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# EXACT RESULTS FOR PERTURBATION TO TOTAL POSITIVITY AND TO TOTAL NONSINGULARITY*

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**Abstract.** A study of the maximum number of equal entries in totally positive and totally nonsingular $m$-by-$n$, matrices for small values of $m$ and $n$, is presented. Equal entries correspond to entries of the totally nonnegative matrix $J$ that are not changed in producing a $TP$ or $TNS$ matrix. It is shown that the maximum number of equal entries in a 7-by-7 totally positive matrix is strictly smaller than that for a 7-by-7 totally non-singular matrix, but, this is the first pair $(m, n)$ for which these maximum numbers differ. Using point-line geometry in the projective plane, a family of values for $(m, n)$ for which these maximum numbers differ is presented. Generalization to the Hadamard core, as well as larger projective planes is also established. Finally, the relationship with $C_4$ free graphs, along with a method for producing symmetric $TP$ matrices with maximal symmetric arrangements of equal entries is discussed.

**Key words.** $C_4$-free graphs, Hadamard core, Matrix perturbation, Projective plane, Total positivity.

**AMS subject classifications.** 15B48, 15B34, 51E15, 05C50, 05C75.

## 1. Introduction.

An $m$-by-$n$ matrix is called totally positive (totally nonnegative, totally nonsingular), $TP$ ($TN$, $TNS$) if every minor of it is positive (nonnegative, nonzero). It is known that any $TN$ matrix may be perturbed to a $TP$ matrix, i.e., that arbitrarily close to any $TN$ matrix is a $TP$ matrix. However, proofs typically change all entries. We are interested in the general problem of what are the minimal collections of entries in a given $TN$ matrix that need be changed (not necessarily a small change) in order that a $TP$ matrix result. This knowledge would likely be useful in the study of $TP$ completion problem [4, 5, 13], but it is fundamental and may well have other applications.

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Here, we consider the special case of what entries of the $TN$ matrix $J$ of all 1's need be changed to produce a $TP$ matrix, and also the related question of what entries need be changed to produce a $TNS$ matrix. As we mentioned in the abstract, this special case is equivalent to finding the maximum number of equal entries in totally positive (and totally nonsingular) matrices. The asymptotic variant of this and related questions was discussed in [6] and [8]. This question also has strong relations to algebraic combinatorial object, such as equal minors in totally nonnegative Grassmanians, and those relations were presented in [7]. In this paper, we focus upon exact results about the number and positions of such collections of equal entries, and together with the works mentioned above we get a more comprehensive understanding of the collections of equal entries in totally positive matrices. In case of perturbing $TN$ to $TP_2$, we generalize our work with $J$ to the set of all $TN$ matrices. This study results in a number of unanticipated and striking combinatorial connections that we discuss here.

In the next section, we lay the groundwork with necessary technical definitions and background results. Then in Section 3, we find the number of entries that need be changed in small cases. Interestingly, in terms of the number (and often the position) of the entries that need be changed, the $TP$ and $TNS$ perturbation problems are the same for $\min\{m,n\} < 7$ (a table is given). However, a difference occurs when $m = n = 7$ and this is documented in Section 4. The key is the second order projective plane. The role of other projective planes is discussed in Section 5. Though the two numbers of entries are again the same for $m = n = 8$, they eventually diverge. There is a natural connection here with $TP_2$ matrices (all 1-by-1 and 2-by-2 minors positive) and this is discussed in connection with perturbation results in the "Hadamard core". The relation of our problem with a classical forbidden subgraph question is discussed (in Section 7). This, for example, allows construction of symmetric $TP$ matrices with maximal arrangements of equal entries. The number of such equal entries is often seen to be a maximum.

**2. Notation and previous results.** As stated in the introduction, an $m$-by-$n$ matrix is called totally positive (totally nonnegative, totally nonsingular), $TP_{m\times n}$ ($TN_{m\times n}$, $TNS_{m\times n}$) if every minor of it is positive (nonnegative, nonzero). An $m$-by-$n$ matrix is called $TP_k$ if every submatrix of it of order $k$-by-$k$ is $TP$. For an $m$-by-$n$ real matrix $A$, the multiplicity of an entry $a$ in the matrix $A$ is the number of occurrences of $a$ in $A$, and we denote by $\#A$ the maximal multiplicity of an entry in $A$.

We define three functions $\nu$, $\pi$, and $\rho$ to be

$$\begin{aligned}\nu(m,n) &= \max\{\#A : A \in TNS_{m\times n}\},\\ \pi(m,n) &= \max\{\#A : A \in TP_{m\times n}\},\\ \rho(m,n) &= \max\{\#A : A \in TP_{2\,m\times n}\}.\end{aligned}$$

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ELA



In the case $m = n$, we simply write $\nu(n)$, $\pi(n)$ and $\rho(n)$ instead of $\nu(m,n)$, $\pi(m,n)$ and $\rho(m,n)$, respectively.

A *configuration* is a (0,1)-matrix that has no 2-by-2 submatrix consisting only of ones. We denote by $C_{m\times n}$ the set of all $m$-by-$n$ configurations, and by $|A|$ the number or weight of ones in a configuration $A$. We call an $m$-by-$n$ configuration $A$ maximum if $|A| = \nu(m,n)$.

