Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 742-778, October 2014

ELA

THE COMBINATORIAL INVERSE EIGENVALUE PROBLEM II: ALL CASES FOR SMALL GRAPHS*

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Abstract. Let G be a simple undirected graph on n vertices and let $\mathcal{S}(G)$ be the class of real symmetric $n \times n$ matrices whose nonzero off-diagonal entries correspond to the edges of G. Given $2n-1$ real numbers $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, and a vertex v of G, the question is addressed of whether or not there exists $A \in \mathcal{S}(G)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $A(v)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$, where $A(v)$ denotes the matrix with vth row and column deleted. A complete solution can be given for the path on n vertices with v a pendant vertex and also for the star on n vertices with v the dominating vertex. The main result is a complete solution to this "λ, μ" problem for all connected graphs on 4 vertices.

Key words. Interlacing inequalities, Inverse eigenvalue problem, Symmetric matrix.

AMS subject classifications. 05C50, 15A42, 15B57.

1. Introduction. Let $G = (V, E)$ be a simple undirected graph with vertex set $V = \{1, 2, \ldots, n\}$. Let $\mathcal{S}(G)$ be the set of all real symmetric $n \times n$ matrices $A = [a_{ij}]$ such that for $i \neq j, a_{ij} \neq 0$ if and only if $ij \in E$. There is no condition on the diagonal entries of A. A fundamental and compelling open problem in combinatorial matrix theory is the inverse eigenvalue problem for graphs. Two of the most actively studied versions of this problem are:

I. Given a connected graph G on n vertices and a list of real numbers $\Lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$, is there an $A \in \mathcal{S}(G)$ with eigenvalues Λ?

Some work on this problem can be found in [1], [2], [3], [6], and [9] (p. 316). We will designate this problem as the Λ problem. If there is an $A \in \mathcal{S}(G)$ with eigenvalues Λ, we say Λ is *realizable* for G.

*Received by the editors on May 30, 2013. Accepted for publication on September 19, 2014. Handling Editor: Bryan L. Shader.

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Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 742-778, October 2014

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II. Given a connected graph G on n vertices, a vertex v of G, and two lists of real numbers

$$\Lambda = \lambda_1, \lambda_2, \ldots, \lambda_n, \qquad U = \mu_1, \mu_2, \ldots, \mu_{n-1},$$

satisfying the interlacing inequalities

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n, \tag{1.1}$$

is there an $A \in \mathcal{S}(G)$ such that A has eigenvalues Λ and $A(v)$ has eigenvalues U?

Some work on this problem appears in [1], [3], [5], [6], [9] and much earlier in the context of tridiagonal matrices in [7], [8]. We will designate this problem as the (Λ, U)-problem. If there is an $A \in \mathcal{S}(G)$ such that A has eigenvalues Λ and $A(v)$ has eigenvalues U, we say (Λ, U) is *realizable* for (G, v).

In [3], we gave a constructive proof of the following result for small graphs.

Theorem 1.1. *Let G be a connected graph on $2 \leq n \leq 4$ vertices, let v be a vertex of G, and let $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n$. Then there exists $A \in \mathcal{S}(G)$ such that the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$ and the eigenvalues of $A(v)$ are $\mu_1, \ldots, \mu_{n-1}$.*

Bryan Shader and his student Keivan Monfared in [12] have shown by means of the Nilpotent-Jacobian method that the above theorem is true for all n.

We will make use of the concepts of the minimum rank and the positive semidefinite minimum rank of a graph defined by

$$\mathrm{mr}(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G)\}$$

and

$$\mathrm{mr}_+(G) = \min\{\mathrm{rank}\, A : A \in \mathcal{S}(G) \text{ and } A \text{ is positive semidefinite}\},$$

respectively.

2. (P_n, pendant) and (S_n, dominating).

The path P_n is the connected graph on n vertices with $n-2$ vertices of degree 2 and 2 pendant vertices (vertices of degree 1). The star S_n is the connected graph on n vertices with one vertex of degree $n-1$ and $n-1$ pendant vertices. We call the vertex of degree $n-1$ the *dominating vertex*. It is well-known that $\mathrm{mr}(P_n) = n-1$ and that $\mathrm{mr}(S_n) = 2$. In this section, we present complete solutions to the (Λ, U)-problem for the cases (P_n, v), v a pendant vertex of P_n and (S_n, v), v the dominating vertex of S_n in terms of inequalities among

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the entries in the lists Λ, U. For P_n, we need the following fundamental result about minimum rank matrices found in [13].

Lemma 2.1. *Let G be a graph on n vertices and let $A \in \mathcal{S}(G)$ such that* $\operatorname{rank} A = \operatorname{mr}(G)$. *Then for any $i \in \{1, 2, \ldots, n\}$,* $\operatorname{rank} A(i) = \operatorname{rank} A$ *or* $\operatorname{rank} A(i) = \operatorname{rank} A - 2$*; i.e.,* $\operatorname{rank} A(i) = \operatorname{rank} A - 1$ *is impossible.*

Theorem 2.2. ((P_n, v), v pendant). *Let $\Lambda = \lambda_1, \ldots, \lambda_n$ and $U = \mu_1, \ldots, \mu_{n-1}$ be lists of real numbers satisfying the interlacing inequalities, and let v be a pendant vertex of the graph P_n. Then (Λ, U) is realizable for (P_n, v) if and only if*

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \cdots > \lambda_{n-1} > \mu_{n-1} > \lambda_n. \tag{2.1}$$

Proof. The reverse implication is well-known and follows from results of Hald [7] and Hochstadt [8] in the 1970's. It is also a special case of Duarte's theorem [5] which states that for any tree T and any vertex v of T, (Λ, U) is realizable for (T, v) whenever all λ_i and μ_i are distinct.

