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# ON THE FIXED-POINT TYPE SYLVESTER MATRIX EQUATIONS OVER COMPLETE COMMUTATIVE DIOIDS*

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**Abstract.** This paper extends the concept of tropical tensor product defined by Butkovič and Fiedler to general idempotent dioids. Then, it proposes an algorithm in order to solve the fixed-point type Sylvester matrix equations of the form $X = A \otimes X \ \oplus \ X \otimes B \ \oplus C$. An application is discussed in efficiently solving the minimum cardinality path problem in Cartesian product graphs.

**Key words.** Dioid, Sylvester matrix equation, Fixed-point type equations, Tensor product, Minimum cardinality path problem.

**AMS subject classifications.** 15A24, 15A80.

## 1. Preliminaries.

Definition 1.1. (Dioid) [1, p. 154] A dioid is a set $\mathcal{D}$ endowed with two operations denoted by $\oplus$ and $\otimes$ (called 'sum' or 'addition' and 'product' or 'multiplication') satisfying the following properties:

1. Associativity of addition.
2. Commutativity of addition.
3. Associativity of multiplication.
4. Distributivity of multiplication with respect to addition.
5. Existence of a neutral element, i.e., $\exists \varepsilon \in \mathcal{D} : \forall a \in \mathcal{D}, a \oplus \varepsilon = a$.
6. Absorbing neutral element, i.e., $\forall a \in \mathcal{D}, a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$.
7. Existence of an identity element, i.e., $\exists e \in \mathcal{D} : \forall a \in \mathcal{D}, a \otimes e = e \otimes a = a$.
8. Idempotency of addition, i.e., $\forall a \in \mathcal{D}, a \oplus a = a$.

Note that the last property is not included in the definition of a diod in some references. See, e.g., [5, 6].

Definition 1.2. [1, p. 155] (Commutative dioid) A dioid is commutative if its

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multiplication is commutative.

We will denote $\underbrace{a \otimes \cdots \otimes a}_{k\ times}$ by $a^k, k \in \mathbb{N}$, and $a^0 = e$.

An order relation is a binary relation (denoted by $\geq$ ) which is reflexive, transitive and antisymmetric, and the order is total if each pair of elements is comparable; otherwise, the order is partial. Also a set endowed with a total or partial order relation is a totally or partially ordered set, respectively.

THEOREM 1.3. [1, p. 160] (Order relation) *In a dioid $\mathcal{D}$, one has the following equivalence:*

$$\forall a, b : a = a \oplus b \Leftrightarrow \exists c : a = b \oplus c.$$

*Moreover, these equivalent statements define a (partial) order relation denoted by $\geq$ as follows:*

$$a \geq b \Leftrightarrow a = a \oplus b.$$

*This order relation is compatible with addition, namely*

$$a \geq b \Rightarrow \{\forall c\ ,\ a \oplus c \geq b \oplus c\},$$

*and multiplication, that is,*

$$a \geq b \Rightarrow \{\forall c\ ,\ a \otimes c \geq b \otimes c\}.$$

*The same result is valid for the left product. Two elements $a$ and $b$ in $\mathcal{D}$ always have an upper bound, namely $a \oplus b$, and $\varepsilon$ is the bottom element of $\mathcal{D}$.*

DEFINITION 1.4. [1, p. 162] (Complete dioid) A dioid is complete if it is closed for infinite sums and Property 4 of Definition 1.1 extends to infinite sums.

In a complete dioid, the top element of the dioid, denoted by T exists and is equal to the sum of all elements in $\mathcal{D}$. The top element is always absorbing for addition since obviously $\forall a, \mathrm{T} \oplus a = \mathrm{T}$. Also $\mathrm{T} \otimes \varepsilon = \varepsilon$, because of Property 6 of Definition 1.1.