We will now extend our discussion on perturbation. A (0,1)-matrix is $TP_2$-*perturbable* ($TP$-*perturbable*) if there is a perturbation of the entries of $J$, corresponding to the 0's, that is $TP_2$ ($TP$). Note that if $A$ is not $TP_2$-perturbable, then $A$ is not $TP$-perturbable. In addition, if $A$ is $TP_2$-perturbable (or $TP$-perturbable), then $A$ must be a configuration. Since multiplying by a positive scalar preserves total positivity and total nonsingularity, $\pi(m,n)$ ($\nu(m,n)$, $\rho(m,n)$) is the maximum number of entries of $J_{m\times n}$ that can stay unchanged when perturbing to $TP$ ($TNS$, $TP_2$). Thus, by knowing the values of those three functions, we know exactly how many entries should be perturbed in a matrix.

When dealing with relations between $\pi$ and $\rho$, the notion of Hadamard product of matrices is useful. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $m$-by-$n$ matrices. The Hadamard product of $A$ and $B$, denoted by $A \circ B$, is the $m$-by-$n$ matrix $C$ where $c_{ij} = a_{ij}b_{ij}$. For an entry-wise nonnegative matrix $A$, let $A^{(t)}$ denote the $t^{th}$ Hadamard power of $A$, that is $A^{(t)} = [a_{ij}^t]$. $A$ is called *eventually* $TP$ if there exists an $N > 0$ such that for each $t > N$, $A^{(t)}$ is $TP$. The following result is from [4]:

THEOREM 2.1. *A matrix $A$ is $TP_2$ if and only if $A$ is eventually $TP$.*

Finally, we present some additional results on $TNS$ matrices. Let $M$ be an $m$-by-$n$ $TNS$ matrix. Then by definition, for each $x \in \mathbb{R}$, the $m$-by-$n$ (0,1)-matrix $M(x) = [m_{ij}(x)]$ defined by

$$m_{ij}(x) = \begin{cases} 1 \text{ if } M_{ij} = x \\ 0 \text{ if } M_{ij} \neq x \end{cases}$$

is a configuration. In [6], the following converse is shown.

THEOREM 2.2. *For each $m$-by-$n$ configuration $A$, there is an $m$-by-$n$ $TNS$ matrix $M$ and a real number $x$ such that $M(x) = A$.*

## 3. Values of $\nu$, $\pi$ and $\rho$ for small $n$ and $m$.

In this section, we introduce several important relations between $\nu$, $\pi$ and $\rho$, and obtain their values for $1 \le m,n \le 6$.

The number of 1's in a configuration has been widely studied [10, 12, 14, 15]. It is known, asymptotically, that this is $O(n^{3/2})$ in the $n$-by-$n$ case [10]. As a corollary

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to Theorem 2.2, we relate $\nu$ to the number of 1's in a configuration.

Corollary 3.1.

$$\nu(m,n) = \max\{|A| : A \in C_{m\times n}\}.$$

The following two corollaries follow directly from the notion of perturbation and Corollary 3.1.

Corollary 3.2.

$$\pi(m,n) = \max\{|A| :\ A \in C_{m\times n} \textit{ is } TP\textit{-perturbable}\}.$$

Corollary 3.3.

$$\pi(m,n) \leq \nu(m,n) \quad \textit{for all } m,n.$$

Finally, using Theorem 2.1 and the fact that Hadamard powering preserves 1's, we get:

Corollary 3.4.

$$\pi(m,n) = \rho(m,n) \quad \textit{for all } m,n.$$

Using corollary 3.1 and [12], we obtain the following table for the values of $\nu(m,n)$, $2 \leq m,n \leq 8$.

| | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 3 | 4 | 6 | 7 | 8 | 9 | 10 | 11 |
| 4 | 5 | 7 | 9 | 10 | 12 | 13 | 14 |
| 5 | 6 | 8 | 10 | 12 | 14 | 15 | 17 |
| 6 | 7 | 9 | 12 | 14 | 16 | 18 | 19 |
| 7 | 8 | 10 | 13 | 15 | 18 | 21 | 22 |
| 8 | 9 | 11 | 14 | 17 | 19 | 22 | 24 |

We are now ready to prove the following theorem:

Theorem 3.5. *For $1 \leq m,n \leq 7$ such that $(m,n) \neq (7,7)$, we have $\nu(m,n) = \pi(m,n) = \rho(m,n)$.*

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 27, pp. 779-797, November 2014

ELA



*Proof.* If $m = 1$ or $n = 1$, then the claim holds, and using Corollary 3.3, it is enough to show that for $2 \leq m, n \leq 6$, $\nu(m,n) = \rho(m,n)$. We start from the case $m = n = 5$. The matrix

$$B = \begin{bmatrix} x^2 & 1 & 1 & x^{-4} & x^{-6} \\ 1 & x^{-1} & 1 & x^{-3} & x^{-4} \\ 1 & 1 & x^2 & 1 & 1 \\ x^{-4} & x^{-3} & 1 & x^{-1} & 1 \\ x^{-6} & x^{-4} & 1 & 1 & x^2 \end{bmatrix}$$