The forward implication also follows from prior results, for example Proposition 3.1 in [10]. We give a simpler proof via the concept of minimum rank.

Suppose $A \in \mathcal{S}(P_n)$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ and $A(v)$, the matrix obtained by deleting the vth row and column of A, has eigenvalues $\mu_1, \ldots, \mu_{n-1}$ satisfying the interlacing inequalities (1.1).

Case 1. Two or more equality signs occur consecutively. Then either $\lambda_i = \lambda_{i+1}$ or $\mu_i = \mu_{i+1}$ for some i. If $\lambda_i = \lambda_{i+1}$, then $A - \lambda_i I_n \in \mathcal{S}(P_n)$ and

$$n - 2 \geq \operatorname{rank}(A - \lambda_i I_n) \geq \operatorname{mr}(P_n) = n - 1,$$

a contradiction. Likewise, if $\mu_i = \mu_{i+1}$ for some i, then $A(v) - \mu_i I_{n-1} \in \mathcal{S}(P_{n-1})$ and

$$n - 3 \geq \operatorname{rank}(A(v) - \mu_i I_{n-1}) \geq \operatorname{mr}(P_{n-1}) = n - 2,$$

a contradiction. So this case cannot occur.

Case 2. Suppose that an equality sign occurs somewhere in the interlacing inequalities (1.1). Then we have either $\mu_{i-1} > \lambda_i = \mu_i > \lambda_{i+1}$ or $\lambda_{i-1} > \mu_{i-1} = \lambda_i > \mu_{i+1}$ for some i. Then $\operatorname{rank}(A - \lambda_i I_n) = n - 1 = \operatorname{mr}(P_n)$ and $\operatorname{rank}(A(v) - \lambda_i I_{n-1}) = n - 2$. This contradicts Lemma 2.1 which requires that $\operatorname{rank}(A(v) - \lambda_i I_{n-1})$ be either $n - 1$ or $n - 3$. So this case cannot occur and 2.1 must hold. $\square$

We now turn to the case (S_n, v), v dominating. A solution to this case is given in Theorem 11 of [9], which also applies to generalized stars. But we wish to give a solution in terms of inequalities among the entries in the lists Λ, U, so we include

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 742-778, October 2014

a proof tailored to that end. We expect that the tools we employ can be applied to other cases of the inverse eigenvalue problem.

We will make use of fundamental results about positive semidefinite matrices and positive semidefinite minimum rank, and a restriction on what (Λ, U) is realizable for any (T, v) in which T is a tree.

The following is a well known result for positive semidefinite matrices.

LEMMA 2.3. (Column inclusion). *If* $A = \begin{bmatrix} B & y \\ y^T & c \end{bmatrix}$ *is positive semidefinite, then* y *is in the column space of* B.

The following result of van der Holst [15] determines the positive semidefinite minimum rank of a tree.

LEMMA 2.4. *Let* G *be a graph on* n *vertices. Then* $\operatorname{mr}_+(G) = n - 1$ *if and only if* G *is a tree.*

LEMMA 2.5. *Let* $\Lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$ *and* $U = \mu_1, \mu_2, \ldots, \mu_{n-1}$ *be lists of real numbers satisfying the interlacing inequalities* (1.1), let T be a tree on n vertices and let v be a vertex of T. If (Λ, U) is realizable for (T, v), then $\lambda_1 > \mu_1$ and $\mu_{n-1} > \lambda_n$.

Proof. Let $A \in \mathcal{S}(T)$ such that A has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $A(v)$ has eigenvalues $\mu_1, \mu_2, \ldots, \mu_{n-1}$. Since we may shift the eigenvalues of A by subtracting $\lambda_n I$, we will assume that $\lambda_n = 0$. Thus, A is positive semidefinite. Since $\operatorname{mr}_+(T) = n - 1$, $\operatorname{rank} A = n - 1$. Assume the last row and column is labeled by v and let $A = \begin{bmatrix} B & x \\ x^T & a \end{bmatrix}$. By column inclusion, x is in the column space of B. Thus, $x = By$ for some vector y. Since $\operatorname{rank} A = \operatorname{mr}_+(T)$, $A = \begin{bmatrix} B & By \\ y^T B & y^T By \end{bmatrix}$ and $\operatorname{rank} B = \operatorname{rank} A = n - 1$.

Thus, B is invertible and $\mu_{n-1} > 0 = \lambda_n$. Similarly $\lambda_1 > \mu_1$. ☐

We also need a result of Mirsky [11] which we cite in the form given by Boley-Golub [4].

LEMMA 2.6. *Given* $2n - 1$ *real numbers* $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$ *such that* $\mu_i \neq \mu_j$ *for all* $i \neq j$, *there exists an* $n \times n$ *bordered matrix* $A = \begin{bmatrix} a & b^T \\ b & M \end{bmatrix}$ *with eigenvalues* $\lambda_1, \ldots, \lambda_n$ *where* $M = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_{n-1})$ *and*

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$$b_i^2 = -\frac{\displaystyle\prod_{j=1}^{n}(\mu_i - \lambda_j)}{\displaystyle\prod_{\substack{j=1\\ j\neq i}}^{n-1}(\mu_i - \mu_j)},\ 1 \le i \le n-1.$$

THEOREM 2.7. ((S_n, v), v dominating). *Let $n \geq 3$, $\Lambda = \lambda_1, \dots, \lambda_n$ and $U = \mu_1, \dots, \mu_{n-1}$ be lists of real numbers satisfying the interlacing inequalities* (1.1), and let v be the dominating vertex of S_n. Then (Λ, U) is realizable for (S_n, v) if and only if $\lambda_1 > \mu_1$, $\mu_{n-1} > \lambda_n$, and for every $k \in \{1, 2, \dots, n-2\}$, either $\mu_k > \lambda_{k+1} > \mu_{k+1}$ or else $\mu_k = \lambda_{k+1} = \mu_{k+1}$.

