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# ON THE RANK-$K$ NUMERICAL RANGE OF MATRIX POLYNOMIALS*

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**Abstract.** This article introduces the notion of the rank-$k$ numerical range $\Lambda_k(L)$ of a matrix polynomial $L(\lambda) = A_m\lambda^m + \cdots + A_1\lambda + A_0$, whose coefficients are $n \times n$ complex matrices. Also, geometric properties are obtained, including the relation to the ordinary numerical range $W(L)$.

**Key words.** Rank-$k$ numerical range, Matrix polynomials, Quantum error correction.

**AMS subject classifications.** 15A60, 15A90, 81P68.

**1. Introduction.** Let $\mathcal{M}_{n_1,n_2}(\mathbb{C})$ be the algebra of all $n_1 \times n_2$ complex matrices, where the case $n_1 = n_2 = n$ is specified by $\mathcal{M}_n$, and let

$$L(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + \cdots + A_1\lambda + A_0 \tag{1.1}$$

be a matrix polynomial with $\lambda \in \mathbb{C}$, $A_i \in \mathcal{M}_n$ $(i = 0, \ldots, m)$ and $A_m \neq 0$, $m \geq 1$. The set of all eigenvalues of $L(\lambda)$, i.e., the *spectrum* of $L(\lambda)$, is defined by

$$\sigma(L) = \{\lambda \in \mathbb{C} : \det L(\lambda) = 0\},$$

and the nonzero solution $x_0 \in \mathbb{C}^n$ of the equation $L(\lambda_0)x = 0$ with $\lambda_0 \in \sigma(L)$ is known as an eigenvector of $L(\lambda)$ associated to $\lambda_0$.

The study of matrix polynomials has attracted special interest, and especially, it has been proved to be very fruitful in many applications on differential equations, linear systems theory and factorization problems [1, 9, 11, 15]. Evenly, this theory has been extended to operator polynomials and analytic operator functions [7].

For a positive integer $k \in \{1, 2, \ldots, n\}$, we define the *rank-k numerical range* of $L(\lambda)$ as

$$\Lambda_k(L) = \{\lambda \in \mathbb{C} : PL(\lambda)P = 0_n \text{ for some } P \in \mathcal{P}_k\}, \tag{1.2}$$

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where $\mathcal{P}_k$ is the set of all orthogonal projections $P$ of $\mathbb{C}^n$ onto any k-dimensional subspace of $\mathbb{C}^n$. Since $P = QQ^*$ with $Q \in \mathcal{M}_{n,k}$ and $Q^*Q = I_k$, we may consider the equivalent definition

$$\Lambda_k(L) = \{\lambda \in \mathbb{C} : Q^*L(\lambda)Q = 0_k \text{ for some } Q \in \mathcal{M}_{n,k}(\mathbb{C})\,,\ Q^*Q = I_k\}. \tag{1.3}$$

In the case $k = 1$, the set reduces to the well known *numerical range* $W(L)$ of $L(\lambda)$ [9], that is,

$$\Lambda_1(L) \equiv W(L) = \{\lambda \in \mathbb{C} : x^*L(\lambda)x = 0 \text{ for some } x \in \mathbb{C}^n, x^*x = 1\}.$$

The set $\Lambda_k(L)$ in (1.2) (or (1.3)) is an interesting generalization of the numerical range $W(L)$, which is utilized in several problems of scientific and engineering applications such as overdamped vibration systems and stability theory [1, 9].

If we consider $L_A(\lambda) = I\lambda - A$, then clearly

$$\Lambda_k(L_A) \equiv \Lambda_k(A) = \{\lambda \in \mathbb{C} : Q^*AQ = \lambda I_k\,,\ Q \in \mathcal{M}_{n,k}(\mathbb{C}),\ Q^*Q = I_k\}, \tag{1.4}$$

namely, it coincides with the *rank-k numerical range* of $A \in \mathcal{M}_n$. The concept of higher rank numerical range of matrices has been studied extensively by Choi *et al* in [4, 5, 8, 16] and later by the authors in [2, 3]. We should note that for $k = 1$, $\Lambda_k(L_A)$ yields the classical *numerical range*

$$F(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

In Section 2, we investigate the non-emptyness of $\Lambda_k(L)$, and in Section 3, we concentrate on algebraic and geometric properties of the set. In particular, we give a description of the set through intersections of numerical ranges of all compressions of the matrix polynomial $L(\lambda)$ to $(n - k + 1)$-dimensional subspaces. This is an extension of an analogous expression for matrices presented in [2], which leads us to investigate the topology of $\Lambda_k(L)$ as well as its relationship with $\Lambda_k(C_L)$, where $C_L$ is the companion pencil of $L(\lambda)$. Further, in Section 4, a connection of the boundary points of $\Lambda_k(L)$ with respect to the boundary points of $W(L)$ is considered and the notion of sharp points is investigated.