is a $TP_2$ perturbation of $J$ for every $x > 1$. In order to prove the statement for $2 \leq m, n \leq 5$, consider the submatrices $B[1,2|1,2]$, $B[1,2|1,2,3]$, $B[2,3|1,2,3,4]$, $B[2,3|1,2,3,4,5]$, $B[1,2,3|1,2,3]$, $B[1,2,3|1,2,3,4]$, $B[1,2,3|1,2,3,4,5]$, $B[1,3,4,5|2,3,4,5]$, $B[1,3,4,5|2,3,4,5]$ and $B[2,3,4,5|1,2,3,4,5]$. In each one of those submatrices the number of unchanged entries correspond to the numbers from the table, and hence, the statement holds for $2 \leq m, n \leq 5$. For $n = m = 6$, the matrix

$$C = \begin{bmatrix} 1 & 1 & 1 & x^{-15} & x^{-27} & x^{-32} \\ 1 & x^{10} & x^{12} & 1 & x^{-10} & x^{-12} \\ 1 & x^{12} & x^{17} & x^7 & 1 & 1 \\ x^{-15} & 1 & x^7 & 1 & x^{-6} & 1 \\ x^{-27} & x^{-10} & 1 & x^{-6} & x^{-8} & 1 \\ x^{-32} & x^{-12} & 1 & 1 & 1 & x^9 \end{bmatrix}$$

is a $TP_2$ perturbation of $J$ for every $x > 1$. In order to prove the statement for $m, n$ that satisfy $\max\{m,n\} = 6, \min\{m,n\} \geq 3$, consider the submatrices $C[1,3,4|1,2,3,4,5,6]$, $C[1,3,4,6|1,2,3,4,5,6]$ and $C[1,2,3,4,6|1,2,3,4,5,6]$. For the 2-by-6 case, consider the matrix

$$K = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & x & x^2 & x^3 & x^4 & x^5 \end{bmatrix}.$$

The 2-by-7 can be handled similarly. For the 6-by-7 and 5-by-7 cases, consider the matrix

$$M = \begin{bmatrix} 1 & 1 & 1 & x^{-23} & x^{-52} & x^{-28} & x^{-61} \\ 1 & x^{18} & x^{20} & 1 & x^{-27} & 1 & x^{-31} \\ 1 & x^{29} & x^{42} & x^{24} & 1 & x^{28} & 1 \\ x^{-32} & 1 & x^{15} & 1 & x^{-23} & x^8 & 1 \\ x^{-52} & x^{-18} & 1 & x^{-14} & x^{-34} & 1 & 1 \\ x^{-61} & x^{-22} & 1 & 1 & 1 & x^{37} & x^{42} \end{bmatrix},$$

and take $M$ and $M[1,2,3,4,5|1,2,3,4,5,6,7]$. For the rest of the cases, consider the

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Volume 27, pp. 779-797, November 2014

ELA



matrix

$$V = \begin{bmatrix} x^{58} & x^{27} & x^{22} & 1 & 1 & 1 & x^{-67} \\ x^{28} & 1 & 1 & x^{-21} & x^{-19} & 1 & x^{-65} \\ 1 & x^{-27} & 1 & x^{-41} & x^{-36} & 1 & x^{-63} \\ 1 & 1 & x^{31} & 1 & x^{17} & x^{57} & 1 \end{bmatrix}.$$

and $V[2,3,4|1,2,3,4,5,6,7]$. □

As a consequence of Theorem 3.5, we get that for every pair $(m,n)$ such that $1 \leq m,n \leq 6$, there exists a maximum configuration that is $TP_2$ and $TP$ perturbable. This surprising property is not true in the case $m = n = 7$ as we will see in the next section. Before continuing to the next section, we will make a remark about "small perturbation". We say that a configuration $A$ enables small $TP_2$ ($TP$) perturbation if for any $\epsilon > 0$, we can perturb $J$ through $A$ to a $TP_2$ ($TP$) matrix such that each perturbed entry differs from 1 by at most $\epsilon$. It is clear that for $1 \leq m,n \leq 6$, all the configurations that were used in the proof enabled small $TP_2$ perturbation. In case of small $TP$ perturbation, the situation is completely different. In fact, through exhaustive check, we get that the configuration that corresponds to matrix $B$ in the proof (and all its sub configurations) enable small $TP$ perturbation. Thus, for $1 \leq m,n \leq 5$, there is no difference between small perturbation to $TP_2$ and small perturbation to $TP$. However, $C$ (a 6-by-6 matrix) does not posses this property. This of course doesn't mean that there is no configuration of order 6 that enables small $TP$ perturbation. We conclude this section with the following conjecture:

Conjecture 3.6. Let $(m,n)$ be a pair for which there exists a configuration $A$ that enables small $TP_2$ perturbation. Then there exists a configuration $P$ for which $|P| = |A|$ that enables small $TP$ perturbation.