Proof. We prove the reverse implication by induction on n. Assume $n = 3$, $\lambda_1 > \mu_1$, and $\mu_2 > \lambda_3$.

If $\mu_1 > \lambda_2 > \mu_2$, then by Lemma 2.6 there exists a bordered matrix $A = \begin{bmatrix} a & b^T \\ b & M \end{bmatrix}$ with eigenvalues λ_1, λ_2, and λ_3 such that $D = \operatorname{diag}\{\mu_1, \mu_2\}$. Since all λ_i's and μ_i's are distinct, then $b_i \neq 0$ for all i. Thus, $A \in \mathcal{S}(S_3)$ and satisfies the spectral conditions for the both the λ_i's and the μ_i's.

If $\mu_1 = \lambda_2 = \mu_2$, then let $A = \begin{bmatrix} r & s & s \\ s & \lambda_2 & 0 \\ s & 0 & \lambda_2 \end{bmatrix}$ where $r = \lambda_1 + \lambda_3 - \lambda_2$ and $s = \sqrt{(1/2)(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)}$. Since $\lambda_1 > \mu_1 = \lambda_2$ and $\lambda_2 = \mu_2 > \lambda_3$, $s \neq 0$. Thus, $A \in \mathcal{S}(S_3)$ and the eigenvalues of $A(1)$ are μ_1 and μ_2. Let $C = A - \lambda_2 I_3$. Then C is singular, $\operatorname{tr} C = \lambda_1 + \lambda_3 - 2\lambda_2$ and the sum of the 2×2 principal minors of C is $(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2)$. Thus, the eigenvalues of C are $\lambda_1 - \lambda_2$, $\lambda_3 - \lambda_2$, and 0, so A has eigenvalues λ_1, λ_2, and λ_3.

Let $\lambda_1 \geq \mu_2 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$ be given such that $\lambda_1 > \mu_1$, $\mu_{n-1} > \lambda_n$ and for every $k \in \{1, 2, \dots, n-2\}$, either $\mu_k > \lambda_{k+1} > \mu_{k+1}$ or $\mu_k = \lambda_{k+1} = \mu_{k+1}$. If $\mu_k > \lambda_{k+1} > \mu_{k+1}$ for every $k \in \{1, 2, \dots, n-2\}$, then we can construct the desired matrix using Lemma 2.6. So assume that $\mu_k = \lambda_{k+1} = \mu_{k+1}$ for some k. Deleting λ_{k+1} and μ_{k+1} from our list of $2n-1$ numbers, we can apply the induction hypothesis to the $2n-3$ remaining numbers. Thus, there exists a matrix $B \in \mathcal{S}(S_{n-1})$ with eigenvalues $\lambda_1, \dots, \lambda_k, \lambda_{k+2}, \dots, \lambda_n$ such that $B(v)$ has eigenvalues $\mu_1, \dots, \mu_k, \mu_{k+2}, \dots, \mu_{n-1}$ where v is the dominating vertex of S_{n-1}.

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A publication of the International Linear Algebra Society
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With an appropriate labeling of the S_{n-1}, let $B = \begin{bmatrix} D & x \\ x^t & y \end{bmatrix}$ such that $D = \operatorname{diag}\{\mu_k, \ldots, \mu_1, \mu_{k+2}, \ldots, \mu_{n-1}\}$. Let $A' = [\mu_{k+1}] \oplus B$. Note that since $\mu_k = \lambda_{k+1} = \mu_{k+1}$, the eigenvalues of A' are $\lambda_1, \ldots, \lambda_n$ and the eigenvalues of $A'(n)$ are $\mu_1, \ldots, \mu_{n-1}$. Let Q_2 be a 2×2 orthogonal matrix with no zero entries and let $Q = Q_2 \oplus I_{n-2}$. Note that Q is an $n \times n$ orthogonal matrix. Let $A = Q^T A' Q$. Since Q is an orthogonal matrix, the eigenvalues of A are $\lambda_1, \ldots, \lambda_n$. Since $a'_{11} = \mu_k = \mu_{k+1} = a'_{22}$, $a_{11} = a_{22} = \mu_k$. Further, $a_{12} = a_{21} = 0$. Since Q_2 is an orthogonal matrix with no zero entries, $a_{1n} = a_{n1} \neq 0$. Thus, $A \in \mathcal{S}(S_n)$.