**2. Non-emptyness of $\Lambda_k(L)$.** In the study of rank-$k$ numerical range of matrix polynomials $\Lambda_k(L)$, there is a significant difference between the case $k = 1$ and $k > 1$. When $k = 1$, for any unit vector $x \in \mathbb{C}^n$, $x^*L(\lambda)x$ is a usual polynomial with complex coefficients and always has roots. However, when $k > 1$, for an $n \times k$ isometry $Q = \begin{bmatrix} q_1 & \cdots & q_k \end{bmatrix}$, the elements of the matrix polynomial $Q^*L(\lambda)Q$, i.e., the $k^2$ scalar polynomials $q_i^*L(\lambda)q_j$, $i, j = 1, \ldots, k$, should have common roots.

In order to obtain that $\Lambda_k(L) \neq \emptyset$ for any matrix polynomial $L(\lambda) = \sum_{l=0}^{m} A_l\lambda^l$ with $A_m \neq 0$, we are led to the common roots of the $k^2 > 1$ scalar polynomials

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$b_{ij}(\lambda, Q) = q_i^* L(\lambda) q_j$, $i, j = 1, \ldots, k$, for some isometries $Q = \begin{bmatrix} q_1 & \cdots & q_k \end{bmatrix} \in \mathcal{M}_{n,k}$. Adapting the notion of the Sylvester matrix $R_s$ appeared in [12] and the discussion therein to the polynomials

$$\begin{aligned} b_{ij}(\lambda, Q) &= q_i^* A_m q_j \lambda^m + \cdots + q_i^* A_l q_j \lambda^l + \cdots + q_i^* A_0 q_j \\ &= b_{ij}^{(m)}(Q)\lambda^m + \cdots + b_{ij}^{(l)}(Q)\lambda^l + \cdots + b_{ij}^{(0)}(Q) \end{aligned} \tag{2.1}$$

for all $i, j = 1, \ldots, k$ and for some $n \times k$ isometry $Q = \begin{bmatrix} q_1 & \cdots & q_k \end{bmatrix}$, we have a condition for the polynomials $b_{ij}(\lambda, Q)$ to share polynomial common factors. Denote by $\sigma \leq m$ the largest degree of the $k^2$ polynomials $b_{ij}(\lambda, Q)$, and let, as in (2.1),

$$b_{i_1,j_1}(\lambda, Q) = b_{i_1,j_1}^{(\sigma)}(Q)\lambda^\sigma + \cdots + b_{i_1,j_1}^{(l)}(Q)\lambda^l + \cdots + b_{i_1,j_1}^{(0)}(Q), \tag{2.2}$$

for some indices $i_1, j_1 \in \{1, \ldots, k\}$. If $\tau \leq \sigma$ is the largest degree of the remaining polynomials, then the generalized Sylvester matrix is

$$R_s(Q) = \begin{bmatrix} R_1(Q) \\ \vdots \\ R_{k^2}(Q) \end{bmatrix}, \tag{2.3}$$

where $R_1(Q)$ is the stripped $\tau \times (\sigma + \tau)$ matrix

$$R_1(Q) = \begin{bmatrix} b_{i_1,j_1}^{(\sigma)}(Q) & b_{i_1,j_1}^{(\sigma-1)}(Q) & & \cdots & b_{i_1,j_1}^{(0)}(Q) & & \mathbf{0} \\ & b_{i_1,j_1}^{(\sigma)}(Q) & b_{i_1,j_1}^{(\sigma-1)}(Q) & & & & \\ & \ddots & & \ddots & & \ddots & \\ \mathbf{0} & & b_{i_1,j_1}^{(\sigma)}(Q) & \cdots & b_{i_1,j_1}^{(\sigma-1)}(Q) & \cdots & b_{i_1,j_1}^{(0)}(Q) \end{bmatrix},$$

and for $p = 2, \ldots, k^2$, $R_p(Q)$ are the following $\sigma \times (\sigma + \tau)$ matrices

$$R_p(Q) = \begin{bmatrix} \mathbf{0} & & b_{i_p,j_p}^{(\tau)}(Q) & \cdot & \cdot & b_{i_p,j_p}^{(0)}(Q) \\ & b_{i_p,j_p}^{(\tau)}(Q) & & & & \\ \cdot & & \cdot & \cdot & \cdot & \\ b_{i_p,j_p}^{(\tau)}(Q) & \cdot & \cdot & b_{i_p,j_p}^{(0)}(Q) & \mathbf{0} & \end{bmatrix},$$

with $i_p, j_p \in \{1, \ldots, k\}$ and $i_p \neq i_1$, $j_p \neq j_1$. Hence, the degree $\delta(Q) \neq 0$ of the greatest common divisor of $b_{ij}(\lambda, Q)$ $(i, j = 1, \ldots, k)$ for some $n \times k$ isometry $Q$ satisfies the relation

$$\operatorname{rank} R_s(Q) = \tau + \sigma - \delta(Q) \leq 2m - \delta(Q), \tag{2.4}$$

and clearly, $\Lambda_k(L) \neq \emptyset$ if and only if there exists an $n \times k$ isometry $Q$ such that $\operatorname{rank} R_s(Q) < 2m$.

