\ &\leq \frac{n-1}{n}\left(\sqrt{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n}\right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}}\right), \end{aligned} \tag{2.16}$$

and so

$$\begin{aligned}&\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}\right| \\ &\leq \sqrt{\frac{n-1}{n}\left(\sqrt{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n}\right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}} - \frac{\operatorname{spd}^2(A)}{2}\right)} \end{aligned} \tag{2.17}$$

for all $\ell \in S$. Since the set S contains $n-2$ numbers, then (2.17) means that at least $n-2$ of the eigenvalues of A lie in the disk given in (2.13). $\square$

REMARK 2.14. Theorem 2.13 guarantees that $n-2$ of the eigenvalues lie in the disk (2.13). The question that arise here is what about the remaining two eigenvalues.

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In fact, it is clear from the proof of Theorem 2.13 that the eigenvalues of the matrix $A \in \mathbb{M}_n(\mathbb{C})$ where the spread is attained do not necessarily lie in the disk (2.13). Moreover, if one of these eigenvalues is not simple, then according to the proof of Theorem 2.13, we can see that this eigenvalue must lie in this disk.

The following lemma enables us to give a new bound for the eigenvalues of matrices. The proof of this lemma follows by direct computations. We leave the details for the interested reader.

Lemma 2.15. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $n \geq 3$. Then:*

(a) $\sum_{j=1}^{n} \left|\lambda_j(A) - \frac{\operatorname{tr} A}{n}\right|^2 = \sum_{j=1}^{n} |\lambda_j(A)|^2 - \frac{|\operatorname{tr} A|^2}{n}$.

(b) $\sum_{\ell=1}^{n} \sum_{j,k \in S_\ell}^{n} |\lambda_j(A) - \lambda_k(A)|^2 = 2(n-2) \sum_{1 \leq j < k \leq n} |\lambda_j(A) - \lambda_k(A)|^2$.

Theorem 2.16. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $n \geq 3$. Then*

$$\sum_{j=1}^{n} |\lambda_j(A)|^2$$
$$\leq (n-1)\sqrt{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n}\right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}} + \frac{|\operatorname{tr} A|^2}{n} - \frac{n-2}{2}\operatorname{spd}^2(A).$$

Proof. Observe that

$$\begin{aligned}
&\sum_{j=1}^{n} |\lambda_j(A)|^2 - \frac{|\operatorname{tr} A|^2}{n} + \frac{n-2}{2}\operatorname{spd}^2(A) \\
&\leq \sum_{j=1}^{n} \left|\lambda_j(A) - \frac{\operatorname{tr} A}{n}\right|^2 + \frac{n-2}{n} \sum_{1 \leq j < k \leq n} |\lambda_j(A) - \lambda_k(A)|^2 \\
&\qquad \text{(by Lemmas 2.9 and 2.15 (a))} \\
&= \sum_{\ell=1}^{n} \left|\lambda_j(A) - \frac{\operatorname{tr} A}{n}\right|^2 + \frac{1}{2n} \sum_{\ell=1}^{n} \sum_{j,k \in S_\ell}^{n} |\lambda_j(A) - \lambda_k(A)|^2 \\
&\qquad \text{(by Lemma 2.15 (b))} \\
&= \sum_{\ell=1}^{n} \left(\left|\lambda_j(A) - \frac{\operatorname{tr} A}{n}\right|^2 + \frac{1}{2n} \sum_{j,k \in S_\ell}^{n} |\lambda_j(A) - \lambda_k(A)|^2 \right)
\end{aligned}$$

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$$\leq \sum_{\ell=1}^{n} \frac{n-1}{n} \left(\sqrt{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n} \right)^2 - \frac{1}{2} \|A^*A - AA^*\|_2^2} \right)$$

(by Theorem 2.6)

$$= (n-1) \sqrt{\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n} \right)^2 - \frac{\|A^*A - AA^*\|_2^2}{2}}. \tag{2.18}$$

Now, the result follows from (2.18). □

3. Eigenvalue localization for normal matrices. In this section, we are interested in estimates for at least $n-2$ of the eigenvalues of a normal matrix $A \in \mathbb{M}_n(\mathbb{C})$. In order to do this, we need the following two lemmas of Bhatia and Sharma [1] and Mirsky [9]. It should be mentioned here that Bhatia and Sharma did not write the first lemma below explicitly in this form, but it can be deduced from their results. Before presenting these lemmas, we need to define two functionals on $\mathbb{M}_n(\mathbb{C})$ as follows: Let

$$\operatorname{dr} A = \sum_{\substack{i,j=1 \\ i \neq j}}^{n} a_{ij}$$

and

$$\operatorname{v}(A) = \frac{\operatorname{dr} A^*A}{n} - \frac{|\operatorname{dr} A|^2}{n^2},$$

where $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$.

