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INEQUALITIES FOR RELATIVE OPERATOR ENTROPIES*

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Abstract. In this paper, operator inequalities are provided for operator entropies transformed by a strictly positive linear map. Some results by Furuichi et al. [S. Furuichi, K. Yanagi, and K. Kuriyama. A note on operator inequalities of Tsallis relative operator entropy. *Linear Algebra Appl.*, 407:19–31, 2005.], Furuta [T. Furuta. Two reverse inequalities associated with Tsallis relative operator entropy via generalized Kantorovich constant and their applications. *Linear Algebra Appl.*, 412:526–537, 2006.], and Zou [L. Zou. Operator inequalities associated with Tsallis relative operator entropy. *Math. Inequal. Appl.*, 18:401–406, 2015.] are extended. In particular, the obtained inequalities are specified for relative operator entropy and Tsallis relative operator entropy. In addition, some bounds for generalized relative operator entropy are established.

Key words. Operator monotone function, f-connection, Operator mean, Relative operator entropy, Tsallis relative operator entropy, Positive linear map.

AMS subject classifications. 94A17, 47A64, 15A39.

1. Introduction.

We start with some notation (see [2, p. 112]).

As usual, the symbol $\mathbb{M}_n(\mathbb{C})$ denotes the C^*-algebra of $n \times n$ complex matrices. For matrices $X, Y \in \mathbb{M}_n(\mathbb{C})$, we write $Y \leq X$ (resp., $Y < X$) if $X - Y$ is positive semidefinite (resp., positive definite).

A linear map $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is said to be *positive* if $0 \leq \Phi(X)$ for $0 \leq X \in \mathbb{M}_n(\mathbb{C})$. If $0 < \Phi(X)$ for $0 < X \in \mathbb{M}_n(\mathbb{C})$ then Φ is said to be *strictly positive*.

A real function $f : J \to \mathbb{R}$ defined on interval $J \subset \mathbb{R}$ is called an *operator monotone function*, if for all Hermitian matrices A and B (of the same order) with spectra in J

$$A \leq B \quad \text{implies} \quad f(A) \leq f(B).$$

Let $f : J \to \mathbb{R}$ be a continuous function on an interval $J \subset \mathbb{R}$. Let A be an $n \times n$ positive definite matrix and B be an $n \times n$ Hermitian matrix such that the spectrum $\operatorname{Sp}(A^{-1/2}BA^{-1/2}) \subset J$. Then the operator σ_f given by

$$A\sigma_f B = A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2} \tag{1.1}$$

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is called f-*connection* (cf. [11, 12]). See [15] for an extension of (1.1).

Note that for the functions $pt + 1 - p$ and t^p, the definition of Eq. (1.1) leads to the arithmetic and geometric operator means (1.2) and (1.3), respectively.

For $A > 0$, $B > 0$ and $p \in [0, 1]$, the p-*arithmetic mean* is defined as follows

$$A \nabla_p B = (1 - p)A + pB. \tag{1.2}$$

For $A > 0$, $B > 0$ and $p \in [0, 1]$, the p-*geometric mean* is defined by (see [12, 17])

$$A \sharp_p B = A^{1/2}(A^{-1/2}BA^{-1/2})^p A^{1/2}. \tag{1.3}$$

We now give definitions of some operator entropies.

For $A > 0$, $B > 0$, the *relative operator entropy* is defined by (see [4])

$$S(A, B) = A^{1/2} \log(A^{-1/2}BA^{-1/2})A^{1/2}. \tag{1.4}$$

For $A > 0$, $B > 0$ and $p \in \mathbb{R}$, the *generalized relative operator entropy* is given by (see [14, 18])

$$S_p(A, B) = A^{1/2}(A^{-1/2}BA^{-1/2})^p \log(A^{-1/2}BA^{-1/2})A^{1/2}. \tag{1.5}$$

For $A > 0$, $B > 0$ and $0 < p \leq 1$, the *Tsallis relative operator entropy* is defined as follows (see [18])

$$T_p(A, B) = \frac{A \sharp_p B - A}{p}. \tag{1.6}$$

It is not hard to check that (1.4), (1.5) and (1.6) are of the form (1.1) for the functions $\log t$, $t^p \log t$ and $\ln_p t = \frac{t^p - 1}{p}$, respectively.

In recent years there has been a growing interest in the study of entropies and means [5, 6, 7, 8, 9, 16, 19].