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In the remainder, we assume that such an isometry exists and $\Lambda_k(L)$ is non-empty.

Investigating the non-emptyness of $\Lambda_k(L)$, we notice that the necessary condition $n \geq 3k-2$ for $\Lambda_k(A) \neq \emptyset$ [8] (for a matrix $A \in \mathcal{M}_n$) fails in general for matrix polynomials. Particularly, if we consider $L(\lambda) = (\lambda - \lambda_0)^m A_m$, where $0 \notin \Lambda_k(A_m)$, then $\Lambda_k(L)$ appears to be non-empty for any $k \leq n$. Clearly, $Q^*L(\lambda)Q = (\lambda - \lambda_0)^m Q^* A_m Q$ for any $n \times k$ isometry $Q$, and due to $0 \notin \Lambda_k(A_m)$, we have that $Q^*A_mQ \neq 0_k$ for all $Q$, and then $\Lambda_k(L) = \{\lambda_0\} \neq \emptyset$.

**3. Geometric properties.** At the beginning of this section, we refer to some properties for $\Lambda_k(L)$, which for $k=1$, have been presented in [9].

PROPOSITION 3.1. *Let $L(\lambda)$ be an $n \times n$ matrix polynomial as in (1.1). Then the following hold:*

**I.** *$\Lambda_k(L)$ is closed in $\mathbb{C}$.*
**II.** *For any $\alpha \in \mathbb{C}$, $\Lambda_k(L(\lambda + \alpha)) = \Lambda_k(L) - \alpha$.*
**III.** *The following are equivalent:*
- **a.** *$\mu \in \Lambda_k(L)$,*
- **b.** *there exists an isometry $M \in \mathcal{M}_{n,k}(\mathbb{C})$ such that $M^*L(\mu)M = 0_k$,*
- **c.** *there exists a $k$-dimensional subspace $\mathcal{K}$ of $\mathbb{C}^n$ such that for any $v \in \mathcal{K}$, we have $v^*L(\mu)v = 0$,*
- **d.** *there exists a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that*

$$U^*L(\mu)U = \begin{bmatrix} 0_k & L_1(\mu) \\ L_2(\mu) & L_3(\mu) \end{bmatrix},$$

*where $L_1(\lambda), L_2(\lambda)$ and $L_3(\lambda)$ are suitable matrix polynomials.*

*Proof.* The arguments (IIIa)–(IIId) are equivalent, since $\mu \in \Lambda_k(L)$ if and only if $0 \in \Lambda_k(L(\mu))$, where $L(\mu)$ is a constant matrix. Thus, we refer to the definition (1.4) and Proposition 1.1 in [4]. □

PROPOSITION 3.2. *Let $L(\lambda)$ be an $n \times n$ matrix polynomial as in (1.1). Then the following hold:*

**I.** *$\Lambda_k(L) \subseteq \Lambda_{k-1}(L) \subseteq \cdots \subseteq \Lambda_1(L)$.*
**II.** *For any positive integer $k \leq n$, $\Lambda_k(\oplus_k L) = W(L)$, where $\oplus_k L$ denotes the direct sum $\underbrace{L \oplus \cdots \oplus L}_{k}$.*

*Proof.* **I.** For any $j \in \{2, \ldots, k\}$, let $\mu_0 \in \Lambda_j(L)$. Obviously, $0 \in \Lambda_j(L(\mu_0)) \subseteq \Lambda_{j-1}(L(\mu_0))$, and consequently, we conclude that $\mu_0 \in \Lambda_{j-1}(L)$.

**II.** Due to (I), $\mu_0 \in \Lambda_k(\oplus_k L) \subseteq W(\oplus_k L)$. Hence, $0 \in F(\oplus_k L(\mu_0)) = F(L(\mu_0))$ or $\mu_0 \in W(L)$, and then we obtain $\Lambda_k(\oplus_k L) \subseteq W(L)$. In addition, $\mu_0 \in W(L) \Rightarrow$

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$0 \in F(L(\mu_0)) \subseteq \Lambda_k(\oplus_k L(\mu_0))$, according to the relation (v) in [5]. Thus, $W(L) \subseteq \Lambda_k(\oplus_k L)$ and the proof is complete. □

The following result sketches the rank-$k$ numerical range of a *matrix* through numerical ranges [2].

Proposition 3.3. *Let $L_A(\lambda) = I\lambda - A$, with $A \in \mathcal{M}_n$. Then*

$$\Lambda_k(L_A) = \bigcap_M F(M^* AM),$$

*where $M$ is any $n \times (n-k+1)$ isometry.*

This expression provides a numerical estimation of $\Lambda_k(L_A)$ through the numerical ranges $F(M^*AM)$ and it also verifies its "*convexity*" in another way of that in [16]. For $k = n$, clearly $\Lambda_n(L_A) = \bigcap_{x \in \mathbb{C}^n, x^*x=1} F(x^*Ax)$ and hence, $\Lambda_n(L_A) \neq \emptyset$ *precisely* when $A$ is scalar.