Lemma 3.1. *Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ be normal. Then*

$$\operatorname{spd}^2(A) \geq \max(\alpha_1, \beta_1), \tag{3.1}$$

where

$$\alpha_1 = \frac{3}{2} \max_{i,j=1,\ldots,n} \left(\sum_{\substack{k=1 \\ k \neq i}}^{n} |a_{ki}|^2 + \sum_{\substack{k=1 \\ k \neq j}}^{n} |a_{kj}|^2 + \frac{|a_{ii} - a_{jj}|^2}{2} \right)$$

and

$$\beta_1 = 3 \left(\frac{\|A\|_2^2}{n} - \frac{|\operatorname{tr} A|^2}{n^2} + \operatorname{v}(A) - \frac{2 \operatorname{Re} \operatorname{tr} A \operatorname{dr} A}{n^2} \right).$$

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LEMMA 3.2. *Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ be normal. Then*

$$\operatorname{spd}(A) \geq \max\left(\alpha_2, \beta_2\right), \tag{3.2}$$

where

$$\alpha_2 = \sqrt{3} \max_{\substack{i,j=1,\ldots,n \\ i\neq j}} |a_{ij}|$$

and

$$\beta_2 = \max_{\substack{i,j=1,\ldots,n \\ i\neq j}} \left(\frac{|a_{ii}-a_{jj}|^2 + \left|(a_{ii}-a_{jj})^2 + 4a_{ij}a_{ji}\right|}{2} + |a_{ij}|^2 + |a_{ij}|^2 \right).$$

THEOREM 3.3. *Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ be normal with $n \geq 3$. Then at least $n-2$ of the eigenvalues of A lie in the disk*

$$\left| z - \frac{\operatorname{tr} A}{n} \right| \leq \sqrt{\frac{n-1}{n}\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n} - \frac{\max\left(\alpha_1, \alpha_2^2, \beta_1, \beta_2^2\right)}{2} \right)}. \tag{3.3}$$

In particular, at least $n-2$ of the eigenvalues of A lie in the disk

$$\left| z - \frac{\operatorname{tr} A}{n} \right| \leq \sqrt{\frac{n-1}{n}\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n} - \frac{3}{2}\max_{\substack{i,j=1,\ldots,n \\ i\neq j}} |a_{ij}|^2 \right)}. \tag{3.4}$$

Proof. Let $\ell \in S$, where S is the set defined in the proof of Theorem 2.13. Then

$$\begin{aligned}
&\left| \lambda_\ell(A) - \frac{\operatorname{tr} A}{n} \right|^2 + \frac{(n-1)}{2n} \max\left(\alpha_1, \alpha_2^2, \beta_1, \beta_2^2\right) \\
&\leq \left| \lambda_\ell(A) - \frac{\operatorname{tr} A}{n} \right|^2 + \frac{n-1}{2n} \operatorname{spd}^2(A) \quad \text{(by Lemmas 3.1 and 3.2)} \\
&\leq \frac{n-1}{n}\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n} \right) \quad \text{(by (2.17))}.
\end{aligned} \tag{3.5}$$

Now, (3.3) follows from Theorem 2.13 and (3.5), while (3.4) is a special case of (3.3). ∎

The constant $\frac{3}{2}$ in (3.4) can be improved if the matrix A is Hermitian. This improvement can be achieved using a result of Mirsky [10] that says if $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ is Hermitian, then

$$\operatorname{spd}^2(A) \geq \max_{\substack{i,j=1,\ldots,n \\ i\neq j}} \left(4\,|a_{ij}|^2 + (a_{ii}-a_{jj})^2 \right). \tag{3.6}$$

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In fact, through Lemma 3.1, Bhatia and Sharma introduced a remarkable improvement of (3.6). This improvement asserts if $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ is Hermitian with $n \geq 3$, then

$$\operatorname{spd}^2(A) \geq 2 \max_{i,j=1,\ldots,n} \left(\sum_{\substack{k=1 \\ k\neq i}}^{n} |a_{ik}|^2 + \sum_{\substack{k=1 \\ k\neq j}}^{n} |a_{kj}|^2 + \frac{(a_{ii} - a_{jj})^2}{2} \right). \tag{3.7}$$

Based on (3.7) and using a proof similar to that given for Theorem 3.3, we have the following result.