Theorem A. (Furuichi et al. [7, Theorem 3.6]) *For $A > 0$, $B > 0$, $1 \geq p > 0$ and $a > 0$, the following inequality holds:*

$$A \sharp_p B - \frac{1}{a} A \sharp_{p-1} B + \frac{1 - a^p}{pa^p} A \leq T_p(A, B) \leq \frac{1}{a} B - \frac{1 - a^p}{pa^p} A \sharp_p B - A. \tag{1.7}$$

The next known double inequalities are consequences of (1.7) (see [5, 7, 8, 19]):

$$A - AB^{-1}A \leq T_p(A, B) \leq B - A,$$

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$$A - AB^{-1}A \leq S(A, B) \leq B - A,$$

and

$$(1 - \log a)A - \frac{1}{a}AB^{-1}A \leq S(A, B) \leq (\log a - 1)A + \frac{1}{a}B \quad \text{for } a > 0.$$

Theorem B. (Zou [19, Theorem 2.2]) *For $A > 0$, $B > 0$, $1 \geq p > 0$ and $a > 0$, the following inequality holds:*

$$-\left(\log a + \frac{1 - a^p}{pa^p}\right) A + a^{-p}T_{-p}(A, B) \leq S(A, B) \leq T_p(A, B) - \frac{1 - a^p}{p} A\sharp_p B - (\log a)A. \tag{1.8}$$

It is easily seen that (1.8) implies a result in [7]:

$$T_{-p}(A, B) \leq S(A, B) \leq T_p(A, B).$$

Theorem C. (Furuta [9, Theorem 2.1]) *Let A and B be $n \times n$ positive definite matrices such that $M_1I \geq A \geq m_1I > 0$ and $M_2I \geq B \geq m_2I > 0$. Put $m = \frac{m_2}{M_1}$, $M = \frac{M_2}{m_1}$, $h = \frac{M}{m} = \frac{M_1M_2}{m_1m_2} > 1$ and $p \in (0, 1]$. Let Φ be normalized positive linear map on $B(H) = \mathbb{M}_n(\mathbb{C})$. Then the following inequalities hold:*

$$\Phi(T_p(A, B)) \leq T_p(\Phi(A), \Phi(B)) \leq \Phi(T_p(A, B)) + \left(\frac{1 - K(p)}{p}\right) \Phi(A)\sharp_p\Phi(B) \tag{1.9}$$

and

$$\Phi(T_p(A, B)) \leq T_p(\Phi(A), \Phi(B)) \leq \Phi(T_p(A, B)) + F(p)\Phi(A), \tag{1.10}$$

where $K(p)$ is the generalized Kantorovich constant defined by

$$K(p) = \frac{h^p - h}{(p - 1)(h - 1)} \left(\frac{(p - 1)(h^p - 1)}{p(h^p - h)}\right)^p$$

and

$$F(p) = \frac{m^p}{p} \left(\frac{h^p - h}{h - 1}\right) (1 - K(p)^{\frac{1}{p-1}}) \geq 0.$$

For a positive concave function $g : J \to \mathbb{R}_+$ defined on an interval $J = [m, M]$ with $m < M$, we define (see [13])

$$a_g = \frac{g(M) - g(m)}{M - m}, \quad b_g = \frac{Mg(m) - mg(M)}{M - m} \quad \text{and} \quad c_g = \min_{t \in J} \frac{a_g t + b_g}{g(t)}. \tag{1.11}$$

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In order to unify our further studies, we introduce the notion of relative g-entropy as follows. Let $g : J \to \mathbb{R}$ be a continuous function defined on an interval $J \subset \mathbb{R}$. For $A > 0$, $B > 0$ with the spectrum of $A^{-1/2}BA^{-1/2}$ in J, we define the *relative g-entropy* of A and B as

$$S_g(A,B) = A\sigma_g B = A^{1/2}g(A^{-1/2}BA^{-1/2})A^{1/2}. \tag{1.12}$$

In the present paper, our aim is to provide some further operator inequalities for entropies and means transformed by a strictly positive linear map Φ.

2. Furuta type inequalities. Throughout $f(t,p)$ is a real function of two variables $t \in J$ and $p \in P = (0, p_0]$, $0 < p_0 \le 1$. We use the notation

$$f_p(t) = f(t,p) \quad \text{for } t \in J \text{ and } p \in P, \tag{2.1}$$

$$g_p(t) = g(t,p) = \frac{f(t,p) - f(t,0)}{p} \quad \text{for } t \in J \text{ and } p \in P. \tag{2.2}$$

If there exist the following limits, then we write

$$f_0(t) = f(t,0) = \lim_{p \to 0^+} f(t,p) \quad \text{for } t \in J, \tag{2.3}$$

$$g_0(t) = g(t,0) = \lim_{p \to 0^+} g(t,p) \quad \text{for } t \in J. \tag{2.4}$$

For example, by substituting $f(t,p) = t^p$ for $t > 0$, $0 < p \le p_0 = 1$, we get $f_0(t) = 1$, $g(t,p) = \ln_p(t)$ and $g_0(t) = \log t$.