Now for the forward implication. Note that S_n is a tree. Thus, by Lemma 2.5, $\lambda_1 > \mu_1$ and $\mu_{n-1} > \lambda_n$. Now suppose by way of contradiction that there exists $A \in \mathcal{S}(S_n)$, such that $\mu_k > \lambda_{k+1} = \mu_{k+1}$ for some $k \in \{1, 2, \ldots, n-2\}$. Label the vertices of S_n so that the dominating vertex corresponds to vertex 1. Let q be the multiplicity of λ_{k+1}. Note that the multiplicity of μ_{k+1} is at most q by the interlacing condition. Consider $B = A - \lambda_{k+1} I_n$. Since the multiplicity of λ_{k+1} is q, $\operatorname{rank} B = n - q$. Since the multiplicity of μ_{k+1} is at most q, there are at most q zero entries on the diagonal of $B(1)$. Thus, $\operatorname{rank} B(1) \geq n - 1 - q$. Let $B = \begin{bmatrix} a_{11} - \lambda_{k+1} & w^T \\ w & B(1) \end{bmatrix}$. Since B is also in $\mathcal{S}(S_n)$ and the first row and column of B correspond to the dominating vertex of S_n, all the entries of w are nonzero. Since there is at least one zero on the diagonal of $B(1)$, w is not a linear combination of the columns of $B(1)$. Similarly, the first row of B is not a linear combination of the other rows of B. Thus, $\operatorname{rank} B = \operatorname{rank} B(1) + 2 \geq n - q + 1$ contradicting that $\operatorname{rank} B = n - q$. We arrive at a similar contradiction if $\mu_k = \lambda_{k+1} > \mu_{k+1}$ for some $k \in \{1, 2, \ldots, n-2\}$. Therefore, either $\mu_k = \lambda_{k+1} = \mu_{k+1}$ or $\mu_k > \lambda_{k+1} > \mu_{k+1}$ for every $k \in \{1, 2, \ldots, n-2\}$. ∎

We conclude this section with an extreme case of Theorem 2.7 for which it is possible to give an explicit form for all matrices achieving the desired eigenvalues.

THEOREM 2.8. *Let $\Lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$ and $U = \mu_1, \mu_2, \ldots \mu_{n-1}$ be lists of real numbers with*

$$\lambda_1 > \mu_1 = \lambda_2 = \mu_2 = \cdots = \lambda_{n-1} = \mu_{n-1} > \lambda_n.$$

Then $A = \begin{bmatrix} a & b^T \\ b & \lambda_2 I_{n-1} \end{bmatrix}$ has eigenvalues Λ if and only if

$$a = \lambda_1 + \lambda_n - \lambda_2 \text{ and } \|b\| = \sqrt{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_n)}.$$

Proof. The matrix A has eigenvalues Λ if and only if $A - \lambda_2 I_n$ has eigenvalues $\lambda_1 - \lambda_2$, $\lambda_n - \lambda_2$, and 0 with multiplicity $n-2$ or equivalently if the characteristic

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 742-778, October 2014

polynomial of $A - \lambda_2 I_n$ is $t^{n-2}[t^2 - (\lambda_1 + \lambda_n - 2\lambda_2)t + (\lambda_1 - \lambda_2)(\lambda_n - \lambda_2)]$. This is equivalent to $\mathrm{tr}(A - \lambda_2 I_n) = \lambda_1 + \lambda_n - 2\lambda_2$ and the sum of the 2×2 principal minors of $A - \lambda_2 I_n$ is $(\lambda_1 - \lambda_2)(\lambda_n - \lambda_2)$. But since $\mathrm{tr}(A - \lambda_2 I_n) = a - \lambda_2$ and the sum of the 2×2 principal minors of $A - \lambda_2 I_n$ is $-\|b\|^2$, this is equivalent to $a = \lambda_1 + \lambda_n - \lambda_2$ and $\|b\| = \sqrt{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_n)}$. □

Note that under the hypothesis of Theorem 2.8, in order for (Λ, U) to be realizable for (G, v), G connected, $G - v$ can have no edges and G must be S_n.

OBSERVATION 2.9. If $\Lambda = \lambda_1, \ldots, \lambda_n$, $U = \mu_1, \ldots, \mu_{n-1}$, and $\mu_1 = \mu_2 = \cdots = \mu_{n-1}$, then (Λ, U) is realizable for (G, v) with G connected only if $G = S_n$ and v is the dominating vertex of S_n.

3. The (Λ, U)-problem for general graphs.

We now resume our discussion in the introduction for any connected graph G. We ask whether or not (Λ, U) satisfying (1.1) is realizable for (G, v), v a vertex of G. The phrase "all cases" in the title of the paper means that for each $\geq$ in (1.1) we consider the two possibilities $=$ and $>$. This means that there are 2^{2n-2} cases to consider. We can reduce this number considerably by means of the following results, the first of which is a slightly modified statement of Lemma 1.2 in [14].

LEMMA 3.1. *Let A be an $n \times n$ real symmetric matrix with eigenvalues $\Lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$ and suppose that $B = A(1)$ has eigenvalues $U = \mu_1, \mu_2, \ldots, \mu_{n-1}$. If U is a sublist of Λ, then $A = a_{11} \oplus B$.*

COROLLARY 3.2. *Let G be a graph with vertex v. Assume the matrix $A \in \mathcal{S}(G)$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ and $A(v)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$. If U is a sublist of Λ, then v is an isolated vertex of G.*

We now explain how Lemma 3.1 rules out a substantial number of these cases.

Let a_n, $n \geq 2$, be the number of cases in (1.1) for which U is a sublist of Λ, and let b_n, $n \geq 2$, be the number of cases in (1.1) for which U is a sublist of Λ and $\lambda_1 > \mu_1$. For $n = 2$, we have the four cases

$$\lambda_1 = \mu_1 = \lambda_2, \qquad \lambda_1 > \mu_1 = \lambda_2, \qquad \lambda_1 = \mu_1 > \lambda_2, \qquad \lambda_1 > \mu_1 > \lambda_2,$$

so $a_2 = 3$ and $b_2 = 1$.

Now assume $n \geq 3$. If $\lambda_1 = \mu_1$, then for either case, $\mu_1 = \lambda_2$, $\mu_1 > \lambda_2$, there are a_{n-1} ways that $\mu_2, \ldots, \mu_n$ can be a sublist of $\lambda_2, \ldots, \lambda_n$ giving $2a_{n-1}$ cases in which U is a sublist of Λ. And if $\lambda_1 > \mu_1$, there are b_n ways that U can be a sublist of Λ. Therefore, $a_n = 2a_{n-1} + b_n$.