REMARK 1.5. The set of $n \times n$ matrices endowed with two operations $\oplus$ and $\otimes$ denoted by $(\mathcal{D}^{n\times n}, \oplus, \otimes)$ is also a dioid. Here, for two given matrices $A, B \in \mathcal{D}^{n\times n}$, $S = A \oplus B$, where for all $i$ and $j$, $S_{ij}$ is defined as:

$$S_{ij} = A_{ij} \oplus B_{ij},$$

and $R = A \otimes B$, where for all i and j, $R_{ij}$ is defined as:

$$R_{ij} = \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}.$$

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The only point that deserves attention is the existence of an identity element. Thanks to Property 6 of Definition 1.1, the usual identity matrix with entries equal to e on the diagonal and to $\varepsilon$ elsewhere is the identity element of $\mathcal{D}^{n\times n}$.This identity matrix will also be denoted by $I$ and the neutral matrix will simply be denoted by $\varepsilon$. Notice that if $\mathcal{D}$ is a commutative dioid, this is not the case for $\mathcal{D}^{n\times n}$ in general. Even if $\mathcal{D}$ is a totally ordered set, $\mathcal{D}^{n\times n}$ is only partially ordered. If $\mathcal{D}$ is complete, $\mathcal{D}^{n\times n}$ is complete too [1, p. 194].

Definition 1.6. [5, p. 93] (Quasi-inverse) We call the quasi-inverse of element $a \in \mathcal{D}$, denoted by $a^*$, the limit, when it exists, of the sequence $a^{(k)}$ where, for every $k \in \mathbb{N}$,

$$a^{(k)} = e \oplus a \oplus a^2 \oplus \cdots \oplus a^k.$$

Theorem 1.7. [5, p. 94] *If $a \in \mathcal{D}$ has a quasi-inverse $a^*$, then $\forall b \in \mathcal{D}$, $a^* \otimes b$ (resp., $b \otimes a^*$) is the minimal solution of the equations:*

$$x = a \otimes x \oplus b \quad (resp.,\ x = x \otimes a \oplus b).$$

Definition 1.8. [5, p. 97] ($p$-stable element) For an integer $p \geq 0$, an element $a$ is said to be $p$-stable if and only if $a^{(p+1)} = a^{(p)}$. We then have

$$a^{(p+2)} = e \oplus a \otimes a^{(p+1)} = e \oplus a \otimes a^{(p)} = a^{(p+1)}.$$

Hence, by induction

$$a^{(p+r)} = a^{(p)}, \quad \text{for all nonnegative integers r.}$$

For each $p$-stable element $a \in \mathcal{D}$, we therefore deduce the existence of $a^*$, the quasi-inverse of $a$, defined as:

$$a^* = \lim_{k\to+\infty} a^{(k)} = a^{(p)},$$

which satisfies the equations

$$a^* = a \otimes a^* \oplus e = a^* \otimes a \oplus e. \tag{1.1}$$

Proposition 1.9. [5, p. 100] *If element $a$ is $p$-stable, then $a^* \otimes b$ is the minimal solution to $x = a \otimes x \oplus b$.*

Remark 1.10. [5, p. 101] Since $\mathcal{D}^{n\times n}$ is a dioid if $\mathcal{D}$ is a dioid, Theorem 1.7 can also be applied to matrix equations of the form:

$$X = A \otimes X \oplus B,$$

for a $p$-stable matrix $A \in \mathcal{D}^{n\times n}$.

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## 2. The main part.

Definition 2.1. For two matrices $Y$ and $Z$ of dimensions $m \times n$ and $r \times s$, respectively, the tensor product of $Y$ and $Z$ over a dioid is the following $mr \times ns$ matrix

$$Y \boxtimes Z := \begin{bmatrix} Y \otimes z_{11} & \cdots & Y \otimes z_{1s} \\ \vdots & \ddots & \vdots \\ Y \otimes z_{r1} & \cdots & Y \otimes z_{rs} \end{bmatrix}.$$

This definition was first introduced by Butkovič and Fiedler [2, p. 3] in the context of the special dioid of max-plus algebra.

In this paper, we consider the fixed-point type Sylvester matrix equations of the form

$$X = A \otimes X \ \oplus \ X \otimes B \ \oplus C, \tag{2.1}$$

where $A \in \mathcal{D}^{m\times m}$, $B \in \mathcal{D}^{n\times n}$ and $C \in \mathcal{D}^{m\times n}$ are given matrices while $X \in \mathcal{D}^{m\times n}$ is unknown. Here, $\mathcal{D}$ is a complete and commutative dioid.

The vec operator stacks the columns of a matrix of size $m \times n$ to obtain a long vector of size $mn \times 1$.