Lemma 2.1. *Let $f(t,p)$ be a real function of two variables $t \in J$ and $p \in P = (0, p_0]$, $0 < p_0 \le 1$, with an interval $J \subset (0,\infty)$. Assume $f(t,0) = 1$, $t \in J$. For $n \times n$ positive definite matrices A and B with spectrum* $\mathrm{Sp}\,(A^{-1/2}BA^{-1/2}) \subset J$, *the following identity holds:*

$$S_{g_p}(A,B) = \frac{S_{f_p}(A,B) - A}{p} \quad \textit{for } p \in P, \tag{2.5}$$

where f_p and g_p are defined by (2.1)–(2.2).

Proof. By (1.12) and (2.2) we establish the equalities

$$\begin{aligned} \frac{S_{f_p}(A,B) - A}{p} &= \frac{A\sigma_{f_p}B - A}{p} \\ &= \frac{A^{1/2}f_p(A^{-1/2}BA^{-1/2})A^{1/2} - A^{1/2}IA^{1/2}}{p} \\ &= A^{1/2}\frac{f_p(A^{-1/2}BA^{-1/2}) - I}{p}A^{1/2}. \end{aligned} \tag{2.6}$$

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Denoting $Z = A^{-1/2}BA^{-1/2}$ and using spectral decomposition of Z, we obtain

$$Z = U^* \mathrm{diag}\,(\mu_1, \mu_2, \dots, \mu_n)U$$

for some $n \times n$ unitary matrix U (i.e., $U^*U = UU^* = I$) with the eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ of Z. Thus, we get

$$\begin{aligned} f_p(Z) = f_p(A^{-1/2}BA^{-1/2}) &= U^* \mathrm{diag}\,(f_p(\mu_1), f_p(\mu_2), \dots, f_p(\mu_n))U \\ &= U^* \mathrm{diag}\,(f(\mu_1, p), f(\mu_2, p), \dots, f(\mu_n, p))U. \end{aligned}$$

Therefore, from (2.6), we derive

$$\begin{aligned} \frac{S_{f_p}(A,B) - A}{p} &= A^{1/2} \frac{f_p(Z) - U^*U}{p} A^{1/2} \\ &= A^{1/2} \frac{U^* \mathrm{diag}\,(f(\mu_1, p), f(\mu_2, p), \dots, f(\mu_n, p))U - U^*IU}{p} A^{1/2} \\ &= A^{1/2} U^* \mathrm{diag}\left(\frac{f(\mu_1, p) - 1}{p}, \frac{f(\mu_2, p) - 1}{p}, \dots, \frac{f(\mu_n, p) - 1}{p} \right) U A^{1/2} \\ &= A^{1/2} U^* \mathrm{diag}\,(g(\mu_1, p), g(\mu_2, p), \dots, g(\mu_n, p)) U A^{1/2} \\ &= A^{1/2} g_p\,(U^* \mathrm{diag}\,(\mu_1, \mu_2, \dots, \mu_n)U) A^{1/2} \\ &= A^{1/2} g_p\,(A^{-1/2}BA^{-1/2}) A^{1/2} = A \sigma_{g_p} B = S_{g_p}(A,B). \end{aligned}$$

This proves (2.5). $\square$

In the forthcoming theorem, we extend Furuta's inequality (1.9) from the functions $t \to t^p$, $p \in (0,1]$, to positive operator monotone functions $t \to f_p(t)$ on $J = [m.M]$, $0 < m < M$.

THEOREM 2.2. *Let $f(t,p)$ be a real function of two variables $t \in J = [m,M]$ with $0 < m < M$, and $p \in P = (0, p_0]$ with $0 < p_0 \le 1$. Let $f(t,0) = 1$, $t \in J$. Assume that $f_p > 0$, $p \in P$, is operator monotone on J. Let A and B be $n \times n$ positive definite matrices such that $mA \le B \le MA$.*

If $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a strictly positive linear map, then

$$S_{g_p}(\Phi(A), \Phi(B)) \le \Phi(S_{g_p}(A,B)) + \frac{1 - c_{f_p}}{p} \Phi(A) \sigma_{f_p} \Phi(B), \tag{2.7}$$

where f_p and g_p, $p \in P$, are defined by (2.1) and (2.2), respectively, and $c_{f_p} = \min\limits_{t \in J} \frac{a_{f_p} t + b_{f_p}}{f_p(t)}$ with $a_{f_p} = \frac{f_p(M) - f_p(m)}{M - m}$ and $b_{f_p} = \frac{M f_p(m) - m f_p(M)}{M - m}$.