**4. The smallest case for which $\nu(m,n) \neq \pi(m,n)$.** In this section, our aim is to show that no maximum 7-by-7 configuration is $TP_2$ perturbable. We begin this section with several definitions. We say that a configuration $A$ is *completely full* if the inner product of any two rows and columns in $A$ is 1. Recall the following theorem from [14]

Theorem 4.1. *Let $N = n^2 + n + 1$. Let $S$ be a square grid of $N^2$ points which are arrayed in $N$ rows and $N$ columns. Let $k = (n+1)(n^2+n+1)$. If there is a set $T$ of $k$ nodes in the grid no four of which form the vertices of a square with sides parallel to the sides of $S$ then each row and each column of $S$ contains exactly $n+1$ nodes of $T$.*

Using this theorem and the table for $\nu$, it is easy to show that any 7-by-7 maximum configuration is completely full and has 3 ones in each row and column. In fact, this is the projective plane of order 2, which is also called the Fano plane.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 779-797, November 2014

ELA



Let $A \in C_{m\times n}$ be a $TP_2$-perturbable configuration. Denote by $M$ the $TP_2$ matrix that is obtained from $J$ using $A$. If $A$ contains a submatrix of the form

$$(\textbf{Type I}) \qquad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

then $M$ has a corresponding submatrix of the form

$$\begin{bmatrix} a & 1 \\ 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ 1 & a \end{bmatrix},$$

where $a > 1$. On the other hand, if $A$ contains a submatrix of the form

$$(\textbf{Type II}) \qquad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

then $M$ has a corresponding submatrix of the form

$$\begin{bmatrix} 1 & a \\ 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix},$$

where $a < 1$. The following definition is motivated by this observation. Let $A \in C_{m\times n}$ be a $TP_2$-perturbable configuration and let $a_{ij}$ be a zero entry of $A$. We say that $a_{ij}$ is *forced up* (*forced down*) if $a_{ij}$ is the zero entry in a submatrix of the form (**Type I**) ( (**Type II**)). Note that no entry of $A$ can be forced both up and down.

Lemma 4.2. *Let $A \in C_{7\times 7}$ be maximum and $TP_2$-perturbable, and let $C_i$ ($R_i$) denote the $i^{th}$ column (row) of $A$. Then the ones in $C_1$, $C_7$, $R_1$ and $R_7$ must be consecutive.*

*Proof.* Without loss of generality, suppose that the ones in $C_1$ are not consecutive. Since $A$ is completely full, it contains one of the following submatrices

$$\begin{bmatrix} 1 & 0 & 1 \\ a_{j1} & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 0 \\ a_{j1} & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

where $a_{j1} = 0$. In either case, $a_{1j}$ is forced both up and down, showing that $A$ is not $TP_2$-perturbable. ∎

We are now ready to prove our main result in this section

Theorem 4.3. *Let $A \in C_{7\times 7}$ be maximum. Then $A$ is not $TP_2$-perturbable.*

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 27, pp. 779-797, November 2014



*Proof.* By Lemma 4.2, there are five options for the first column of $A$:

$$v_1 = \begin{bmatrix}1\\1\\1\\0\\0\\0\\0\end{bmatrix}, \quad v_2 = \begin{bmatrix}0\\1\\1\\1\\0\\0\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}0\\0\\1\\1\\1\\0\\0\end{bmatrix}, \quad v_4 = \begin{bmatrix}0\\0\\0\\1\\1\\1\\0\end{bmatrix}, \quad v_5 = \begin{bmatrix}0\\0\\0\\0\\1\\1\\1\end{bmatrix}.$$

By symmetry, we have three cases:

1. The first column is $v_1$ or $v_5$.
2. The first column is $v_2$ or $v_4$.
3. The first column is $v_3$.

We start from Case 1. Without loss of generality, assume that the first column of $A$ is $v_1$. Then by Lemma 4.2, we get that the first row and column of $A$ are of the form

$$\begin{bmatrix}1&1&1&0&0&0&0\\1&&&&&&\\1&&&&&&\\0&&&&&&\\0&&&&&&\\0&&&&&&\\0&&&&&&\end{bmatrix}.$$

In order to avoid having a 2-by-2 submatrix of ones, we must have

$$\begin{bmatrix}1&1&1&0&0&0&0\\1&0&0&&&&\\1&0&0&&&&\\0&&&&&&\\0&&&&&&\\0&&&&&&\\0&&&&&&\end{bmatrix}.$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 27, pp. 779-797, November 2014





Thus, by Lemma 4.2, the only way to fill in the last row and column is

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & & & & 0 \\ 1 & 0 & 0 & & & & 1 \\ 0 & & & & & & 1 \\ 0 & & & & & & 1 \\ 0 & & & & & & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}. \tag{4.1}$$

Note that the last column in this case is $v_3$. Thus, from symmetry, we don't need to treat Case 3 separately. Now, we wish to fill in the (4,2) and (4,3) entries of the matrix in 4.1. There are two cases:

*Case 1a:* The (4,2) entry is a one, and hence, the $(4,3)$ entry is a zero. Thus, $A$ looks like:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & & & & 0 \\ 1 & 0 & 0 & & & & 1 \\ 0 & 1 & 0 & & & & 1 \\ 0 & & & & & & 1 \\ 0 & & & & & & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Now, the (5,2) entry cannot be a one, or else columns 2 and 7 would share two ones. Furthermore, each column and row must contain three ones. In addition, $A$ is completely full, and hence, it must be of the form

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & & & & 0 \\ 1 & 0 & 0 & & & & 1 \\ 0 & 1 & 0 & & & & 1 \\ 0 & 0 & 1 & & & 1 & 1 \\ 0 & 1 & 0 & & & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

The (5,2) entry is forced both up and down and we get a contradiction.

