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A NOTE ON THE SPECTRAL RADIUS OF A PRODUCT OF COMPANION MATRICES*

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Abstract. Conditions are given on the coefficients of the characteristic polynomials of a set of k companion matrices to ensure that the spectral radius of their product is bounded by t^k where $0 < t < 1$.

Key words. Companion matrices, Matrix products, Spectral radius.

AMS subject classifications. 15A42, 15B99.

1. Introduction. In a recent paper [2] on population dynamics, A. Blumenthal and B. Fernandez used a bound on the spectral radius of a finite product of companion matrices [2, Lemma 5.5]. In light of the authors' work [4] on products of companion matrices, B. Fernandez inquired of the authors if they could supply a proof of their Lemma 5.5, which we have done in Theorem 1 in Section 3 below. In Section 2, we point out that a special case of our result is connected to the well-known Eneström-Kakaya theorem [1, Theorem 1.2] on the location of zeros of a polynomial.

2. The Eneström-Kakaya theorem. Consider the n by n companion matrix

$$C = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{2.1}$$

Its characteristic polynomial is

$$p(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n.$$

By the Eneström-Kakaya theorem [1, Theorem 1.2], the assumption

$$1 \ge a_1 \ge a_2 \ge \cdots \ge a_n \ge 0. \tag{2.2}$$

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implies that all zeros λ of $p(\lambda)$ satisfy $|\lambda| \le 1$. Therefore, under assumption (2.2), the spectral radius $\rho(C)$ of C is at most 1.

We can prove this result using matrix notation as follows. Let A be the $n+1$ by $n+1$ matrix

$$A = \begin{bmatrix} C & 0 \\ u & 1 \end{bmatrix}, \tag{2.3}$$

where $u = (0, 0, \ldots, 0, 1)$ is a row vector of dimension n. Let B be the $n+1$ by $n+1$ companion matrix

$$B = \begin{bmatrix} 1-a_1 & a_1-a_2 & a_2-a_3 & \cdots & a_{n-1}-a_n & a_n \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \tag{2.4}$$

and let L be the $n+1$ by $n+1$ matrix with 1's on the main diagonal, -1's on the super diagonal and 0 entries everywhere else. By calculation, we verify that

$$AL = LB$$

so

$$L^{-1}AL = B. \tag{2.5}$$

Therefore, A, B are similar and so $\rho(A) = \rho(B)$. We obtain

$$\rho(C) \le \rho(A) = \rho(B) = 1,$$

where B is a stochastic matrix [3, page 526], that is, a nonnegative matrix whose row sums are all 1.

3. An extension. We use the same idea to prove the following lemma.

LEMMA 3.1. *Let C_i, $i = 1, \ldots, k$, be companion matrices of the form (2.1) with first rows $-(a_{i1}, a_{i2}, \ldots, a_{in})$, respectively. Suppose that*

$$1 \ge a_{i1} \ge a_{i2} \ge \cdots \ge a_{in} \ge 0 \ \textit{ for } \ i = 1, 2, \ldots, k. \tag{3.1}$$

Then

$$\rho(C_1 C_2 \cdots C_k) \le 1.$$

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Proof. We form the matrices A_i, B_i as before and note that

$$\rho(C_1C_2\cdots C_k) \le \rho(A_1A_2\cdots A_k) = \rho(B_1B_2\cdots B_k) = 1$$

since $B_1B_2\cdots B_k$ is a row stochastic matrix. $\square$

We now obtain our main result.

THEOREM 3.2. *Let C_i, $i = 1,\ldots,k$, be companion matrices of the form (2.1) with first rows $-(a_{i1}, a_{i2},\ldots,a_{in})$, respectively. Suppose that*

$$a_{i0} := 1 > a_{i1} > a_{i2} > \cdots > a_{in} \ge 0 \ \text{ for } \ i = 1,2,\ldots,k. \tag{3.2}$$

Define

$$t = \max_{i=1}^{k} \max_{j=1}^{n} \frac{a_{i,j}}{a_{i,j-1}} < 1.$$

Then

$$\rho(C_1C_2\cdots C_k) \le t^k < 1.$$

Proof. We define $\tilde{a}_{i,j} = t^{-j}a_{i,j}$ and corresponding companion matrices $\tilde{C}_i$. Let $W = \mathrm{diag}(1,t^{-1},\ldots,t^{-n+1})$. Then

$$C_i = tW\tilde{C}_iW^{-1}.$$

Therefore,

$$\rho(C_1C_2\cdots C_k) = t^k\rho(\tilde{C}_1\tilde{C}_2\cdots\tilde{C}_k) \le t^k,$$

where we applied Lemma 3.1 to the matrices $\tilde{C}_i$. $\square$

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