If in addition $\frac{1 - c_{f_p}}{p} \to d$ as $p \to 0$, then

$$S_{g_0}(\Phi(A), \Phi(B)) \le \Phi(S_{g_0}(A,B)) + d\, \Phi(A), \tag{2.8}$$

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where f_0 and g_0 are defined by (2.3) and (2.4), respectively.

Proof. It is not hard to verify that the assertion of [13, Corollary 3.4] can be extended to the case $0 < mA \leq B \leq MA$. In consequence, since $f_p > 0$ is operator monotone on J, the following inequality is met (cf. [13, Corollary 3.4]):

$$c_{f_p}\Phi(A)\sigma_{f_p}\Phi(B) \leq \Phi(A\sigma_{f_p}B). \tag{2.9}$$

In addition, $\Phi(A)\sigma_{f_0}\Phi(B) = \Phi(A)$, because $f_0 \equiv 1$. So, it follows from (2.5) and (2.9) that

$$\begin{aligned}\Phi(A)\sigma_{g_p}\Phi(B) - \frac{1-c_{f_p}}{p}\Phi(A)\sigma_{f_p}\Phi(B) &= \frac{c_{f_p}\Phi(A)\sigma_{f_p}\Phi(B) - \Phi(A)}{p} \\ &\leq \frac{\Phi(A\sigma_{f_p}B) - \Phi(A)}{p} \\ &= \Phi\left(\frac{A\sigma_{f_p}B - A}{p}\right) = \Phi(A\sigma_{g_p}B).\end{aligned}$$

Therefore, we have

$$\Phi(A)\sigma_{g_p}\Phi(B) \leq \Phi(A\sigma_{g_p}B) + \frac{1-c_{f_p}}{p}\Phi(A)\sigma_{f_p}\Phi(B). \tag{2.10}$$

Now, the inequality (2.7) can be deduced from (2.10) via (1.12).

By passing to the limit in (2.7) as $p \to 0$, we get $\Phi(A)\sigma_{f_p}\Phi(B) \to \Phi(A)\sigma_{f_0}\Phi(B)$, $A\sigma_{g_p}B \to A\sigma_{g_0}B$ and $\Phi(A)\sigma_{f_p}\Phi(B) \to \Phi(A)\sigma_{f_0}\Phi(B) = \Phi(A)$. Thus, (2.7) leads to (2.8). This completes the proof of Theorem 2.2. ☐

For $A > 0$, $B > 0$ and $p, q \geq 0$, $p + q \leq 1$, the (p, q)-*generalized relative operator entropy* is defined by

$$S_{p,q}(A, B) = A^{1/2}(A^{-1/2}BA^{-1/2})^p(\log(A^{-1/2}BA^{-1/2}))^q A^{1/2}. \tag{2.11}$$

Notice that for $q = 0$ one has $S_{p,q}(A, B) = A\sharp_p B$, and for $q = 1$ and $p = 0$, $S_{p,q}(A, B) = S(A, B)$.

It is worth emphasing that the function $J \ni t \to t^p(\log t)^q$, $p, q \geq 0$, $p + q \leq 1$, is operator monotone on any interval $J = [m, M]$, $1 < m < M$ (see [1, Corollary 2.7]).

Below we give an interpretation of statement (2.7) for the (p, q)-generalized relative operator entropy.

Corollary 2.3. *Let A and B be $n \times n$ positive definite matrices such that $mA \leq B \leq MA$, $1 < m < M$.*

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If $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a strictly positive linear map, then

$$S_{p,q}(\Phi(A),\Phi(B)) \leq \Phi(S_{p,q}(A,B)) + \frac{1-c_{f_{p,q}}}{p}\Phi(A)\sigma_{f_{p,q}}\Phi(B), \tag{2.12}$$

where $p, q \geq 0$, $p+q \leq 1$, and $S_{p,q}$ is the (p,q)-generalized relative operator entropy defined by (2.11), and $c_{f_{p,q}}$ is defined by (1.11).

Proof. Apply Theorem 2.2 to the functions $f_{p,q}(t) = pt^p(\log t)^q + 1$, $f_{0,q}(t) = 1$, and $g_{p,q}(t) = t^p(\log t)^q$, $t \in [m, M]$ with fixed q and $p \in [0, p_0]$, $p_0 = 1-q$. ◻

3. Extending Furuichi et al. and Zou's results. In this section, we develop some results due to Furuichi et al. [7] and Zou [19]. To do so, we involve star-shaped functions.

Remind that a real nonnegative function F on $[0, p_0)$, $0 < p_0 \leq \infty$, with $F(0) = 0$ is said to be *star-shaped* if $F(\alpha p) \leq \alpha F(p)$ for $p \in [0, p_0]$ and $0 \leq \alpha \leq 1$.