We can explicitly determine b_n. Since $\lambda_1 > \mu_1$ and U is a sublist of Λ, $\mu_i = \lambda_{i+1}$ for $i = 1, \ldots, n-1$. But for each $\lambda_i \geq \mu_i$, $i = 2, \ldots, n-1$, the $\geq$ can be either $=$ or $>$.

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A publication of the International Linear Algebra Society
Volume 27, pp. 742-778, October 2014

So if $\lambda_1 > \mu_1$, there are exactly 2^{n-2} cases in which U is a sublist of Λ; i.e., $b_n = 2^{n-2}$. We now have $a_n = 2a_{n-1} + 2^{n-2}$ and $a_2 = 3$. Solving this difference equation by the standard method, we have $a_n = (n+1)2^{n-2}$. So the number of realizable cases in (1.1) when G is connected is at most $2^{2n-2} - (n+1)2^{n-2}$.

We can reduce the number of cases that must be considered yet further. We begin with an example for $n = 3$. Suppose:

- G is a connected graph on 3 vertices,
- v is a vertex of G,
- $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3$,
- and $\Lambda = \lambda_1, \lambda_2, \lambda_3$ and $U = \mu_1, \mu_2$ are realizable for (G, v).

Then $-\Lambda = -\lambda_3, -\lambda_2, -\lambda_1$ and $-U = -\mu_2, -\mu_1$ are realizable for (G, v).

To see this, if A is a matrix for which (Λ, U) is realizable for (G, v), then $-A$ is a matrix for which $(-\Lambda, -U)$ is realizable for (G, v), and $-\lambda_3 = -\mu_2 = -\lambda_2 > -\mu_2 > -\lambda_1$.

Thus, when considering the (Λ, U)-problem for $n = 3$, it suffices to consider only one of the strings $>>==$ and $==>>$.

In general, for an arbitrary positive integer n, we use the notation

$$f = f_1\, f_2\, \ldots f_{2n-2},$$

where each f_i is either $=$ or $>$, and call this a $(2n-2)$–EI string or EI string, where EI is an abbreviation for equality/inequality. Given

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \lambda_n$$

let $f_1\, f_2 \ldots f_{2n-2}$ be the unique EI string such that

$$\lambda_1\, f_1\, \mu_1\, f_2\, \lambda_2\, f_3\, \mu_2\, f_4 \ldots f_{2n-3}\, \mu_{n-1}\, f_{2n-2}\, \lambda_n.$$

Then $(-\lambda_n)\, f_{2n-2}\, (-\mu_{n-1})\, f_{2n-3} \ldots (-\mu_2)\, f_3\, (-\lambda_2)\, f_2\, (-\mu_1)\, f_1\, (-\lambda_1)$. By the same argument as above, it suffices to consider only one of the EI strings:

$$f_1\, f_2 \cdots f_{2n-1}\, f_{2n-2} \qquad f_{2n-2}\, f_{2n-1} \cdots f_2\, f_1$$

when considering the (Λ, U)-problem for a pair (G, v), where G is a connected graph on n vertices. We call $f_{2n-2}\, f_{2n-2} \ldots f_2\, f_1$ the *reversal* of $f_1\, f_2 \ldots f_{2n-1}\, f_{2n-2}$ and the pair of EI strings a *symmetric pair*. If an EI string f and its reversal are the same we call f a *palindromic* EI string.

We are now ready to carry out our main goal, to identify all realizable EI strings for connected graphs on 4 vertices.

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4. The connected 4-vertex graphs. We now restrict to the inverse eigenvalue problem:

Given a connected graph G on 4 vertices, a vertex v of G, and two lists of real numbers

$$\Lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4, \qquad U = \mu_1, \mu_2, \mu_3,$$

satisfying the interlacing inequalities

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4 \tag{4.1}$$

is there a matrix $A \in \mathcal{S}(G)$ such that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the eigenvalues of A and μ_1, μ_2, μ_3 are the eigenvalues of $A(v)$?

As discussed in section 3, there are $2^{2n-2} = 64$ possible EI strings for (4.1) and $2^{2n-2} - (n+1)2^{n-2} = 44$ strings for which U is not a sublist of Λ. It is clear that the number of palindromic EI strings for $n = 4$ is $2^3 = 8$. For two of these, $======$, $=>==>=$, U is a sublist of Λ. Thus, 6 of the 44 strings for which U is not a sublist of Λ are palindromic leaving 38 which are not. These 38 EI strings constitute 19 symmetric pairs, and as explained at the end of the last section, we need consider only one of each pair. Therefore, it suffices to consider $19 + 6 = 25$ potentially realizable EI strings. By Observation 2.9, the string $>====>$ is only realizable for $G = S_4$ and v the vertex of degree 3. Since this case has been solved for all n in Theorem 2.7, we can disregard it. Finally, we know from Theorem 1.1 that the string $>>>>>>$ is always realizable, so we also need not consider it. This leaves 23 potentially realizable EI strings to investigate, which are:

