Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 821-826, December 2014

ELA

A DETERMINANTAL INEQUALITY FOR POSITIVE DEFINITE MATRICES*

MINGHUA LIN†

Abstract. Let A, B, C be $n \times n$ positive semidefinite matrices. It is known that

$$\det(A+B+C)+\det C \geq \det(A+C)+\det(B+C),$$

which includes

$$\det(A+B) \geq \det A+\det B$$

as a special case. In this article, a relation between these two inequalities is proved, namely,

$$\det(A+B+C)+\det C-(\det(A+C)+\det(B+C)) \geq \det(A+B)-(\det A+\det B).$$

Key words. Determinantal inequality, Positive definite matrix.

AMS subject classifications. 15A45.

1. Introduction. Let A, B be $n \times n$ positive semidefinite matrices. It is well known that

$$\det(A+B) \geq \det A+\det B. \tag{1.1}$$

There are many generalizations and extensions of (1.1) in the literature. For example, (1.1) is a simple consequence of the Minkowski inequality [3, p. 510]:

$$(\det(A+B))^{1/n} \geq (\det A)^{1/n}+(\det B)^{1/n}.$$

Haynsworth [2] and later Hartfiel [1] obtained a refinement of (1.1) as follows:

$$\begin{aligned}\det(A+B) \geq &\left(1+\sum_{k=1}^{n-1}\frac{\det B_k}{\det A_k}\right)\det A+\left(1+\sum_{k=1}^{n-1}\frac{\det A_k}{\det B_k}\right)\det B\\ &+(2^n-2n)\sqrt{\det AB},\end{aligned}$$

where $A_k, B_k,\ k=1,\ldots,n-1$, denote the k-th leading principal submatrices of positive definite matrices A and B, respectively.

*Received by the editors on July 3, 2014. Accepted for publication on October 19, 2014. Handling Editor: Bryan L. Shader.

†Department of Mathematics and Statistics, University of Victoria, Victoria, BC, V8W 2Y2, Canada (mlin87@ymail.com). Research supported by a PIMS postdoctoral fellowship.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 821-826, December 2014

ELA

Another attractive extension of (1.1) is the following (assuming C is also an $n\times n$ positive semidefinite matrix)

$$\det(A+B+C)+\det C \geq \det(A+C)+\det(B+C). \tag{1.2}$$

Inequality (1.2) can be found e.g., in [8, p. 215, Problem 36].

Setting $C=0$, (1.2) reduces to (1.1). But that is not the only relation between (1.1) and (1.2). In this article, we shall reveal one more connection between them. That is, the difference in (1.1) is dominated by the difference in (1.2). More precisely, we have the following result.

Theorem 1.1. *Let A,B,C be $n\times n$ positive semidefinite matrices. Then*

$$\begin{aligned}\det(A+B+C)+\det C-(\det(A+C)+\det(B+C))&\\ \geq \det(A+B)-(\det A+\det B).&\end{aligned} \tag{1.3}$$

Inequality (1.3), to some extent, can be regarded as an analogue of the classical Hlawka's inequality ([6, p. 171]): Let $\mathcal{V}$ be an inner product space and let $x,y,z\in\mathcal{V}$. Then

$$\|x+y+z\|+\|x\|+\|y\|+\|z\| \geq \|x+y\|+\|y+z\|+\|z+x\|,$$

where the norm $\|\cdot\|$ denotes the norm induced by the inner product.