A characterization of $\Lambda_k(L)$, extending the previous expression, is demonstrated in the next proposition.

Proposition 3.4. *Suppose $L(\lambda)$ is an $n \times n$ matrix polynomial as in (1.1). Then*

$$\Lambda_k(L) = \bigcap_M W(M^*LM) = \bigcup_N \Lambda_k(N^*LN),$$

*where $M \in \mathcal{M}_{n,n-k+1}(\mathbb{C})$, $N \in \mathcal{M}_{n,k}(\mathbb{C})$ are isometries.*

*Proof.* Obviously, by Proposition 3.3,

$$\mu_0 \in \Lambda_k(L) \Leftrightarrow 0 \in \Lambda_k(L(\mu_0)),$$

or equivalently,

$$0 \in \bigcap_M F(M^*L(\mu_0)M) \Leftrightarrow \mu_0 \in \bigcap_M W(M^*LM).$$

Considering the equation $\Lambda_k(A) = \bigcup_N \Lambda_k(N^*AN)$ [2], we have

$$\mu_0 \in \Lambda_k(L) \Leftrightarrow 0 \in \Lambda_k(L(\mu_0)),$$

or equivalently,

$$0 \in \bigcup_N \Lambda_k(N^*L(\mu_0)N) \Leftrightarrow \mu_0 \in \bigcup_N \Lambda_k(N^*LN). \quad \square$$

The first equality of Proposition 3.4 will be used to provide a numerical algorithm for a graphical estimation of the set $\Lambda_k(L)$ through the numerical ranges $W(M^*LM)$,

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giving an interesting illustration of what $\Lambda_k(L)$ for $k > 1$ approximately looks like (Fig. 3.1a). In addition, it will constitute a useful technical tool for proving many of our results. Next we apply Proposition 3.4 to derive a relation between the rank-$k$ numerical range of $L(\lambda)$ and its corresponding $mn \times mn$ companion pencil

$$C_L(\lambda) = \begin{bmatrix} I_n & 0 & 0 & \cdots & 0 \\ 0 & I_n & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & & 0 \\ 0 & & \cdots & & A_m \end{bmatrix} \lambda - \begin{bmatrix} 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & & I_n \\ -A_0 & & \cdots & & -A_{m-1} \end{bmatrix},$$

well known as *companion linearization* of $L(\lambda)$, since there exist suitable matrix polynomials $E(\lambda)$ and $F(\lambda)$ with constant nonzero determinants such that

$$\begin{bmatrix} L(\lambda) & 0 \\ 0 & I_{n(m-1)} \end{bmatrix} = E(\lambda) C_L(\lambda) F(\lambda).$$

PROPOSITION 3.5. *Let $L(\lambda)$ be an $n \times n$ matrix polynomial as in* (1.1). *Then* $\Lambda_k(L) \cup \{0\} \subseteq \Lambda_k(C_L)$.

*Proof.* By Proposition 3.4 and the relationship $W(L) \cup \{0\} \subseteq W(C_L)$ in [13], we have

$$\Lambda_k(L) \cup \{0\} = \left( \bigcap_M W(M^* L M) \right) \cup \{0\} \subseteq \bigcap_M W(C_{M^*LM}), \tag{3.1}$$

where $M \in \mathcal{M}_{n,n-k+1}(\mathbb{C})$ is an isometry and $C_{M^*LM}(\lambda)$ is the companion linearization of the matrix polynomial $M^* L(\lambda) M$. Since,

$$\begin{aligned} C_{M^*LM}(\lambda) &= (I_m \otimes M)^* \begin{bmatrix} \lambda I_n & -I_n & 0 & \cdots & 0 \\ 0 & \lambda I_n & -I_n & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & & -I_n \\ A_0 & & \cdots & & A_m\lambda + A_{m-1} \end{bmatrix} (I_m \otimes M) \\ &= (I_m \otimes M)^* C_L(\lambda) (I_m \otimes M), \end{aligned}$$

considering the expanded isometry $Q = \begin{bmatrix} I_m \otimes M & V \end{bmatrix} \in \mathcal{M}_{mn,mn-k+1}(\mathbb{C})$, we have

$$\begin{aligned} \bigcap_M W(C_{M^*LM}) &= \bigcap_M W((I_m \otimes M)^* C_L (I_m \otimes M)) \\ &\subseteq \bigcap_Q W(Q^* C_L Q) = \Lambda_k(C_L), \end{aligned} \tag{3.2}$$

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where the latter equality is confirmed by Theorem 2.2 in [2]. Thus, by (3.1) and (3.2), the proof is completed. $\square$

Following, we present a statement concerning the boundedness of $\Lambda_k(L)$.