1. $> \; > \; = \; = \; = \; =$
2. $> \; = \; = \; > \; = \; =$
3. $= \; = \; > \; > \; = \; =$
4. $> \; > \; > \; = \; = \; =$
5. $> \; > \; = \; > \; = \; =$
6. $> \; > \; = \; = \; > \; =$
7. $> \; > \; = \; = \; = \; >$
8. $> \; = \; > \; > \; = \; =$
9. $> \; = \; > \; = \; = \; >$
10. $> \; = \; = \; > \; > \; =$
11. $= \; > \; > \; > \; = \; =$
12. $> \; > \; > \; > \; = \; =$
13. $> \; > \; > \; = \; > \; =$
14. $> \; > \; = \; > \; > \; =$
15. $> \; = \; > \; > \; > \; =$
16. $= \; > \; > \; > \; > \; =$
17. $> \; > \; > \; = \; = \; >$
18. $> \; > \; = \; > \; = \; >$
19. $> \; = \; > \; > \; = \; >$
20. $> \; > \; = \; = \; > \; >$
21. $> \; > \; > \; > \; > \; =$
22. $> \; > \; > \; > \; = \; >$
23. $> \; > \; > \; = \; > \; >$

It remains to consider the 23 strings above for each connected graph G on 4 vertices and each vertex v of G. Since we treated the case K_n for all n in [3], and (P_n, v) and (S_n, v) for a pendant and dominating vertex, respectively in section 2, it remains to deal with the following cases;

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- (P_4, v), v degree 2
- (S_4, v), v pendant
- (paw, v), v pendant
- (paw, v), v degree 2
- (paw, v), v dominating
- $(diamond, v)$, v degree 2
- $(diamond, v)$, v dominating
- (C_4, v)

4.1. The case (P_4, v), v degree 2.

We first consider which of the 23 EI strings are possible. If $\lambda_i = \lambda_{i+1}$ for some i, then $2 \geq \text{rank}(A - \lambda_i) \geq \text{mr}(P_4) = 3$, a contradiction. Consequently, the EI strings 1–8, 11–12, 20 cannot occur. If a solitary $=$ sign occurs in an EI string (one not preceded nor followed by an $=$ sign), we have $\lambda_i = \mu_j$ for $j = i-1$ or i with the other two μ_k distinct from μ_j. Then $\text{rank}(A - \lambda_i I) = 3 = \text{mr}(P_4)$ while $\text{rank}(A(v) - \lambda_i I) = 2$ contradicting Lemma 2.1. Therefore the EI strings 9–10, 13–16, 18–19, 21–23 also cannot occur. This leaves only EI string 17 and we now show that any (Λ, U) satisfying 17 is realizable. By Theorem 2.7, there exists

$$A = \begin{bmatrix} \mu_1 & 0 & 0 & a \\ 0 & \mu_2 & 0 & b \\ 0 & 0 & \mu_3 & c \\ a & b & c & d \end{bmatrix} \in \mathcal{S}(S_4)$$

such that Λ is the list of eigenvalues of A. Let $Q_2 = \dfrac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} b & a \\ -a & b \end{bmatrix}$, $Q = Q_2 \oplus I_2$, and $B = Q^T A Q$. Since $\mu_1 > \mu_2$,

$$B = \begin{bmatrix} r & s & 0 & 0 \\ s & t & 0 & \sqrt{a^2+b^2} \\ 0 & 0 & \mu_3 & c \\ 0 & \sqrt{a^2+b^2} & c & d \end{bmatrix}$$

where $s \neq 0$. Then $B \in \mathcal{S}(P_4)$, Λ is the list of eigenvalues of B, and U is the list of eigenvalues of $B(4)$ where 4 is a vertex of degree 2. Thus, (Λ, U) is realizable for (P_4, v), v degree 2.

Incorporating the case of distinct λ_i, μ_i and the reversal of EI string 17 we have the following theorem.

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THEOREM 4.1. $((P_4, v),\ v$ degree 2). *Given any* $\Lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ *and* $U = \mu_1, \mu_2, \mu_3$ *satisfying the interlacing inequalities, and* v *a vertex of* P_4 *of degree 2, then* (Λ, U) *is realizable for* (P_4, v) *if and only if either*

- $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$,
- $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$,
  or
- $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$.

This completes the two cases involving P_4.

Before proceeding with the cases involving the other graphs, we consider (Λ, U) corresponding to EI string 1; i.e.,

$$\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3 = \mu_3 = \lambda_4. \tag{4.2}$$

If A is any 4×4 matrix with eigenvalues Λ, $\mathrm{rank}(A - \lambda_2 I) = 1$. But it is well known that each of S_4, the paw, C_4, and the diamond have minimum rank 2. Thus if G is any of these graphs, (4.2) is not realizable for (G, v), so we need no longer consider this string.

4.2. The case (S_4, v), v pendant.

THEOREM 4.2. $((S_4, v),\ v$ pendant). *Given any* $\Lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ *and* $U = \mu_1, \mu_2, \mu_3$ *satisfying the interlacing inequalities, and* v *a pendant vertex of* S_4*, then* (Λ, U) *is realizable for* (S_4, v) *if and only if either*

- $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 > \mu_3 > \lambda_4$ *and*
$$\mu_2 = \frac{\mu_1\mu_3(\lambda_1 + \lambda_2 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1\lambda_2\lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_2) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_2 - \mu_3)(\mu_3 - \lambda_4)},$$
- $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$ *and*
$$\mu_2 = \frac{\mu_1\mu_3(\lambda_1 + \lambda_3 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1\lambda_3\lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_3) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_3 - \mu_3)(\mu_3 - \lambda_4)},$$
- $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3 > \mu_3 > \lambda_4$ *and* $\lambda_1 + \lambda_4 = \mu_1 + \mu_3$,
  or
- $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$.

Remark: On examination of the list of 23 EI strings near the beginning of Section 4, (Λ, U) is realizable for (S_4, v), v pendant if and only if either all (Λ, U) are distinct or else (Λ, U) satisfies either the string 20, 23 or its reversal with additional restrictions.

Proof. $(\Leftarrow)$ The sufficiency of the last case follows from Theorem 1.1.