THEOREM 3.1. *With the definitions (2.1)–(2.4) for a real function $f(t,p)$ of two variables $t \in J \subset (0,\infty)$ with an interval J and $p \in P = [0,1]$, assume that for each $t \in J$ the function $p \to f(t,p) - f(t,0)$, $p \in P$, is positive and star-shaped. Let $\varphi : J \to J$, i.e., $\varphi(t) \in J$ for $t \in J$. Let A and B be $n \times n$ positive definite matrices such that the spectrum $\mathrm{Sp}\,(A^{-1/2}BA^{-1/2}) \subset J$. Then for any $p \in (0,1]$, the following two inequalities hold:*

$$S_{g_p}(A,B) \leq S_{g_1}(A, S_\varphi(A,B)) - S_{h_p}(A,B), \tag{3.1}$$

$$S_{g_0}(A,B) \leq S_{g_p}(A, S_\varphi(A,B)) - S_{h_0}(A,B), \tag{3.2}$$

where

$$h_p(t) = h(t,p) = g(\varphi(t),p) - g(t,p) \quad \textit{for } t \in J, \tag{3.3}$$

$$h_0(t) = h(t,0) = g(\varphi(t),0) - g(t,0) \quad \textit{for } t \in J. \tag{3.4}$$

Proof. The function $(0,1] \ni p \to \frac{f(t,p)-f(t,0)}{p} = g(t,p)$ is nondecreasing [3, Lemma 3], i.e.,

$$0 < p_1 \leq p_2 \leq 1 \quad \text{implies} \quad \frac{f(t,p_1)-f(t,0)}{p_1} \leq \frac{f(t,p_2)-f(t,0)}{p_2}.$$

Hence,

$$g(t,0) = \lim_{p_1 \to 0^+} \frac{f(t,p_1)-f(t,0)}{p_1} \leq \frac{f(t,p_2)-f(t,0)}{p_2} \quad \text{for any } 0 < p_2 \leq 1.$$

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Consequently, the following double inequality is valid:

$$g(t,0) \le g(t,p) \le g(t,1) \quad \text{for any } 0 < p \le 1. \tag{3.5}$$

To prove (3.1), we employ the inequality $g(t,p) \le g(t,1)$ for $t \in J$, $0 < p \le 1$ (see (3.5)). Since $\varphi(t) \in J$ for $t \in J$, we obtain

$$g(\varphi(t),p) \le g(\varphi(t),1) \quad \text{for } t \in J,$$

or, equivalently,

$$g(t,p) \le g(\varphi(t),1) - [g(\varphi(t),p) - g(t,p)] \quad \text{for } t \in J.$$

So, by (3.3), we find that

$$g(t,p) \le g(\varphi(t),1) - h(t,p) \quad \text{for } t \in J.$$

In other words, we have

$$g_p(t) \le g_1(\varphi(t)) - h_p(t) \quad \text{for } t \in J. \tag{3.6}$$

By denoting $Z = A^{-1/2}BA^{-1/2}$ and making use of (3.6), we get

$$g_p(Z) \le g_1(\varphi(Z)) - h_p(Z).$$

Hence,

$$A^{1/2}g_p(Z)A^{1/2} \le A^{1/2}g_1(\varphi(Z))A^{1/2} - A^{1/2}h_p(Z)A^{1/2},$$

which means

$$A\sigma_{g_p}B \le A\sigma_{g_1\circ\varphi}B - A\sigma_{h_p}B. \tag{3.7}$$

However, we can show that

$$A\sigma_{g_1\circ\varphi}B = A\sigma_{g_1}(A\sigma_\varphi B). \tag{3.8}$$

Indeed, by using (1.1), we derive

$$\begin{aligned} A\sigma_{g_1\circ\varphi}B &= A^{1/2}(g_1\circ\varphi)(A^{-1/2}BA^{-1/2})A^{1/2} = A^{1/2}g_1(\varphi(A^{-1/2}BA^{-1/2}))A^{1/2} \\ &= A^{1/2}g_1(A^{-1/2}A^{1/2}\varphi(A^{-1/2}BA^{-1/2})A^{1/2}A^{-1/2})A^{1/2} \\ &= A^{1/2}g_1(A^{-1/2}(A\sigma_\varphi B)A^{-1/2})A^{1/2} = A\sigma_{g_1}(A\sigma_\varphi B), \end{aligned}$$

completing the proof of (3.8).

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So, by virtue of (3.7)–(3.8), we infer that

$$S_{g_p}(A,B) \le S_{g_1}(A, A\sigma_\varphi B) - S_{h_p}(A,B),$$

which proves (3.1).