2. Auxiliary results and proofs. The first lemma coincides with Theorem 1.1 in the case where A,B,C are diagonal.

Lemma 2.1. *Let $a_i,b_i,c_i\geq 0$, $i=1,\ldots,n$. Then*

$$\begin{aligned}\prod_{i=1}^n(a_i+b_i+c_i)+\prod_{i=1}^n c_i-\left(\prod_{i=1}^n(a_i+c_i)+\prod_{i=1}^n(b_i+c_i)\right)&\\ \geq \prod_{i=1}^n(a_i+b_i)-\left(\prod_{i=1}^n a_i+\prod_{i=1}^n b_i\right).&\end{aligned} \tag{2.1}$$

Proof. There is no loss of generality to assume $c_i>0$, $i=1,\ldots,n$. Firstly, we show a special case of (2.1) by assuming $c_i=1$, $i=1,\ldots,n$. The proof is by induction. The base case $n=1$ is trivial. Assume that for $n=m\geq 2$, it holds that

$$\begin{aligned}\prod_{i=1}^m(a_i+b_i+1)+1-\left(\prod_{i=1}^m(a_i+1)+\prod_{i=1}^m(b_i+1)\right)&\\ \geq \prod_{i=1}^m(a_i+b_i)-\left(\prod_{i=1}^m a_i+\prod_{i=1}^m b_i\right).&\end{aligned}$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 821-826, December 2014

ELA

For $n = m + 1$, compute

$$\begin{aligned}
\prod_{i=1}^{m+1}(a_i + b_i + 1) &= (a_{m+1} + b_{m+1} + 1)\prod_{i=1}^{m}(a_i + b_i + 1) \\
&\geq (a_{m+1} + b_{m+1} + 1) \\
&\quad \times \left(\prod_{i=1}^{m}(a_i + b_i) + \prod_{i=1}^{m}(a_i + 1) + \prod_{i=1}^{m}(b_i + 1) - \prod_{i=1}^{m} a_i - \prod_{i=1}^{m} b_i - 1\right) \\
&= \prod_{i=1}^{m+1}(a_i + b_i) + \prod_{i=1}^{m+1}(a_i + 1) + \prod_{i=1}^{m+1}(b_i + 1) - \prod_{i=1}^{m+1} a_i - \prod_{i=1}^{m+1} b_i - 1 \\
&\quad + \left\{\prod_{i=1}^{m}(a_i + b_i) + b_{m+1}\prod_{i=1}^{m}(a_i + 1) + a_{m+1}\prod_{i=1}^{m}(b_i + 1)\right. \\
&\quad \left. -(a_{m+1} + 1)\prod_{i=1}^{m} b_i - (b_{m+1} + 1)\prod_{i=1}^{m} a_i - (a_{m+1} + b_{m+1})\right\} \\
&\geq \prod_{i=1}^{m+1}(a_i + b_i) + \prod_{i=1}^{m+1}(a_i + 1) + \prod_{i=1}^{m+1}(b_i + 1) - \prod_{i=1}^{m+1} a_i - \prod_{i=1}^{m+1} b_i - 1 \\
&\quad + \left\{\prod_{i=1}^{m} a_i + \prod_{i=1}^{m} b_i + b_{m+1}\Big(1 + \prod_{i=1}^{m} a_i\Big) + a_{m+1}\Big(1 + \prod_{i=1}^{m} b_i\Big)\right. \\
&\quad \left. -(a_{m+1} + 1)\prod_{i=1}^{m} b_i - (b_{m+1} + 1)\prod_{i=1}^{m} a_i - (a_{m+1} + b_{m+1})\right\} \\
&= \prod_{i=1}^{m+1}(a_i + b_i) + \prod_{i=1}^{m+1}(a_i + 1) + \prod_{i=1}^{m+1}(b_i + 1) - \prod_{i=1}^{m+1} a_i - \prod_{i=1}^{m+1} b_i - 1.
\end{aligned}$$

We have thus proved

$$\begin{aligned}
\prod_{i=1}^{n}(\widehat{a}_i + \widehat{b}_i + 1) + 1 - \left(\prod_{i=1}^{n}(\widehat{a}_i + 1) + \prod_{i=1}^{n}(\widehat{b}_i + 1)\right)& \\
\geq \prod_{i=1}^{n}(\widehat{a}_i + \widehat{b}_i) - \left(\prod_{i=1}^{n}\widehat{a}_i + \prod_{i=1}^{n}\widehat{b}_i\right)&
\end{aligned} \tag{2.2}$$

for any $\widehat{a}_i, \widehat{b}_i \geq 0$, $i = 1, \ldots, n$.