*Case 1b:* The (4,2) entry is zero. Similarly to Case 1a, we can deduce that $A$ is of

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**Volume 27, pp. 779-797, November 2014**



the form

$$\left[\begin{array}{ccccccc} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & & & & 0 \\ 1 & 0 & 0 & & & & 1 \\ 0 & 0 & 1 & & & & 1 \\ 0 & 1 & 0 & & & & 1 \\ 0 & 1 & 0 & & & & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right],$$

which is again a contradiction (the (4,2) entry is forced both up and down).

Consider Case 2, and assume without loss of generality that the first column of $A$ is $v_2$. Hence, $A$ must be of the form

$$\left[\begin{array}{ccccccc} 0 & & & & & & 0 \\ 1 & & & & & & 0 \\ 1 & & & & & & 0 \\ 1 & & & & & & 1 \\ 0 & & & & & & 1 \\ 0 & & & & & & 1 \\ 0 & & & & & & 0 \end{array}\right].$$

By Lemma 4.2, there are only two ways to fill in the first and the last row:

$$\left[\begin{array}{ccccccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & & & & & & 0 \\ 1 & & & & & & 0 \\ 1 & & & & & & 1 \\ 0 & & & & & & 1 \\ 0 & & & & & & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{ccccccc} 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & & & & & & 0 \\ 1 & & & & & & 0 \\ 1 & & & & & & 1 \\ 0 & & & & & & 1 \\ 0 & & & & & & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right].$$

We start from the first case. Since $A$ is completely full, it must look like

$$\left[\begin{array}{ccccccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & * & * & 0 & & & 0 \\ 1 & & & 0 & & & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & & & 0 & & & 1 \\ 0 & & & 0 & & & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

Electronic Journal of Linear Algebra ISSN 1081-3810
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Volume 27, pp. 779-797, November 2014



and exactly one of the starred entries equals to 1. Thus, the $(2,4)$ entry is forced both up and down, and we get a contradiction. Consider now the second case. Similarly to the first case, $A$ must look like

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & & & 0 & & & 0 \\ 1 & & & 0 & & & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & & & 0 & & & 1 \\ 0 & & & 0 & & & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since $A$ is completely full, each one of the four "empty squares" must be either $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It is easy to see that the only option in which no entry is forced both up and down is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since it contains a 2-by-2 submatrix of 1's, we get a contradiction. □

Corollary 4.4. *$\pi(7,7) < \nu(7,7)$. In particular, $\pi(7,7) = 20$ whereas $\nu(7,7) = 21$.*

*Proof.* The first statement follows from Theorem 4.3. The second part follows from the table of $\nu$, and the following matrix

$$\begin{bmatrix} 1 & 1 & 1 & x^{-23} & x^{-52} & x^{-28} & x^{-61} \\ 1 & x^{18} & x^{20} & 1 & x^{-27} & 1 & x^{-31} \\ 1 & x^{29} & x^{42} & x^{24} & 1 & x^{28} & 1 \\ x^{-32} & 1 & x^{15} & 1 & x^{-23} & x^{8} & 1 \\ x^{-52} & x^{-18} & 1 & x^{-14} & x^{-34} & 1 & 1 \\ x^{-37} & 1 & x^{19} & x^{17} & 1 & x^{35} & x^{37} \\ x^{-61} & x^{-22} & 1 & 1 & 1 & x^{37} & x^{42} \end{bmatrix}. \quad \square$$

Although $\pi(7,7) < \nu(7,7)$, surprisingly, we still have $\pi(8,8) = \nu(8,8)$. This can

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Volume 27, pp. 779-797, November 2014





be demonstrated using the matrix

$$U = \begin{bmatrix} x^{58} & x^{27} & x^{49} & x^{22} & 1 & 1 & 1 & x^{-67} \\ x^{28} & 1 & x^{24} & 1 & x^{-21} & x^{-19} & 1 & x^{-65} \\ 1 & x^{-27} & 1 & x^{-23} & x^{-41} & x^{-36} & 1 & x^{-63} \\ 1 & x^{-9} & x^{20} & 1 & x^{-16} & 1 & x^{37} & x^{-23} \\ 1 & 1 & x^{31} & x^{13} & 1 & x^{17} & x^{57} & 1 \\ x^{-21} & x^{-19} & x^{15} & 1 & x^{-12} & x^{8} & x^{55} & 1 \\ x^{-42} & x^{-37} & 1 & x^{-14} & x^{-23} & 1 & x^{53} & 1 \\ x^{-47} & x^{-39} & 1 & 1 & 1 & x^{24} & x^{80} & x^{29} \end{bmatrix}.$$

From [6] it is known that asymptotically $\pi(n,n)$ behaves like $O(n^{4/3})$, but we don't from which point $\pi(n,n) < \nu(n,n)$ for all $n$.

**5. Certain projective planes are not $TP$-perturbable.** A *projective plane of order* $n$, denoted $\mathcal{P}_n(P,L,I)$ consists of two sets, points and lines, with an incidence relation between them satisfying the following conditions:

1. Any two distinct points are incident with one line.
2. Any two distinct lines contain exactly one point in common.
3. There is a set of four distinct points, no three of which are on the same line.
4. There is a line which contains exactly $n+1$ points.

From these axioms of the projective plane, it is easy to deduce that the number of points is $n^2+n+1$. This is also the number of lines. Furthermore, one can deduce that the total number of incidences in the plane is $(n+1)(n^2+n+1)$.