Lemma 2.2. *For matrices $A, B, C$ and $D$ of compatible sizes, where the entries are from a commutative dioid, we have:*

1. $\mathrm{vec}(A \otimes B \otimes C) = (A \boxtimes C^T) \otimes \mathrm{vec}(B)$.
2. $(A \boxtimes B) \otimes (C \boxtimes D) = (A \otimes C) \boxtimes (B \otimes D)$.

*Proof.* The proof of parts 1 and 2 are similar to those of Theorem 7 and Theorem 3 of Butkovič and Fiedler [2, p. 4], respectively. □

As a result of the first part of the above lemma, we deduce that the fixed-point type Sylvester matrix equation (2.1) can be written in the following form of a fixed-point type linear system of equations where the entries are from a commutative dioid.

$$x = P \otimes x \oplus c, \ \text{where } P := A \boxtimes I \oplus I \boxtimes B^T, \ c := \mathrm{vec}(C), \ x := \mathrm{vec}(X). \tag{2.2}$$

Here, $P$ is a matrix of size $mn \times mn$ and $c$ and $x$ are vectors of length $mn$.

Remark 2.3. By the second part of Lemma 2.2 for matrices $A, B$ of compatible sizes with entries over a commutative dioid, we have

$$(A \boxtimes I) \otimes (I \boxtimes B) = A \boxtimes B = (I \boxtimes B) \otimes (A \boxtimes I).$$

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LEMMA 2.4. *Let $(\mathcal{D}, \oplus, \otimes)$ be a commutative dioid and $A, B \in \mathcal{D}^{n\times n}$. Then,*

$$B^T \otimes A^T = (A \otimes B)^T.$$

*Proof.* The proof of this result is easy by consulting the corresponding result in any standard linear algebra textbook. ◻

LEMMA 2.5. *Suppose that $A$ and $B$ are square matrices with entries over a commutative dioid. Then,*

1. $(I \boxtimes A)^k = I \boxtimes A^k$,
2. $(B \boxtimes I)^k = B^k \boxtimes I$.

*Proof.* It is trivial that $(I \boxtimes A)^1 = I \boxtimes A^1$. Now, by induction on the power $k$, let $(I \boxtimes A)^{k-1} = I \boxtimes A^{k-1}$. So,

$$(I \boxtimes A)^{k-1} \otimes (I \boxtimes A) = I \boxtimes A^{k-1} \otimes (I \boxtimes A).$$

Therefore, by the second part of Lemma 2.2 we have $(I \boxtimes A)^k = I \boxtimes A^k$ and the proof is complete. The proof of part 2 is similar. ◻

Here, we provide a sufficient condition for the existence of a solution to equation (2.1). Let $G(A)$ and $G(B)$ be graphs associated with the matrices $A$ and $B$, respectively. Suppose that the weight of all the circuits in $G(A)$ and $G(B)$ are $p$-stable. Then, $A^*$ and $B^*$ exist [5, p. 126]. For example, in the $(\min, +)$ dioid the aforementioned condition is equivalent to $G(A)$ and $G(B)$ having no circuit with a negative weight [5, p. 98]. Our main result is the following.

THEOREM 2.6. *Suppose that $A \in \mathcal{D}^{m\times m}$, $B \in \mathcal{D}^{n\times n}$ and $C \in \mathcal{D}^{m\times n}$ are given matrices, where $\mathcal{D}$ is a complete, commutative dioid and $A^*$ and $B^*$ exist. Then, the minimal solution to the fixed-point type Sylvester matrix equation*

$$X = A \otimes X \ \oplus \ X \otimes B \ \oplus C$$

*is $A^* \otimes C \otimes B^*$.*

*Proof.* In order to find the minimal solution to (2.1), it is sufficient to find the minimal solution to (2.2), that is, $P^* \otimes c$, where $P = A \boxtimes I \oplus I \boxtimes B^T$. See, e.g., [6, p. 103] and [5, pp. 127–128]. We have

$$\begin{aligned} P^* \otimes \mathrm{vec}(C) &= (A \boxtimes I \oplus I \boxtimes B^T)^* \otimes \mathrm{vec}(C) \\ &= \left[ \bigoplus_{k=0}^{\infty} (A \boxtimes I \oplus I \boxtimes B^T)^k \right] \otimes \mathrm{vec}(C) \end{aligned}$$