The general case follows from (2.2) by taking $\widehat{a}_i = \dfrac{a_i}{c_i}$, $\widehat{b}_i = \dfrac{b_i}{c_i}$, $i = 1, \ldots, n$, and then multiplying both sides of (2.2) with $\prod_{i=1}^{n} c_i$. $\square$

For a vector $x \in \mathbb{R}^n$, we denote by $x^{\downarrow} = (x_1^{\downarrow}, \ldots, x_n^{\downarrow}) \in \mathbb{R}^n$ the vector with the same components as x, but sorted in descending order. Given $x, y \in \mathbb{R}^n$, we say that

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 821-826, December 2014

ELA

x majorizes y, written as $x \succ y$, if

$$\sum_{i=1}^{k} x_i^{\downarrow} \geq \sum_{i=1}^{k} y_i^{\downarrow} \quad \text{for } k = 1, \ldots, n-1,$$

and the equality holds at $k = n$.

A real-valued function ϕ defined on a set $\mathcal{A} \subset \mathbb{R}^n$ is said to be Schur-convex on $\mathcal{A}$ if

$$x \succ y \ \text{ on } \ \mathcal{A} \Longrightarrow \phi(x) \geq \phi(y).$$

ϕ is Schur-concave if $-\phi$ is Schur-convex.

Let $\mathcal{I} \subset \mathbb{R}$ be an open interval and let $\phi : \mathcal{I}^n \to \mathbb{R}$ be continuously differentiable. The well known Schur's condition ([4, p. 84]) says that ϕ is Schur-convex on $\mathcal{I}^n$ if and only if ϕ is symmetric on $\mathcal{I}^n$ and

$$(x_1 - x_2)\left(\frac{\partial \phi}{\partial x_1} - \frac{\partial \phi}{\partial x_2}\right) \geq 0.$$

Lemma 2.2. *The function $f(x) = \prod_{i=1}^{n}(1+x_i) - \prod_{i=1}^{n} x_i$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$, is Schur concave.*

Proof. Clearly, $f(x)$ is symmetric. Moreover,

$$\begin{aligned}
&(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \\
&= (x_1 - x_2)\left(\left(\frac{1}{1+x_1} - \frac{1}{1+x_2}\right)\prod_{i=1}^{n}(1+x_i) - \left(\frac{1}{x_1} - \frac{1}{x_2}\right)\prod_{i=1}^{n} x_i\right) \\
&= -(x_1 - x_2)^2\left(\prod_{i=3}^{n}(1+x_i) - \prod_{i=3}^{n} x_i\right) \leq 0. \qquad \square
\end{aligned}$$

We also need a classical result of Fan. For an $n \times n$ Hermitian matrix X, we denote the vector of eigenvalues of X by $\lambda(X) = (\lambda_1(X), \ldots, \lambda_n(X))$ with $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$.

Lemma 2.3. [8, p. 356] *Let X, Y be $n \times n$ Hermitian matrices. Then*

$$\lambda(X) + \lambda(Y) \succ \lambda(X + Y).$$

Now we are ready to present:

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 821-826, December 2014

ELA

Proof of Theorem 1.1. By a standard continuity argument, we may assume C to be positive definite. We may further assume $C = I_n$ (the $n \times n$ identity matrix) by pre- and post-multiplying both sides of (1.3) with $\det C^{-1/2}$.

Thus, we need to show

$$\begin{aligned}\det(A+B+I_n) + 1 - (\det(A+I_n) + \det(B+I_n))\\ \geq \det(A+B) - (\det A + \det B).\end{aligned}$$