Let $\mathcal{P}_n$ be a projective plane of order $n$. Set $N = n^2+n+1$ and let $\{p_i\}_{i=1}^N$ and $\{l_i\}_{i=1}^N$ denote the set of points and lines, respectively. The *incidence matrix of the projective plane* $\mathcal{P}_n$ is the $N \times N$ matrix $A = [a_{ij}]$ defined by

$$a_{ij} = \begin{cases} 1, & p_i \text{ is incident with } l_j \\ 0, & \text{otherwise} \end{cases}.$$

The following is from [14].

Theorem 5.1. *Suppose a projective plane of order $n$ exists and let $N = n^2 + n + 1$. Let $A$ be an incidence matrix of a projective plane of order $n$. Then $A$ is a configuration. Moreover, the matrix $A$ satisfies $|A| = \nu(N)$.*

In [9], it is mentioned that if $n$ is a power of 2, then there is a projective plane of order $n$ which contains as a subplane the Fano plane. Therefore, the following

**Electronic Journal of Linear Algebra ISSN 1081-3810**
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**Volume 27, pp. 779-797, November 2014**



corollary is an immediate consequence of Theorem 5.1.

COROLLARY 5.2. *For every $n$ that is power of 2, there exists a maximum $n$-by-$n$ configuration that is not $TP$-perturbable.*

**6. A generalization to the Hadamard core.** In our discussion on perturbing $TN$ matrices to $TP$ matrices, we have concentrated on the matrix $J$. In this section, we extend our work to a wider family of $TN$ matrices- matrices in the Hadamard core that have positive entries. The Hadamard core of the $m$-by-$n$ $TN$ matrices is the set

$$CTN_{m,n} := \{A \in TN : B \in TN \implies A \circ B \in TN\}. \tag{6.1}$$

The following lemma and theorem can be found in [4].

LEMMA 6.1. *An $m$-by-$n$ $TN$ matrix $A$ is in the Hadamard core if and only if every submatrix of $A$ is in the corresponding Hadamard core.*

THEOREM 6.2. *Let $A \in CTN_{m,n}$, and let $B$ be an $n$-by-$n$ $TN$ matrix. Then*

$$\det(A \circ B) \geq \det B \prod_{i=1}^{n} a_{ii}. \tag{6.2}$$

Using the lemma and theorem mentioned above, we prove the following corollary:

COROLLARY 6.3. *Let $A \in CTN_{m,n}$ be a matrix with positive entries, and let $P$ be an $m$-by-$n$ $TP$ matrix. Then $A \circ P$ is $TP$.*

*Proof.* Let $A'$ be a square submatrix of $A$ and let $P'$ be the corresponding square submatrix of $P$. By Lemma 6.1, $A'$ is in the Hadamard core, and since $A$ is entrywise positive, the diagonal entries of $A'$ are positive as well. $P$ is $TP$, and hence, $\det(P') > 0$. Applying Theorem 6.2, we get $\det(A' \circ P') > 0$, and hence, $A \circ P$ is $TP$. ∎

As an immediate consequence, we get the following:

COROLLARY 6.4. *Let $A \in CTN_{m,n}$ be a matrix with positive entries, and let $P$ be a $TP$ perturbation for $J$. Then $A \circ P$ is a $TP$ perturbation for $A$.*

Thus, the number and positionings of entries in $J$ that must be perturbed in order to obtain a $TP$ matrix stays the same when we choose any matrix in the Hadamard core instead of $J$. With regards to $TP_2$ perturbability, we now extend our work with $J$ to the whole class of $TN$ matrices with positive entries. Let $A$ and $B$ be two matrices with positive entries such that $A$ is $TN$ and $B$ is $TP_2$. It is easy to show that $A \circ B$ is $TP_2$. Thus, we obtain the following corollary:

COROLLARY 6.5. *Let $A$ be an $m$-by-$n$ $TN$ matrix with positive entries and let $P$*

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 779-797, November 2014





*be a $TP_2$ $J$-perturbed matrix. Then $A \circ P$ is a $TP_2$ perturbation of $A$ that is obtained using the same configuration.*

**7. Graphs without four cycles.** Let $G$ be a graph that does not contain $C_4$ as a subgraph. We call such $G$ a $C_4$-free graph. For such $G$, the adjacency matrix of $G$, $A(G)$, is a configuration. Here, we deal with maximum $C_4$-free graphs (i.e., $C_4$-free graphs with a maximum number of edges). These have been widely studied [11, 3, 2]. Since the adjacency matrix of a graph is symmetric, the set of adjacency matrices of $C_4$-free graphs of order $n$ form a proper subset of $C_{n\times n}$. Moreover, if $G$ is maximum $C_4$-free, it does not imply that $A(G)$ is maximum configuration (note that $|A(G)| = 2|E(G)|$). However, during our work on this subject, we observed a method that, in many cases, enabled us to obtain a maximum configuration from a maximum $C_4$-free graph. We describe here the method, and show how to obtain an $n$-by-$n$ maximum configuration from a maximum $C_4$-free graph for all $1 \leq n \leq 20$. There has been related work in this area; it can be found in [1].