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$$= I \otimes \mathrm{vec}C \oplus (A \boxtimes I \oplus I \boxtimes B^T) \otimes \mathrm{vec}(C)$$
$$\oplus (A \boxtimes I \oplus I \boxtimes B^T)^2 \otimes \mathrm{vec}(C) \oplus \cdots .$$

By Remark 2.3, we know that $(A \boxtimes I) \otimes (I \boxtimes B^T) = (I \boxtimes B^T) \otimes (A \boxtimes I)$. Therefore, since $\oplus$ is idempotent,

$$P^* \otimes \mathrm{vec}(C) = \left( \bigoplus_{q=0}^{\infty} \bigoplus_{k=0}^{q} (A \boxtimes I)^{q-k} \otimes (I \boxtimes B^T)^k \right) \otimes \mathrm{vec}(C).$$

By the second part of Lemma 2.2 and Lemma 2.5, it is easy to see that

$$P^* \otimes \mathrm{vec}(C) = \left( \bigoplus_{q=0}^{\infty} \bigoplus_{k=0}^{q} A^{q-k} \boxtimes (B^T)^k \right) \otimes \mathrm{vec}(C).$$

According to Lemma 2.4, we have

$$P^* \otimes \mathrm{vec}(C) = \left( \bigoplus_{q=0}^{\infty} \bigoplus_{k=0}^{q} A^{q-k} \boxtimes (B^k)^T \right) \otimes \mathrm{vec}(C),$$

which, by the first part of Lemma 2.2, means that

$$\begin{aligned} P^* \otimes \mathrm{vec}(C) &= \bigoplus_{q=0}^{\infty} \bigoplus_{k=0}^{q} \mathrm{vec}(A^{q-k} \otimes C \otimes B^k) \\ &= \mathrm{vec}(\bigoplus_{q=0}^{\infty} \bigoplus_{k=0}^{q} A^{q-k} \otimes C \otimes B^k) \\ &= \mathrm{vec}(A^* \otimes C \otimes B^*). \end{aligned}$$

The last equality is based on the definition of the product of two power series ([1, p. 198]), where $A^*$ and $B^*$ exist, $A^* \otimes C = \bigoplus_{k=0}^{\infty} A^k \otimes C$ and $B^* = \bigoplus_{k=0}^{\infty} B^k$. □

Corollary 2.7. *Suppose that $A \in \mathcal{D}^{m\times m}$, $B \in \mathcal{D}^{n\times n}$ and $C \in \mathcal{D}^{n\times m}$ are given matrices, where $\mathcal{D}$ is a complete, commutative dioid. The minimal solution to the fixed-point type Sylvester matrix equation*

$$X = X \otimes A \;\oplus\; B \otimes X \;\oplus C \tag{2.3}$$

*is $B^* \otimes C \otimes A^*$.*

*Proof.* Similar to (2.2), the fixed-point type Sylvester matrix equation (2.3) can be written in the following form of a fixed-point type linear system of equations with entries over a commutative dioid.

$$x = P \otimes x \oplus c, \text{ where } P := I \boxtimes A^T \oplus B \boxtimes I,\ c := \mathrm{vec}(C),\ x := \mathrm{vec}(X). \tag{2.4}$$

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Some tedious manipulations similar to those in the proof of Theorem 2.6 yield $B^* \otimes C \otimes A^*$ as the minimal solution. ◻

Cohen et al. [3, Theorem 17] proved that the minimal solution to the fixed-point type Sylvester matrix equation (2.1) is $A^* \otimes C \otimes B^*$, where $A, B, C \in \mathcal{D}^{n\times n}$ and $(D, \oplus, \otimes)$ is a complete dioid. Here, we have considered the case where $A \in \mathcal{D}^{m\times m}$, $B \in \mathcal{D}^{n\times n}$, $C \in \mathcal{D}^{m\times n}$ and $(D, \oplus, \otimes)$ is a commutative dioid. Note that, here, the sizes of matrices are not necessarily the same. This means that $A$, $B$ and $C$ can be from different dioids. We have actually proved a result stronger than [3, Theorem 17] using a different approach, namely, the tensor product of matrices over dioids.