We shall show (3.2). According to the inequality $g(t,0) \le g(t,p)$ for $t \in J$, $0 < p \le 1$ (see (3.5)), we get

$$g(\varphi(t),0) \le g(\varphi(t),p) \quad \text{for } t \in J,$$

because $\varphi(t) \in J$ for $t \in J$. So, by (3.4), we have

$$g(t,0) \le g(t,p) - [g(t,p) - g(\varphi(t),p)] - h(t,0) \quad \text{for } t \in J,$$

which means

$$g_0(t) \le g_p(t) - [g_p(t) - g_p(\varphi(t))] - h_0(t) \quad \text{for } t \in J. \tag{3.9}$$

With the notation $Z = A^{-1/2}BA^{-1/2}$, inequality (3.9) gives

$$g_0(Z) \le g_p(Z) - [g_p(Z) - g_p(\varphi(Z))] - h_0(Z).$$

Next, by pre- and post-multiplying by $A^{1/2}$ we obtain

$$A^{1/2}g_0(Z)A^{1/2} \le A^{1/2}g_p(Z)A^{1/2} - [A^{1/2}g_p(Z)A^{1/2} - A^{1/2}g_p(\varphi(Z))A^{1/2}] - A^{1/2}h_0(Z)A^{1/2}.$$

This amounts to

$$A\sigma_{g_0}B \le A\sigma_{g_p}B - [A\sigma_{g_p}B - A\sigma_{g_p\circ\varphi}B] - A\sigma_{h_0}B. \tag{3.10}$$

Similarly as in (3.8), we have

$$A\sigma_{g_p\circ\varphi}B = A\sigma_{g_p}(A\sigma_\varphi B). \tag{3.11}$$

Therefore, (3.10)–(3.11) lead to

$$S_{g_0}(A,B) \le S_{g_p}(A,B) - [S_{g_p}(A,B) - S_{g_p}(A, A\sigma_\varphi B)] - S_{h_0}(A,B),$$

completing the proof of (3.2). □

Remark 3.2. According to [3, Theorem 5], Theorem 3.1 remains valid if the star-shapedness of the function $p \to f(t,p) - f(t,0)$, is replaced by convexity or convexity on the average.

Remark 3.3. **(i)**. It is not hard to verify that Theorem 3.1, Eq. (3.1), reduces to Theorem A, with the following specification

$$f_p(t) = t^p\ , \quad g_p(t) = \ln_p t = \frac{t^p - 1}{p}\ , \quad g_1(t) = t - 1\ , \quad \varphi(t) = \frac{t}{a}\ ,\ a > 0.$$

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(ii). Likewise, Theorem 3.1, Eq. (3.2), becomes Theorem B, whenever

$$f_p(t) = t^p \ , \quad g_0(t) = \log t \ , \quad g_p(t) = \frac{t^p - 1}{p} \ , \quad \varphi(t) = at \ , \ a > 0.$$

In the next corollary, we provide analogs of Theorem A and Theorem B for the generalized relative operator entropy defined by (1.5).

Corollary 3.4. *Let A and B be $n \times n$ positive definite matrices such that the spectrum* $\mathrm{Sp}\,(A^{-1/2}BA^{-1/2}) \subset (1,\infty)$. *Then for any $p \in P = (0,1]$ and $a \geq 1$, the following two inequalities hold:*

$$S_p(A,B) \leq a^{1-p}(\log a)B + a^{1-p}S_1(A,B) - (\log a)A\sharp_p B, \tag{3.12}$$

$$a^{-p}S(A,B) + (a^{-p}\log a)A - (\log a)A\sharp_p B \leq S_p(A,B), \tag{3.13}$$

where S_p is the generalized relative operator entropy defined by (1.5), and S is the relative operator entropy defined by (1.4).

Proof. We apply Theorem 3.1 to the functions $f(t,p) = pt^p \log t$, $f(t,0) = 0$, $g_p(t) = g(t,p) = t^p \log t$, $g_0(t) = g(t,0) = \log t$, and $\varphi(t) = at$, $a \geq 1$, for $t \in J = (1,\infty)$ and $p \in (0,1]$. So, it is easily seen that $S_\varphi(A,B) = aB$.