Proposition 3.6. *Let $L(\lambda)$ be an $n \times n$ matrix polynomial as in (1.1). If $0 \notin \Lambda_k(A_m)$, then $\Lambda_k(L)$ is bounded.*

*Conversely, let $\operatorname{rank} R_s(Q) < 2m$ for all isometries $Q \in \mathcal{M}_{n,k}$ such that $Q^*A_mQ = zI_k$ $(z \in \mathbb{C}\backslash\{0\})$. If $\Lambda_k(A_m) \neq \{0\}$ and $\Lambda_k(L)$ is bounded, then $0 \notin \Lambda_k(A_m)$.*

*Proof.* For $\Lambda_k(L) \neq \emptyset$, let $\Lambda_{k-1}(L)$ be unbounded. If $0 \notin \Lambda_k(A_m)$, then by Proposition 3.3, there exists an $n\times(n-k+1)$ isometry $M_0$ such that $0 \notin F(M_0^*A_mM_0)$. Hence, $W(M_0^*LM_0)$ is bounded [9], and by Proposition 3.4, as $\Lambda_k(L) \subseteq W(M_0^*LM_0)$, we conclude that $\Lambda_k(L)$ is bounded.

For the converse, suppose that $\Lambda_k(A_m) \neq \{0\}$ and $\Lambda_k(L)$ is bounded. It is clear that $\Lambda_1(L)$ may be either a bounded or an unbounded set. If $\Lambda_1(L)$ is bounded, then $0 \notin \Lambda_1(A_m)$ [9], which infers $0 \notin \Lambda_k(A_m)$ for any $k > 1$. On the other hand, if $\Lambda_1(L)$ is unbounded, we consider $k_0 > 1$ to be the minimum positive integer such that $\Lambda_{k_0}(L)$ is bounded. Hence, it is enough to prove our argument for $k = k_0$, keeping in mind that the $k_0^2m \times 2m$ Sylvester matrix $R_s(Q)$ has rank less than $2m$ for any isometry $Q \in \mathcal{M}_{n,k_0}$ such that $Q^*A_mQ = zI_{k_0}$.

Assume that $0 \in \Lambda_{k_0}(A_m)$. Since $\Lambda_{k_0}(A_m)$ is a convex set not degenerating to the singleton $\{0\}$, we may find a nonzero sequence $\{z_\nu\} \subseteq \Lambda_{k_0}(A_m)$ such that $\lim_{\nu\to\infty} z_\nu = 0$. Consequently, a sequence of $n \times k_0$ isometries $\{Q_\nu\}$ which correspond to the points $z_\nu \in \Lambda_{k_0}(A_m)$, i.e., $Q_\nu^*A_mQ_\nu = z_\nu I_{k_0} \to 0_{k_0}$. Due to the compactness of the group of $n \times k_0$ isometries, there is a subsequence $\{Q_\rho\}$ of $\{Q_\nu\}$ such that $\lim_{\rho\to\infty} Q_\rho = Q_0$, with $Q_0 \in \mathcal{M}_{n,k_0}$ being an isometry. That is, eventually all terms of $\{Q_\rho\}$ are contained in any neighborhood $\mathcal{U}(Q_0)$ of $Q_0$ and should be $Q_\rho^*A_mQ_\rho = z_\rho I_{k_0}$, where $\{z_\rho\}$ is a subsequence of $\{z_\nu\}$. By continuity, $\lim_{\rho\to\infty} Q_\rho^*A_mQ_\rho = Q_0^*A_mQ_0 = 0_{k_0}$.

Moreover, there exists an index $j \neq m$ such that $Q_0^*A_jQ_0 \neq 0_{k_0}$ (otherwise, $\Lambda_{k_0}(L) \equiv \mathbb{C}$), whereupon $\|Q_\rho^*A_jQ_\rho\| > \varepsilon$ for some $\varepsilon > 0$ and sufficiently large $\rho$. Also, for the coefficients of $Q_\rho^*L(\lambda)Q_\rho = z_\rho(I_{k_0}\lambda^m + \cdots + \frac{1}{z_\rho}Q_\rho^*A_jQ_\rho\lambda^j + \cdots + \frac{1}{z_\rho}Q_\rho^*A_0Q_\rho)$, we have [6, Th.4.2]

$$(-1)^{j-m}\frac{1}{z_\rho}Q_\rho^*A_jQ_\rho = \sum_{1\le i_1<\cdots<i_s\le m} \widetilde{\lambda}_{i_1}\widetilde{\lambda}_{i_2}\cdots\widetilde{\lambda}_{i_s}I_{k_0},$$

where $\widetilde{\lambda}_{i_1}, \widetilde{\lambda}_{i_2}, \ldots, \widetilde{\lambda}_{i_s}$ are the roots of the equation $Q_\rho^*L(\lambda)Q_\rho = 0$, i.e., the common roots of the $k_0^2$ scalar polynomials, elements of the matrix $Q_\rho^*L(\lambda)Q_\rho$, which are always