We start by introducing some notation:

$$t(n) = \max\{|E(G)| : |V(G)| = n \text{ and } G \text{ is a } C_4\text{-free graph}\}, \tag{7.1}$$

$$T(n) = \{G : |V(G)| = n, |E(G)| = t(n), \text{ and } G \text{ is a } C_4\text{-free graph}\}. \tag{7.2}$$

It is known ([2]) that asymptotically,

$$t(n) = \Theta(\nu(n)). \tag{7.3}$$

However, Table 1 shows that in many cases the adjacency matrix of a maximum $C_4$-free graph is not a maximum configuration. Given a $C_4$-free graph $G$, one of the ways to increase the number of ones in $A(G) = [a_{ij}]$ is to place some ones on the diagonal (where there were 0's). Note that if the vertex $v_i$ belongs to a three-cycle in $G$, then $a_{ii}$ must stay zero (since if we change $a_{ii}$ to 1 the resulting matrix will not be a configuration). In addition, if $\{v_i, v_j\} \in E(G)$ and we set $a_{jj} = 1$, then $a_{ii}$ must stay zero. As long as both of these conditions are satisfied, we can continue to add ones on the diagonal. Thus, for $G \in T(n)$ we look for maximum collections of vertices that can be changed to 1 under the constraints just mentioned. By applying this argument to all of the graphs in $T(n)$, we obtain configurations with various weights. We then select the subcollection of those configurations with the maximum weight. These are then candidates for maximum configurations.

By using the previous process for graphs on $n$ vertices, $2 \leq n \leq 21$, we managed to obtain maximum configurations for all of them. As a corollary to our findings, we

**Electronic Journal of Linear Algebra ISSN 1081-3810**
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**Volume 27, pp. 779-797, November 2014**





get that there are symmetric maximum configurations for each square size up to and including 21-by-21. Before describing the configurations, we need further notation.

A pair of vertices $u, v$ in a graph $G$ is **hot** if there exists a vertex $w$ such that $\{u, w\}, \{v, w\} \in E(G)$. The pair $u, v$ is **cold** if it is not hot. A set $S$ of vertices of $G$ is cold if all pairs of vertices in $S$ are cold. Let $G \in T(n)$. Let $\{V_i(G)\}_{i\in I}$ denote a collection of sets, in which each $V_i(G)$ is a set of vertices such that no vertex is a member of a three-cycle and no vertex is adjacent to another vertex in the set. Define

$$H(G) = \max\{|V_i(G)| : i \in I\}. \tag{7.4}$$

That is, $H(G)$ is the maximum number of diagonal entries that can be changed to 1 in the matrix $A(G)$. Define

$$k(n) = \max\{H(G) : G \in T(n)\}, \tag{7.5}$$

And let $K(n)$ to be

$$K(n) = \{G : G \in T(n) \text{ and } H(G) = k(n)\}. \tag{7.6}$$

Note that using this notations we have $\nu(n, n) \geq 2t(n) + k(n)$, and hence, for a fixed $n$, $2 \leq n \leq 21$, it is enough to show the process described above results in $\nu(n, n) = 2t(n) + k(n)$.

We now describe the process of obtaining maximum configurations for $2 \leq n \leq 21$. We follow closely the ideas presented in [3] in order to construct maximum $C_4$-free graphs.

Before we prove our result, however, we require some notation that is useful in the proof.

Theorem 7.1. *For $n$ satisfying $2 \leq n \leq 21$, we have*

$$\nu(n, n) = 2t(n) + k(n). \tag{7.7}$$

The values of $\nu(n, n), t(n)$, and $k(n)$ that make this equality hold are given in Table 7.1 (the values of $t(n)$ and $\nu(n, n)$ can be found in [3]).

*Proof.* The graphs for $n = 2, 3, 4$ are easily constructed and given in Figure 7.1 below. The dark vertices are the ones whose corresponding diagonal entries can be changed to one. Note also that the number of dark vertices is $k(n)$.

Using the results from [3], the graphs in $K(5)$, $K(6)$, and $K(7)$ are easily found and are displayed in Figure 7.2.

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 779-797, November 2014





TABLE 7.1

| $n$ | $\nu(n,n)$ | $t(n)$ | $k(n)$ | $n$ | $\nu(n,n)$ | $t(n)$ | $k(n)$ | n | $\nu(n,n)$ | $t(n)$ | $k(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 3 | 1 | 1 | 9 | 29 | 13 | 3 | 16 | 67 | 33 | 1 |
| 3 | 6 | 3 | 0 | 10 | 34 | 16 | 2 | 17 | 74 | 36 | 2 |
| 4 | 9 | 4 | 1 | 11 | 39 | 18 | 3 | 18 | 81 | 39 | 3 |
| 5 | 12 | 6 | 0 | 12 | 45 | 21 | 3 | 19 | 88 | 42 | 4 |
| 6 | 16 | 7 | 2 | 13 | 52 | 24 | 4 | 20 | 96 | 46 | 4 |
| 7 | 21 | 9 | 3 | 14 | 56 | 27 | 2 | 21 | 105 | 50 | 5 |
| 8 | 24 | 11 | 2 | 15 | 60 | 30 | 0 | | | | |

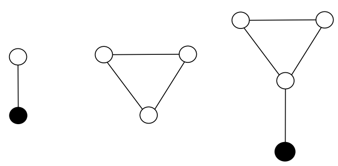

FIG. 7.1. *The graph from each $K(2)$, $K(3)$, and $K(4)$, respectively.*

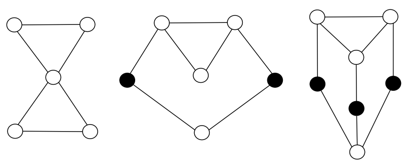

FIG. 7.2. *The graph in each $K(5)$, $K(6)$, and $K(7)$, respectively.*

The set $K(8)$ is slightly special, since it has three non-isomorphic graphs. These graphs are displayed in Figure 7.3. Note that set of three graphs in $K(8)$ gives rise to a set of three maximum configurations of size $8 \times 8$, no two of which are permutation equivalent to each other.