Compute

$$\begin{aligned}&\det(A+B+I_n) - \det(A+B)\\ &= \prod_{i=1}^n \lambda_i(A+B+I_n) - \prod_{i=1}^n \lambda_i(A+B)\\ &= \prod_{i=1}^n \Big(1+\lambda_i(A+B)\Big) - \prod_{i=1}^n \lambda_i(A+B)\\ &\geq \prod_{i=1}^n \Big(1+\lambda_i(A)+\lambda_i(B)\Big) - \prod_{i=1}^n \Big(\lambda_i(A)+\lambda_i(B)\Big)\\ &\geq \prod_{i=1}^n \Big(1+\lambda_i(A)\Big) + \prod_{i=1}^n \Big(1+\lambda_i(B)\Big) - \prod_{i=1}^n \lambda_i(A) - \prod_{i=1}^n \lambda_i(B) - 1\\ &= \det(A+I_n) + \det(B+I_n) - (\det A + \det B + 1),\end{aligned}$$

in which the first inequality is by Lemma 2.2 and Lemma 2.3; and the second inequality is by Lemma 2.1. This completes the proof. ∎

3. Concluding remarks.

(i) Let G be a subgroup of the symmetric group S_n, and let χ be an irreducible character of G. The generalized matrix function (also known as immanant) afforded by G and χ is defined by

$$d_\chi^G(A) = \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^n a_{i\sigma(i)},$$

where $A = (a_{ij})$ is an $n \times n$ complex matrix.

If $G = S_n$ and χ is the signum function with values ± 1, then the generalized matrix function becomes the usual matrix determinant; setting $\chi \equiv 1$ defines the permanent of the matrix.

The following remarkable extension of (1.1) is known (e.g., [5, p. 228]): Let A, B be $n \times n$ positive semidefinite matrices. Then

$$d_\chi^G(A+B) \geq d_\chi^G(A) + d_\chi^G(B).$$

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 821-826, December 2014

Recently, Paksoy, Turkmen, and Zhang [7] showed that (1.2) also possessed such an extension (assuming C is also an $n \times n$ positive semidefinite matrix)

$$d_\chi^G(A+B+C) + d_\chi^G(C) \geq d_\chi^G(A+C) + d_\chi^G(B+C).$$

It is natural to ask whether Theorem 1.1 is true for any generalized matrix function.

(ii) Let $\mathcal{A} \subset \mathbb{C}^{n\times n}$, the set of all $n \times n$ complex matrices. A function $f : \mathcal{A} \to \mathbb{R}$ is superadditive if

$$f(A+B) \geq f(A) + f(B)$$

and strongly superadditive if

$$f(A+B+C) + f(C) \geq f(A+C) + f(B+C)$$

for all $A, B, C \in \mathcal{A}$.

Thus, Paksoy, Turkmen, and Zhang's result tells that the generalized matrix function is strongly superadditive on the cone of positive semidefinite matrices. A more general question would be: For what kind of strongly superadditive function f, one has

$$f(A+B+C) + f(C) - (f(A+C) + f(B+C)) \geq f(A+B) - (f(A) + f(B)),$$

where A, B, C are $n \times n$ positive semidefinite matrices. This of course deserves further investigation.

REFERENCES

[1] D.J. Hartfiel. An extension of Haynsworth's determinant inequality. *Proc. Amer. Math. Soc.*, 41:463–465, 1973.
[2] E.V. Haynsworth. Applications of an inequality for the Schur complemment. *Proc. Amer. Math. Soc.*, 24:512–516, 1970.
[3] R.A. Horn and C.R. Johnson. *Matrix Analysis*, second edition. Cambridge University Press, Cambridge, 2013.
[4] A.W. Marshall, I. Olkin, and B. Arnold. *Inequalities: Theory of Majorization and Its Applications*, second edition. Springer, New York, 2011.
[5] R. Merris. *Multilinear Algebra*. Gordon & Breach, Amsterdam, 1997.
[6] D.S. Mitrinović and P.M. Vasić. *Analytic Inequalities*. Springer-Verlag, Berlin, 1970.
[7] V. Paksoy, R. Turkmen, and F. Zhang. Inequalities of generalized matrix functions via tensor products. *Electron. J. Linear Algebra*, 27:332–341, 2014.
[8] F. Zhang. *Matrix Theory: Basic Results and Techniques*, second edition. Springer, New York, 2011.