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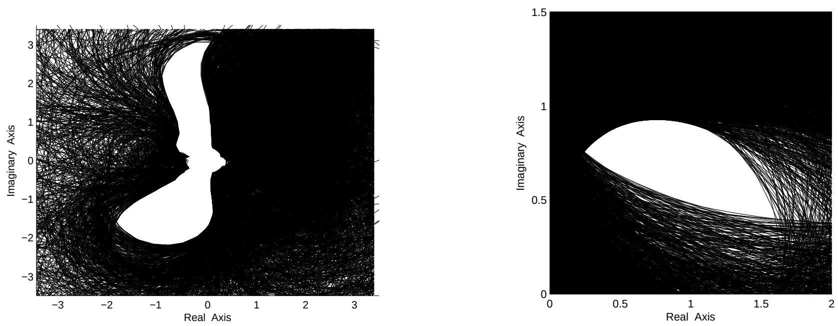
guaranteed by the condition of the Sylvester. Clearly, $\pm\frac{1}{z_\rho}Q_\rho^* A_j Q_\rho$ is not bounded as $\rho \to \infty$, concluding that $\Lambda_{k_0}(L)$ is not bounded. This contradicts the assumption, and the proof is complete. $\square$

Obviously, if $L(\lambda)$ is a monic matrix polynomial, then $\Lambda_k(L)$ is always bounded. Next, we present an illustrative example of Proposition 3.6.

EXAMPLE 3.7. Let the matrix polynomial $L(\lambda) = A_2\lambda^2 + A_1\lambda + A_0$, where

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{i} & 0 & 0 \\ 2 & \mathbf{i} & 0 & 2 \\ -\mathbf{i} & 0 & -2 & 8 \end{bmatrix},\ A_1 = \begin{bmatrix} \mathbf{i} & 2 & \mathbf{i} & 3 \\ 3 & 0 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ \mathbf{i} & 0 & \mathbf{i} & 0 \end{bmatrix} \text{ and } A_0 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \end{bmatrix}.$$

The unshaded area in Fig.3.1a approximates the set $\Lambda_2(L)$, which is bounded, although $\Lambda_1(L) = \mathbb{C}$. The set $\Lambda_2(A_2)$ of the leading coefficient $A_2$ is illustrated by the unshaded area in Fig.3.1b, where we observe that $0 \notin \Lambda_2(A_2)$.




a b

FIG. 3.1. *The graphical implementation of the sets $\Lambda_2(L)$ (part a) and $\Lambda_2(A_2)$ (part b) has been achieved due to Propositions 3.4 and 3.3, respectively.*

Should we point out that in Fig.3.1b the set $\Lambda_2(A_2)$ has been determined via the numerical ranges $F(M^*A_2M)$ for $4 \times 3$ isometries $M$ according to Proposition 3.3. It is quite interesting to note that sketching $\Lambda_2(A_2)$ likewise the classical numerical range using the expression proved in [10]

$$\Lambda_2(A_2) = \bigcap_{\theta\in[0,2\pi)} e^{-\mathrm{i}\theta}\{z \in \mathbb{C} : \operatorname{Re} z \le \lambda_2(H(e^{\mathrm{i}\theta}A_2))\},$$

where $\lambda_2(H(\cdot))$ denotes the second largest eigenvalue of the Hermitian part $H(\cdot)$ of a matrix, then $\Lambda_2(A_2)$ appears to have additional "wings" at the corners and seen in the next Fig. 3.2. This is due to the fact that the line $l_\theta = \{z \in \mathbb{C} : \operatorname{Re} z = \lambda_2(H(e^{i\theta}A_2))\}$ is not tangential to $\Lambda_2(e^{i\theta}A_2)$, for some $\theta \in [0, 2\pi)$ and $x_2(\theta)^*A_2x_2(\theta)$ does not lie on

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the boundary of $\Lambda_2(A_2)$, where $x_2(\theta)$ is a unit eigenvector of $H(e^{i\theta}A_2)$ corresponding to $\lambda_2(H(e^{i\theta}A_2))$.

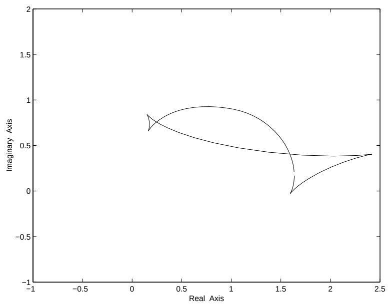


FIG. 3.2. *The boundary of $\Lambda_2(A_2)$ is plotted by the points $x_2(\theta)^* A_2 x_2(\theta)$, $\theta \in [0, 2\pi)$, where $x_2(\theta)$ is a unit eigenvector corresponding to the second largest eigenvalue of $H(e^{i\theta}A_2)$. In this way, two additional "wings" appear at the corners of the figure.*