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A publication of the International Linear Algebra Society
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As for the first two cases, we need only consider the first one because they are a symmetric pair. Let $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 > \mu_3 > \lambda_4$. By Theorem 2.2, there exists $C = \begin{bmatrix} a & x & 0 \\ x & b & y \\ 0 & y & c \end{bmatrix}$, $xy \neq 0$, such that $\lambda_1, \lambda_2, \lambda_4$ are the eigenvalues of C and μ_1, μ_3 are the eigenvalues of $\begin{bmatrix} a & x \\ x & b \end{bmatrix}$. We show in the Appendix that in this circumstance, a is determined by $\lambda_1, \lambda_2, \lambda_4, \mu_1$, and μ_3. By Theorem A.1 in the Appendix and the condition in Theorem 4.2,

$$a = \frac{\mu_1\mu_3(\lambda_1 + \lambda_2 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1\lambda_2\lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_2) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_2 - \mu_3)(\mu_3 - \lambda_4)} = \mu_2.$$

Let

$$B = \begin{bmatrix} \mu_2 & 0 & 0 & 0 \\ 0 & a & x & 0 \\ 0 & x & b & y \\ 0 & 0 & y & c \end{bmatrix}.$$

Then B has eigenvalues Λ and $B(4)$ has eigenvalues U. Let $P = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ be an orthogonal matrix with $pq \neq 0$. Then $A = (P^T \oplus I_2)B(P \oplus I_2) \in \mathcal{S}(S_4)$, A has eigenvalues Λ and $A(4)$ has eigenvalues U.

If $\lambda_1 > \mu_1 > \lambda_2 = \mu_2 = \lambda_3 > \mu_3 > \lambda_4$, we choose $C = \begin{bmatrix} a & x & 0 \\ x & b & y \\ 0 & y & c \end{bmatrix}$, $xy \neq 0$ such that $\lambda_1, \mu_2, \lambda_4$ are the eigenvalues of C and μ_1, μ_3 are the eigenvalues of $\begin{bmatrix} a & x \\ x & b \end{bmatrix}$. Applying Theorem A.1 again to determine a, a calculation then shows that the simpler condition $\lambda_1 + \lambda_4 = \mu_1 + \mu_3$ forces $a = \mu_2$. The rest of the proof is the same as the previous case.

($\Rightarrow$) Assume

$$A = \begin{bmatrix} d_1 & 0 & a & 0 \\ 0 & d_2 & b & 0 \\ a & b & d_3 & c \\ 0 & 0 & c & d_4 \end{bmatrix} \in \mathcal{S}(S_4)$$

such that Λ is the list of eigenvalues of A and U is the list of eigenvalues of $A(4)$. Furthermore, assume there is at least one equality among Λ and U. We will show that equality can only occur around μ_2 and that the restriction on μ_2 must be satisfied.

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A publication of the International Linear Algebra Society
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Let Q_2 and Q be as in the case (P_4, v), v degree 2. Then

$$B = Q^T A Q = \begin{bmatrix} r & s & 0 & 0 \\ s & t & \sqrt{a^2+b^2} & 0 \\ 0 & \sqrt{a^2+b^2} & d_3 & c \\ 0 & 0 & c & d_4 \end{bmatrix}.$$

By Theorem 2.2, we must have $s = 0$. Then

$$Q_2^T \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} Q_2 = \begin{bmatrix} r & 0 \\ 0 & t \end{bmatrix}$$

which implies $d_1 = d_2$, $r = t$, and

$$B = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & r & \sqrt{a^2+b^2} & 0 \\ 0 & \sqrt{a^2+b^2} & d_3 & c \\ 0 & 0 & c & d_4 \end{bmatrix}.$$

Moreover, Λ is the list of eigenvalues of B and U is the list of eigenvalues of $B(4)$.

Note that the r in the first row and column implies $r \in \Lambda$ and $r \in U$.

If $r = \mu_3$, then μ_1, μ_2 are the eigenvalues of $\begin{bmatrix} r & \sqrt{a^2+b^2} \\ \sqrt{a^2+b^2} & d_3 \end{bmatrix}$, which means $r > \mu_2$. However, $\mu_2 \geq \mu_3 = r$, a contradiction.

Similarly, a contradiction arises if $r = \mu_1$. Thus, $r = \mu_2$.

Case 1. $r = \mu_2 = \lambda_3$. Since $B(1) = \begin{bmatrix} r & \sqrt{a^2+b^2} & 0 \\ \sqrt{a^2+b^2} & d_3 & c \\ 0 & c & d_4 \end{bmatrix} \in \mathcal{S}(P_3)$, by Theorems 2.2 and A.1, we have $\lambda_1 > \mu_1 > \lambda_2 > \mu_3 > \lambda_4$ and

$$\mu_2 = r = \frac{\mu_1\mu_3(\lambda_1 + \lambda_2 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1\lambda_2\lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_2) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_2 - \mu_3)(\mu_3 - \lambda_4)}$$

as desired.