THEOREM 3.4. *Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ be Hermitian with $n \geq 3$. Then at least $n - 2$ of the eigenvalues of A lie in the disk*

$$\left| z - \frac{\operatorname{tr} A}{n} \right| \leq \sqrt{\frac{n-1}{n} \left(\frac{\|A\|_2^2}{2} - \frac{|\operatorname{tr} A|^2}{n} - 2 \max_{i,j=1,\ldots,n} \left(\sum_{\substack{k=1 \\ k\neq i}}^{n} |a_{ik}|^2 + \sum_{\substack{k=1 \\ k\neq j}}^{n} |a_{kj}|^2 + \frac{(a_{ii} - a_{jj})^2}{2} \right) \right)}. \tag{3.8}$$

An application of (3.4) can be seen as follows. Applications of (3.3) and (3.8) can be deduced by a similar argument.

COROLLARY 3.5. *Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ be normal with $n \geq 2$. Then all the eigenvalues of A lie in the disk*

$$\left| z - \frac{\operatorname{tr} A}{n} \right| \leq \sqrt{\frac{2n-1}{n} \left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n} - \frac{3}{4} \max_{\substack{i,j=1,\ldots,n \\ i\neq j}} |a_{ij}|^2 \right)}. \tag{3.9}$$

Proof. Let $B = [b_{ij}] = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$. Then $B \in \mathbb{M}_{2n}(\mathbb{C})$ is normal and the eigenvalues of B are the same as those of A with duplicate multiplicities. It follows from (3.4), applied to the matrix B, that the disk

$$\left| z - \frac{\operatorname{tr} B}{2n} \right| \leq \sqrt{\frac{2n-1}{2n} \left(\|B\|_2^2 - \frac{|\operatorname{tr} B|^2}{2n} - \frac{3}{2} \max_{\substack{i,j=1,\ldots,n \\ i\neq j}} |b_{ij}|^2 \right)} \tag{3.10}$$

contains at least $2n - 2$ of the eigenvalues of B. Since the eigenvalues of B are not simple, it follows from Remark 2.14 that these eigenvalues lie in the disk (3.10). In

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particular, the eigenvalues of A lie in this disk. Now, the result follows in view of the facts that $\|B\|_2^2 = 2\|A\|_2^2$, $\operatorname{tr} B = 2 \operatorname{tr} A$, and $\max_{\substack{i,j=1,\dots,n \\ i\neq j}} |b_{ij}|^2 = \max_{\substack{i,j=1,\dots,n \\ i\neq j}} |a_{ij}|^2$. □

REMARK 3.6. Theorems 3.3, 3.4, and Corollary 3.5 are based on the lower bounds for the spreads of normal matrices mentioned in this paper. Related localization results can be obtained using further lower bounds for the spreads of normal matrices (see, e.g., [5], [6], and [8]).

In the following result, we utilize Theorem 2.6 and the spectral theorem for normal matrices to investigate the equality conditions of (1.1).

COROLLARY 3.7. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $n \geq 3$. Then*

$$\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}\right| = \sqrt{\frac{n-1}{n}\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n}\right)}. \tag{3.11}$$

for some $\ell \in \{1,\dots,n\}$ if and only if A is a scalar matrix or A is normal with only two distinct eigenvalues one of multiplicity $n-1$ and the other is of multiplicity one.

Our final result follows from Theorem 2.13, Corollary 3.7, and Remark 2.14.

COROLLARY 3.8. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $n \geq 3$. If*

$$\left|\lambda_\ell(A) - \frac{\operatorname{tr} A}{n}\right| = \sqrt{\frac{n-1}{n}\left(\|A\|_2^2 - \frac{|\operatorname{tr} A|^2}{n}\right)} \tag{3.12}$$

for some $\ell \in \{1,\dots,n\}$, then A is normal and at least $n-1$ of the eigenvalues of A lie in the disk (2.13).

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