EXAMPLE 3.8. Consider the $4 \times 4$ matrix polynomial $L(\lambda) = I_2 \otimes (B\lambda + I_2)$, with $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\Lambda_2(I_2 \otimes B) \neq \{0\}$ and additionally, $0 \in \Lambda_2(I_2 \otimes B)$. In this case, for any $4 \times 2$ isometry $Q$ such that $Q^*(I_2 \otimes B)Q = zI_2 \neq 0_2$, the Sylvester matrix in (2.3) is $R_s(Q) = \begin{bmatrix} 1 & 1/z \\ 1 & 1/z \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ with $\operatorname{rank} R_s(Q) = 1 < 2$. Since, $0 \in F(A_2)$, the range $W(L)$ is unbounded as well as $\Lambda_2(L \oplus L)$ (Proposition 3.2, II). This was expected by the converse part in Proposition 3.6.

**4. Sharp points.** In this section, we define the notion of sharp points in analogy with [13]. Particularly, $z_0 \in \partial\Lambda_k(L)$ is called to be a *sharp point* if for a connected component $\Lambda_k^{(s)}(L)$ of $\Lambda_k(L)$, there exist a disc $S(z_0, \varepsilon)$, with $\varepsilon > 0$, and two angles $\theta_1 < \theta_2$, with $\theta_1, \theta_2 \in [0, 2\pi)$, such that

$$\operatorname{Re}(e^{\mathbf{i}\theta} z_0) = \max\{\operatorname{Re} z : e^{-\mathbf{i}\theta} z \in \Lambda_k^{(s)}(L) \cap S(z_0, \varepsilon)\}, \quad \forall\ \theta \in (\theta_1, \theta_2).$$

Following, we present a condition for an eigenvalue lying on the boundary $\partial W(L)$ to be a boundary point of $\Lambda_k(L)$, as well.

PROPOSITION 4.1. *Let an $n \times n$ matrix polynomial $L(\lambda)$ as in (1.1). If $\gamma \in \sigma(L) \cap \partial W(L)$ with algebraic multiplicity $k$, then for $j = 2, \ldots, k$, it follows*

$$\gamma \in \partial\Lambda_j(L).$$

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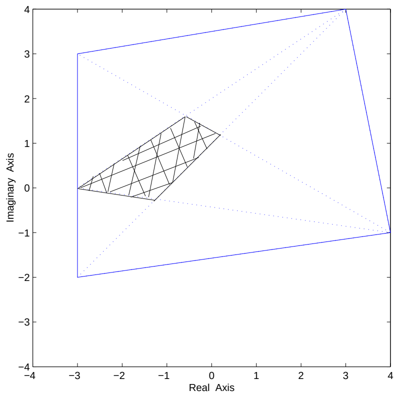
*Proof.* By hypothesis, $\gamma$ is a seminormal eigenvalue of the matrix polynomial $L(\lambda)$ of multiplicity $k$ [7, Theorem 6]. That is, there exists a unitary matrix $U$ such that

$$U^*L(\gamma)U = 0_k \oplus R(\gamma),$$

where $R(\lambda)$ is an $(n-k)\times(n-k)$ matrix polynomial and $\gamma \notin \mathrm{int}W(R)$. Hence, by Propositions 3.1, IIId, and 3.2, I, it is implied that $\gamma \in \Lambda_j(L) \subseteq \Lambda_{j-1}(L)$ for $j = 2,\ldots,k$, and due to $\gamma \notin \mathrm{int}W(L)$ ($\equiv \mathrm{int}\Lambda_1(L)$), we obtain $\gamma \in \partial\Lambda_j(L)$, for $j = 2,\ldots,k$. □

The converse of Proposition 4.1 is not true, as it is illustrated in the next example.

Example 4.2. Let $A = diag(3+4\mathbf{i}, 4-\mathbf{i}, -3-2\mathbf{i}, -3, -3+3\mathbf{i})$ and $L_A(\lambda) = I\lambda - A$. The outer polygon of the figure below is $W(L_A)$, whereas the inner shaded polygon is $\Lambda_2(L_A)$, which is the intersection of all $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ convex combinations of the eigenvalues $\lambda_{j_1}, \lambda_{j_2}, \lambda_{j_3}, \lambda_{j_4}$ of $A$, with $1 \le j_1 \le \cdots \le j_4 \le 5$. Notice that the simple eigenvalue of matrix $A$, $\lambda_0 = -3$, lies on $\partial W(L_A) \cap \partial\Lambda_2(L_A)$. In addition, $\Lambda_3(L_A) = \emptyset$.




In view of the definition of sharp points, for a pencil $A\lambda - B$, we have the next proposition.