Next, we shall show the identity

$$S_q(A,aB) = (a^q \log a)A\sharp_q B + a^q S_q(A,B) \quad \text{for } q \in (0,1]. \tag{3.14}$$

Indeed, we have

$$S_q(A,aB) = A^{1/2}g_q(A^{-1/2}(aB)A^{-1/2})A^{1/2} = A^{1/2}g_q(aA^{-1/2}BA^{-1/2})A^{1/2}.$$

By denoting $Z = A^{-1/2}BA^{-1/2}$, we write $Z = U^*(\mathrm{diag}\,\mu_i)U$ with unitary U and the eigenvalues μ_i, $i = 1,\dots,n$, of Z. Hence,

$$\begin{aligned} g_q(aZ) &= g_q(U^*(\mathrm{diag}\,a\mu_i)U) = U^*(\mathrm{diag}\,g_q(a\mu_i))U \\ &= U^*(\mathrm{diag}\,((a\mu_i)^q \log(a\mu_i)))U \\ &= U^*(\mathrm{diag}\,(a^q\mu_i^q \log a + a^q\mu_i^q \log \mu_i))U \\ &= U^*(\mathrm{diag}\,((a^q \log a)\mu_i^q))U + U^*(\mathrm{diag}\,(a^q\mu_i^q \log \mu_i))U \\ &= (a^q \log a)U^*(\mathrm{diag}\,\mu_i^q)U + a^qU^*(\mathrm{diag}\,(\mu_i^q \log \mu_i))U \\ &= (a^q \log a)Z^q + a^q g_q(Z). \end{aligned}$$

By pre- and post-multiplying by $A^{1/2}$ we obtain

$$\begin{aligned} S_q(A,aB) = S_{g_q}(A,aB) &= A^{1/2}g_q(aZ)A^{1/2} \\ &= (a^q \log a)A^{1/2}Z^qA^{1/2} + a^qA^{1/2}g_q(Z)A^{1/2} \\ &= (a^q \log a)A\sharp_q B + a^q S_q(A,B), \end{aligned}$$

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completing the proof of (3.14).

(i). In order to prove (3.12), we derive

$$\begin{aligned} h_p(t) = h(t,p) &= g(\varphi(t),p) - g(t,p) \\ &= (at)^p \log(at) - t^p \log t \\ &= (a^p \log a)t^p + (a^p - 1)t^p \log t \quad \text{for } t \in J. \end{aligned}$$

For this reason,

$$S_{h_p}(A,B) = (a^p \log a)A\sharp_p B + (a^p - 1)S_p(A,B).$$

By employing (3.14) with $q = 1$, we establish

$$S_1(A,aB) = (a \log a)B + aS_1(A,B). \tag{3.15}$$

In fact, for $q = 1$, we have $A\sharp_q B = B$. Consequently, (3.14) quarantees (3.15).

Now, by utilizing (3.1) we conclude that

$$S_p(A,B) \le (a \log a)B + aS_1(A,B) - (a^p \log a)A\sharp_p B - (a^p - 1)S_p(A,B),$$

which gives (3.12).

(ii). We shall show (3.13). By virtue of (3.2) we get

$$S(A,B) \le S_p(A,aB) - S_{h_0}(A,B). \tag{3.16}$$

Putting $q = p$ into (3.14) yields

$$S_p(A,aB) = (a^p \log a)A\sharp_p B + a^p S_p(A,B).$$

It is obvious that

$$h_0(t) = h(t,0) = g(at,0) - g(t,0) = \log(at) - \log t = \log a \quad \text{for } t \in J.$$

Hence, $S_{h_0}(A,B) = (\log a)A$.

It now follows from (3.16) that

$$S(A,B) \le a^p S_p(A,B) + (a^p \log a)A\sharp_p B - (\log a)A.$$

Thus we obtain (3.13), as desired. □

We are now in a position to show a complement to Furuta type inequality (2.7).

THEOREM 3.5. *With the definitions (2.1)–(2.4) for a real function $f(t,p)$ of two variables $t \in J = (0,\infty)$ and $p \in P = (0,1]$, assume that for each $t \in J$ the function*

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$p \to f(t,p) - f(t,0)$, $p \in P$, is positive and star-shaped. Let $\varphi : J \to J$ be such that $\varphi(t) = at \in J$, $a > 0$, for $t \in J$. Suppose that $g_1(t) = \alpha t + \beta$, $\alpha > 0$, is an affine function, and that g_2 is an concave function with its chord function $t \to a_{g_2}t + b_{g_2}$, $t \in J$, $a_{g_2} > 0$ (see (1.11)). Let A and B be $n \times n$ positive definite matrices.

If $\Phi : \mathbb{M}_n(\mathbb{C}) \to \mathbb{M}_k(\mathbb{C})$ is a strictly positive linear map, then for any $p \in P$,

$$S_{g_p}(\Phi(A), \Phi(B)) \leq \frac{\alpha a}{a_{g_2}} \Phi(S_{g_2}(A,B)) - S_{h_p}(\Phi(A), \Phi(B)) + \left(\beta - \alpha a \frac{b_{g_2}}{a_{g_2}}\right) \Phi(A), \tag{3.17}$$

where $h_p(t) = h(t,p) = g(at,p) - g(t,p)$ for $t \in J$.