**3. An application in solving the minimum cardinality path problem in Cartesian product graphs.** To find the path with the smallest number of arcs, we consider the following structure: $D = \mathbb{N} \cup \{\infty\}$, $\oplus = min$, $\otimes = +$, $\varepsilon = \infty$ and $e = 0$. See, e.g., [5, p. 159]. Let $G$ be an undirected graph that has $n$ vertices $g_1, g_2, \ldots, g_n$. The set of arcs in $G$ is denoted by $E(G)$. We define the adjacency matrix $A = A(G)$ associated with $G$ as follows:

$$a_{ij} = \begin{cases} \infty, & \text{if arc (i,j) does not exist;} \\ 1, & \text{otherwise.} \end{cases}$$

The following properties are valid ([5, pp. 125,160] and [6, p. 97]).

- $A^*$ exists and $A^* = \bigoplus_{k=0}^{n-1} A^k$, where $A^0 = I$.
- $A^*_{ij}$ represents the number of arcs in the minimum cardinality path between $i$ and $j$.

Recall that the Cartesian product of $G$ and $H$ is a graph, denoted as $G \square H$, whose vertex set is $V(G) \times V(H)$, where $\times$ is the Cartesian product. Two vertices $(g, h)$ and $(g', h')$ are adjacent precisely if $g = g'$ and $hh' \in E(H)$, or $gg' \in E(G)$ and $h = h'$. Thus,

$$V(G \square H) = \{(g, h) | g \in V(G) \text{ and } h \in V(H)\},$$

$$E(G \square H) = \{(g, h)(g', h') | g = g', hh' \in E(H), \text{or } gg' \in E(G), h = h'\}.$$

See e.g., Chapters 4 and 5 in [7] for details and examples. See also [8, 9] for applications of fixed-point type Sylvester matrix equations over semirings for modeling large product graphs arising from real-life problems.

Lemma 3.1. *Let $G$ and $H$ be finite graphs with sets of vertices*

$$V(G) = \{g_1, g_2, \ldots, g_m\} \quad \textit{and} \quad V(H) = \{h_1, h_2, \ldots, h_n\},$$

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*respectively. Also, let the vertices of $G\Box H$ be ordered as*

$$V(G\Box H) = \{(g_1,h_1),\dots,(g_1,h_n),(g_2,h_1),\dots,(g_2,h_n),\dots,(g_m,h_1),\dots,(g_m,h_n)\}.$$

*Then,*

$$A(G\Box H) = I_n \boxtimes A(G) \oplus A(H) \boxtimes I_m,$$

*where $I_n$ denotes the $n\times n$ identity matrix and $\boxtimes$ is the tensor product of the matrices.*

*Proof.* Partition $A(G\Box H)$ into $n \times n$ block matrices as follows:

$$A(G\Box H) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix}.$$

For $i = 1,2,\dots,m$, we must count an edge in entry $jk$ of $A_{ii}$ if there exists an edge between $h_j$ and $h_k$. Counting only these edges we have $A(H)\boxtimes I_m$. For $i = 1,2,\dots,n$, we must count an edge in entry $ii$ of $A_{jk}$ if there exists an edge from $g_j$ to $g_k$. Counting only these edges, we have $I_n \boxtimes A(G)$. Thus, we have accounted for all edges in $A(G\Box H)$ and counting them all together, we add ($\oplus$) the two expressions above to get the result. This is justified because $\varepsilon = \infty$ is the neutral element of $\oplus$. $\Box$

Remark 3.2. On the basis of the above comments, we see that computing $A^*$ is needed in order to solve the minimum cardinality path problem between every two nodes of graph $G$. The time complexity of computing the quasi-inverse of $A$, i.e., $A^*$, is $\mathcal{O}(n^3)$, e.g., by the generalized escalator method [5, p. 156] or the Floyd-Warshall algorithm [4, pp. 629,633]. This means that solving the same problem for the Cartesian product of two graphs $G$ and $H$ needs computing $P^*$ where $P = A(G\Box H)$. This is especially useful for graphs which are known a priori to be the Cartesian product of other graphs, like lattice or grid graphs. Note that the number of vertices in $G\Box H$ is $mn$, i.e., $P$ is a matrix of size $mn \times mn$. Therefore, roughly, we have a problem with solution algorithms of the onerous computational complexity $\mathcal{O}(m^3n^3)$. Our proposed approach, based on Corollary 2.7, reduces this cost to $\mathcal{O}(m^3+n^3)$. The reason is that we only need to compute $A(G)^*$, $A(H)^*$ and the product of the three matrices in the min-plus dioid, and these all involve a cubic time complexity.