Proposition 4.3. *Consider a pencil $L(\lambda) = A\lambda - B \in \mathcal{M}_n(\mathbb{C})$ and let $z_0$ be a sharp point of $W(A\lambda - B)$, which is an eigenvalue of $A\lambda - B$ of algebraic multiplicity $k$. Then $z_0$ is also a sharp point of $\Lambda_j(A\lambda - B)$, for $j = 2,\ldots,k$.*

*Proof.* Since, by hypothesis, the sharp point $z_0$ of $W(A\lambda - B)$ is an eigenvalue of the pencil $A\lambda - B$, with algebraic multiplicity $k$, Proposition 4.1 implies that $z_0 \in \partial\Lambda_j(A\lambda - B)$ for $j = 2,\ldots,k$. It only suffices to prove that for any disc $S(z_0, \varepsilon)$

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with $\varepsilon > 0$, $z_0$ satisfies the equality

$$\mathrm{Re}(e^{\mathbf{i}\theta} z_0) = \max\{\mathrm{Re}\, z : e^{-\mathbf{i}\theta} z \in \Lambda_j(A\lambda - B) \cap S(z_0, \varepsilon)\},$$

or equivalently, due to Proposition 3.4,

$$\mathrm{Re}(e^{\mathbf{i}\theta} z_0) = \max\left\{\mathrm{Re}\, z : z \in \bigcap_M \left(W(e^{\mathbf{i}\theta} M^*(A\lambda - B)M) \cap S(e^{\mathbf{i}\theta} z_0, \varepsilon)\right)\right\}$$

for every angle $\theta \in (\theta_1, \theta_2)$ with $0 \leq \theta_1 < \theta_2 < 2\pi$.

The inclusion $W(M^*(A\lambda - B)M) \subseteq W(A\lambda - B)$ for any $n \times (n - j + 1)$ isometry $M$, $j = 2, \ldots, k$, verifies the inequality

$$\max_{\bigcap\limits_M (W(e^{\mathbf{i}\theta} M^*(A\lambda - B)M) \cap S(e^{\mathbf{i}\theta} z_0, \varepsilon))} \mathrm{Re}\, z \leq \max_{W(e^{\mathbf{i}\theta}(A\lambda - B)) \cap S(e^{\mathbf{i}\theta} z_0, \varepsilon)} \mathrm{Re}\, z = \mathrm{Re}(e^{\mathbf{i}\theta} z_0) \tag{4.1}$$

for any disc $S(e^{\mathbf{i}\theta} z_0, \varepsilon)$ and every $\theta \in (\theta_1, \theta_2)$. Moreover, $\ker(Az_0 - B) \cap \mathrm{Im}(MM^*) \neq \emptyset$, since $\dim\ker(Az_0 - B) + \dim\mathrm{Im}(MM^*) = k + n - j + 1 \geq n + 1$. Hence, for an eigenvector $x_0 \in \mathbb{C}^n$ of $A\lambda - B$ corresponding to $z_0$, there exists a vector $y_0 \in \mathbb{C}^n$ such that $x_0 = MM^* y_0$. Obviously, $M^* y_0 \in \mathbb{C}^{n-j+1}$ is an eigenvector of $M^*(A\lambda - B)M$ corresponding to $z_0$, yielding $z_0 \in \sigma(M^*(A\lambda - B)M) \subseteq W(M^*(A\lambda - B)M)$ for any $n \times (n - j + 1)$ isometry $M$.

Thus, $z_0 \in \bigcap\limits_M W(M^*(A\lambda - B)M)$, i.e., $\mathrm{Re}\, z_0 \in \mathrm{Re}\left(\bigcap\limits_M W(M^*(A\lambda - B)M)\right)$, whereupon we confirm the inequality

$$\mathrm{Re}(e^{\mathbf{i}\theta} z_0) \leq \max_{\bigcap\limits_M (W(e^{\mathbf{i}\theta} M^*(A\lambda - B)M) \cap S(e^{\mathbf{i}\theta} z_0, \varepsilon))} \mathrm{Re}\, z \tag{4.2}$$

for any disc $S(e^{\mathbf{i}\theta} z_0, \varepsilon)$ and every $\theta \in (\theta_1, \theta_2)$. Therefore, by (4.1) and (4.2),

$$\mathrm{Re}(e^{\mathbf{i}\theta} z_0) = \max\left\{\mathrm{Re}\, z : z \in \bigcap_M \left(W(e^{\mathbf{i}\theta} M^*(A\lambda - B)M) \cap S(e^{\mathbf{i}\theta} z_0, \varepsilon)\right)\right\}$$

for any disc $S(e^{\mathbf{i}\theta} z_0, \varepsilon)$ and every $\theta \in (\theta_1, \theta_2)$, establishing the assertion. □

By the previous results, we have the following corollary concerning the sharp points of the rank-$k$ numerical range of a matrix $A \in \mathcal{M}_n(\mathbb{C})$.

Corollary 4.4. *Let $A \in \mathcal{M}_n(\mathbb{C})$ and $z_0$ be a sharp point of $F(A)$, which is an eigenvalue of $A$ of algebraic multiplicity $k$, then $z_0$ is also a sharp point of $\Lambda_j(A)$, for $j = 2, \ldots, k$.*

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