Proof. As in the proof of Theorem 3.1 (see (3.5)), we have $g_p(t) \leq g_1(t)$ for $t \in J$, and

$$g_p(t) \leq g_1(at) - h_p(t) = \alpha at + \beta - h_p(t) \quad \text{for } t \in J.$$

Since g_2 is concave with its chord function $t \to a_{g_2}t + b_{g_2}$, $t \in J$, $a_{g_2} > 0$ (see (1.11)), we get

$$a_{g_2}t + b_{g_2} \leq g_2(t) \quad \text{for } t \in J.$$

By using the last two inequalities, we obtain

$$g_p(t) + h_p(t) - \beta \leq \alpha\varphi(t) = \alpha at = \frac{\alpha a}{a_{g_2}} a_{g_2} t \leq \frac{\alpha a}{a_{g_2}} \left(g_2(t) - b_{g_2}\right) \quad \text{for } t \in J.$$

In consequence, for $Z = A^{-1/2}BA^{-1/2}$ and $W = C^{-1/2}DC^{-1/2}$ with $C = \Phi(A)$ and $D = \Phi(B)$, we find that

$$\begin{aligned} & g_p(W) + h_p(W) - \beta I \leq \alpha a W, \\ & \alpha a Z \leq \frac{\alpha a}{a_{g_2}} \left(g_2(Z) - b_{g_2} I\right). \end{aligned} \tag{3.18}$$

Thus, we obtain

$$\begin{aligned} & C^{1/2} g_p(W) C^{1/2} + C^{1/2} h_p(W) C^{1/2} - \beta C \leq \alpha a C^{1/2} W C^{1/2}, \\ & \alpha a A^{1/2} Z A^{1/2} \leq \frac{\alpha a}{a_{g_2}} \left(A^{1/2} g_2(Z) A^{1/2} - b_{g_2} A\right). \end{aligned}$$

That is,

$$\begin{aligned} & C\sigma_{g_p} D + C\sigma_{h_p} D - \beta C \leq \alpha a D, \\ & \alpha a B \leq \frac{\alpha a}{a_{g_2}} \left(A\sigma_{g_2} B - b_{g_2} A\right). \end{aligned} \tag{3.19}$$

Hence,

$$\alpha a \Phi(B) \leq \frac{\alpha a}{a_{g_2}} \left(\Phi(A\sigma_{g_2} B) - b_{g_2} \Phi(A)\right). \tag{3.20}$$

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But (3.19) can be rewritten as

$$\Phi(A)\sigma_{g_p}\Phi(B)+\Phi(A)\sigma_{h_p}\Phi(B)-\beta\Phi(A)\le\alpha a\Phi(B). \tag{3.21}$$

Now, by combining (3.21) and (3.20), we establish

$$\Phi(A)\sigma_{g_p}\Phi(B)+\Phi(A)\sigma_{h_p}\Phi(B)-\beta\Phi(A)\le\frac{\alpha a}{a_{g_2}}\left(\Phi(A\sigma_{g_2}B)-b_{g_2}\Phi(A)\right).$$

So, we infer that

$$\Phi(A)\sigma_{g_p}\Phi(B)\le\frac{\alpha a}{a_{g_2}}\Phi(A\sigma_{g_2}B)-\Phi(A)\sigma_{h_p}\Phi(B)+\left(\beta-\alpha a\frac{b_{g_2}}{a_{g_2}}\right)\Phi(A),$$

which is equivalent to (3.17). □

Corollary 3.6. *With the assumptions of Theorem 3.5, if in addition $g_2=g_1$ then (3.17) reduces to*

$$S_{g_p}(\Phi(A),\Phi(B))\le a\Phi(S_{g_1}(A,B))-S_{h_p}(\Phi(A),\Phi(B))+\beta\,(1-a)\,\Phi(A). \tag{3.22}$$

If additionally Φ is the identity, then (3.22) yields

$$S_{g_p}(A,B)\le aS_{g_1}(A,B)-S_{h_p}(A,B)+\beta\,(1-a)\,A. \tag{3.23}$$

Remark 3.7. By letting $\alpha=1$, $\beta=-1$ and

$$f_p(t)=t^p\ ,\quad g_p(t)=\ln_p t=\frac{t^p-1}{p}\ ,\quad \varphi(t)=\frac{t}{a}\ ,\quad h_p(t)=t^p\ln_p\frac{1}{a}\ ,\ 1\ge a>0,$$

inequality (3.23) becomes the result (1.7) due to Furuichi et al. (see Theorem A).

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