Example 3.3. Consider the graphs $G$, $H$ and $G\Box H$ in Figure 3.1.

$$A(G) = \begin{bmatrix} \infty & 1 & \infty \\ 1 & \infty & 1 \\ \infty & 1 & \infty \end{bmatrix} \quad \text{and} \quad A(H) = \begin{bmatrix} \infty & 1 \\ 1 & \infty \end{bmatrix}$$

are the adjacency matrices of graphs $G$ and $H$, respectively. We are interested in finding the length of the paths with the minimum number of arcs from all nodes of

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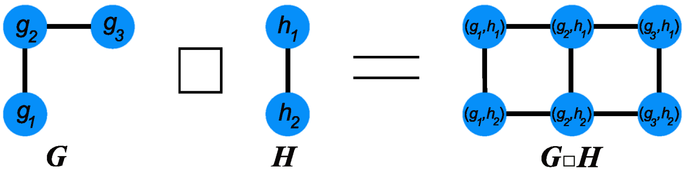


FIG. 3.1. *Graphs $G$, $H$ and their Cartesian product in Example 1.*

$G\square H$ to node $(g_1, h_1)$. By Lemma 3.1, the adjacency matrix of $A(G\square H)$ is $P = I \boxtimes A(G)^T \oplus A(H) \boxtimes I$, where $A(G)^T = A(G)$ (since the adjacency matrix of an undirected graph is symmetric).

$$I \boxtimes A(G)^T = \begin{bmatrix} \infty & \infty & 1 & \infty & \infty & \infty \\ \infty & \infty & \infty & 1 & \infty & \infty \\ 1 & \infty & \infty & \infty & 1 & \infty \\ \infty & 1 & \infty & \infty & \infty & 1 \\ \infty & \infty & 1 & \infty & \infty & \infty \\ \infty & \infty & \infty & 1 & \infty & \infty \end{bmatrix}, \; A(H) \boxtimes I = \begin{bmatrix} \infty & 1 & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & 1 & \infty & \infty \\ \infty & \infty & 1 & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty & \infty & 1 \\ \infty & \infty & \infty & \infty & 1 & \infty \end{bmatrix},$$

and

$$P = \begin{bmatrix} \infty & 1 & 1 & \infty & \infty & \infty \\ 1 & \infty & \infty & 1 & \infty & \infty \\ 1 & \infty & \infty & 1 & 1 & \infty \\ \infty & 1 & 1 & \infty & \infty & 1 \\ \infty & \infty & 1 & \infty & \infty & 1 \\ \infty & \infty & \infty & 1 & 1 & \infty \end{bmatrix}.$$

Finding the paths with the minimum number of arcs from all nodes of $G\square H$ to $(g_1, g_2)$ is equivalent to finding the minimal solution to the fixed-point type equation $x = P \otimes x \oplus c$ in $(\mathbb{N} \cup \{\infty\}, min, +)$ where $c = (0\ \infty\ \infty\ \infty\ \infty\ \infty)^T$. So, we need to find $P^* \otimes c$. By Corollary 2.7

$$P^* \otimes c = \mathrm{vec}(A(H)^* \otimes C \otimes A(G)^*).$$

We have

$$A(G)^* = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \quad A(H)^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 0 & \infty & \infty \\ \infty & \infty & \infty \end{bmatrix}.$$

Note that $A(H)$ is 1-stable so that by the comments following Definition 1.8 we see that $A(H)^* = A(H)^{(1)}$. Similarly, $A(G)$ is 2-stable and we have $A(G)^* = A(G)^{(2)}$.

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We therefore have

$$\mathrm{vec}(A(H)^* \otimes C \otimes A(G)^*) = \mathrm{vec}\left(\left[\begin{array}{ccc} 0 & 1 & 2 \\ 1 & 2 & 3 \end{array}\right]\right) = \left[\begin{array}{cccccc} 0 & 1 & 1 & 2 & 2 & 3 \end{array}\right]^T.$$

So, the number of arcs of paths with minimum cardinality from $(g_1, h_1)$ to $(g_1, h_1)$, $(g_1, h_2)$, $(g_2, h_1)$, $(g_2, h_2)$, $(g_3, h_1)$ and $(g_3, h_2)$ are 0, 1, 1, 2, 2 and 3, respectively.

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