For $n$ larger than 8, we no longer picture all of the graphs in $K(n)$. It suffices to give only one example of a graph in each $K(n)$, thereby verifying the theorem in case $n$. Thus, we have given an example of a graph in $K(n)$ for $n$ satisfying $9 \leq n \leq 14$ in the figures below.

Consider now the case $n = 15$. From [3], we know that there exists a graph in $K(15)$ that has five disjoint sets of three cold vertices. This graph ($L_{15}$) presented in Figure 7.6. We have denoted the different sets by the capital letters A - E.

Consider now the case $n = 16$. In order to provide a construction, we take $L_{15}$ and add a new vertex. Note that since $t(16) = t(15) + 3$, this new vertex must be

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 779-797, November 2014



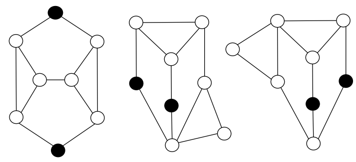

FIG. 7.3. *The three graphs in $K(8)$.*

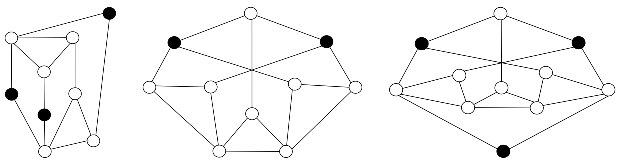

FIG. 7.4. *A graph from each $K(9)$, $K(10)$, and $K(11)$, respectively.*

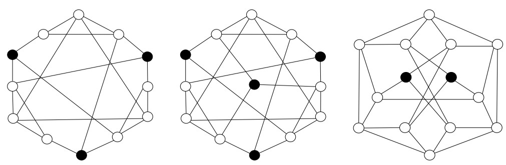

FIG. 7.5. *A graph from each $K(12)$, $K(13)$, and $K(14)$, respectively.*

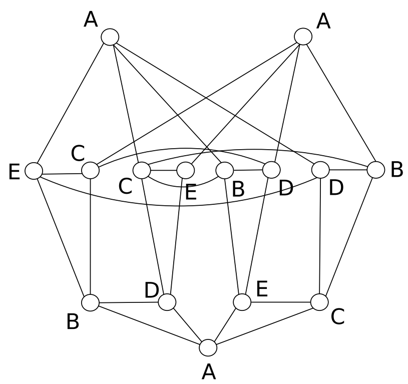


FIG. 7.6. *A graph $L_{15}$ from $K(15)$. The five sets of three cold vertices are grouped by letter.*

of degree 3, so we attach it to some set of three cold vertices. We can color this new vertex, since it is attached to a set of cold vertices, none of which are adjacent. Therefore, we obtain a graph in $K(16)$ (Figure 7.7).

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 779-797, November 2014



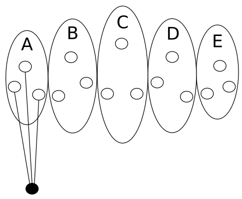


FIG. 7.7. *A graph from $K(16)$. Each lettered group represents a collection of three cold vertices from $L_{15}$.*

The graphs in $K(17)$ through $K(21)$ are constructed in a similar manner to the graph in Figure 7.7. We have included the rest of the graphs in the same notational style.

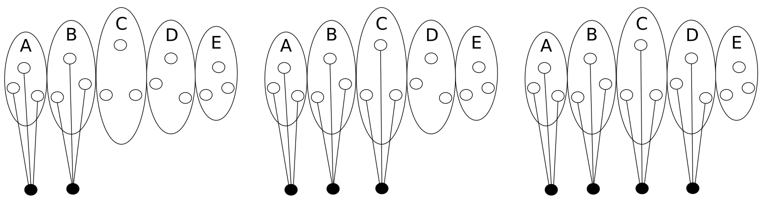


FIG. 7.8. *A graph from $K(17)$, $K(18)$, and $K(19)$, respectively.*

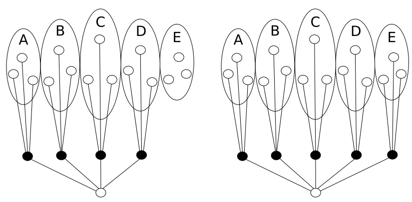


FIG. 7.9. *A graph from $K(20)$ and one from $K(21)$.*

The proof is complete. □

This theorem suggests the following conjecture:

CONJECTURE 7.2. For all $n \geq 2$, there is a symmetric matrix $C \in C_{n\times n}$.

Finally, we pose a conjecture regarding symmetric perturbation:

CONJECTURE 7.3. For all $n \geq 2$, there exists a symmetric configuration $A$ for which $|A| = \pi(n, n)$.

**Electronic Journal of Linear Algebra ISSN 1081-3810**
**A publication of the International Linear Algebra Society**
**Volume 27, pp. 779-797, November 2014**



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