Case 2. $r = \mu_2 = \lambda_2$. Interchanging the roles of λ_2 and λ_3 we have $\lambda_1 > \mu_1 > \lambda_3 > \mu_3 > \lambda_4$ and

$$\mu_2 = r = \frac{\mu_1\mu_3(\lambda_1 + \lambda_3 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1\lambda_3\lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_3) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_3 - \mu_3)(\mu_3 - \lambda_4)}.$$

Case 3. $r = \mu_2 = \lambda_2 = \lambda_3$. Then the condition in either Case 1 or Case 2 becomes

$$\mu_2[(\lambda_1 - \mu_1)(\mu_1 - \mu_2) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\mu_2 - \mu_3)(\mu_3 - \lambda_4)]$$

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$$= \mu_1\mu_3(\lambda_1 + \mu_2 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1\mu_2\lambda_4$$

which can be rewritten

$$(\lambda_1 + \lambda_4 - \mu_1 - \mu_3)(\mu_1 - \mu_2)(\mu_2 - \mu_3) = 0,$$

so $\lambda_1 + \lambda_4 = \mu_1 + \mu_3$. □

4.3. The case (paw, v), v pendant.

THEOREM 4.3. ((paw, v), v pendant). *Given any $\Lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $U = \mu_1, \mu_2, \mu_3$ satisfying the interlacing inequalities, and v a pendant vertex of paw, then (Λ, U) is realizable for (paw, v) if and only if either $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$ or else Λ can be partitioned into $\alpha_1, \alpha_2, \alpha_3, \gamma$ and U can be partitioned into β_1, β_2, γ such that*

- $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \alpha_3$,
- *if $\mu_1 > \lambda_2 = \gamma = \mu_2 > \lambda_3$, then*
$$\mu_2 \neq \frac{\mu_1\mu_3(\lambda_1 + \lambda_3 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1\lambda_3\lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_3) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_3 - \mu_3)(\mu_3 - \lambda_4)},$$
- *if $\lambda_2 > \mu_2 = \gamma = \lambda_3 > \mu_3$, then*
$$\mu_2 \neq \frac{\mu_1\mu_3(\lambda_1 + \lambda_2 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1\lambda_2\lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_2) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_2 - \mu_3)(\mu_3 - \lambda_4)},$$
- *if $\lambda_2 = \mu_2 = \lambda_3$, then $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$.*

Remark: On examination of the list of 23 EI strings near the beginning of Section 4, the partitioning is possible for exactly any (Λ, U) satisfying one of the strings 12, 17, 20–23 or its reversal, where 20, 23 require some additional restrictions on the specific value of the λ's and μ's.

Proof. ($\Leftarrow$) The case $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$ follows from Theorem 1.1. As for the other cases, we use the following construction. Since $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \alpha_3$, by Theorem 2.2, there exists $C = \begin{bmatrix} a & x & 0 \\ x & b & y \\ 0 & y & c \end{bmatrix}$, $xy \neq 0$, such that $\alpha_1, \alpha_2, \alpha_3$ are the eigenvalues of C and β_1, β_2 are the eigenvalues of $\begin{bmatrix} a & x \\ x & b \end{bmatrix}$. Let

$$B = \begin{bmatrix} \gamma & 0 & 0 & 0 \\ 0 & a & x & 0 \\ 0 & x & b & y \\ 0 & 0 & y & c \end{bmatrix}.$$

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A publication of the International Linear Algebra Society
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Then B has eigenvalues Λ and $B(4)$ has eigenvalues U.

We claim $\gamma \neq a$:

If $\gamma = \mu_3$, then $\beta_1 = \mu_1$ and $\beta_2 = \mu_2$. Thus, $\gamma = \mu_3 \leq \mu_2 < a$.

If $\gamma = \mu_1$, then $\beta_1 = \mu_2$ and $\beta_2 = \mu_3$. Thus, $\gamma = \mu_1 \geq \mu_2 > a$.

If $\mu_1 > \lambda_2 = \gamma = \mu_2 > \lambda_3$, then by hypothesis and Theorem A.1,

$$\gamma = \mu_2 \neq \frac{\mu_1\mu_3(\lambda_1 + \lambda_3 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1\lambda_3\lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_3) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_3 - \mu_3)(\mu_3 - \lambda_4)} = a.$$

Similarly if $\lambda_2 > \mu_2 = \gamma = \lambda_3 > \mu_3$, then

$$\gamma = \mu_2 \neq \frac{\mu_1\mu_3(\lambda_1 + \lambda_2 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1\lambda_2\lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_2) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_2 - \mu_3)(\mu_3 - \lambda_4)} = a.$$

If $\gamma = \lambda_2 = \mu_2 = \lambda_3$, then essentially the same calculation as Case 3 of the previous theorem shows that $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$ implies

$$\gamma = \mu_2 \neq \frac{\mu_1\mu_3(\lambda_1 + \lambda_2 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1\lambda_2\lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_2) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_2 - \mu_3)(\mu_3 - \lambda_4)} = a.$$

Let $P = \begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ be an orthogonal matrix with $pq \neq 0$. Then

$$A = (P^T \oplus I_2)B(P \oplus I_2) = \begin{bmatrix} r & s & qx & 0 \\ s & t & px & 0 \\ qx & px & b & y \\ 0 & 0 & y & c \end{bmatrix} \in \mathcal{S}(paw)$$

since $s \neq 0$. Moreover, A has eigenvalues Λ and $A(4)$ has eigenvalues U.

$(\Rightarrow)$ Assume

$$A = \begin{bmatrix} d_1 & f & a & 0 \\ f & d_2 & b & 0 \\ a & b & d_3 & c \\ 0 & 0 & c & d_4 \end{bmatrix} \in \mathcal{S}(paw)$$

such that Λ is the list of eigenvalues of A and U is the list of eigenvalues of $A(4)$. Furthermore, assume there is at least one equality among Λ and U. We will show that it must be the case that Λ and U can be partitioned in the desired way.

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A publication of the International Linear Algebra Society
Volume 27, pp. 742-778, October 2014

Let Q_2 and Q be as in the case (P_4, v), v degree 2. Then

$$B = Q^T AQ = \begin{bmatrix} r & s & 0 & 0 \\ s & t & \sqrt{a^2+b^2} & 0 \\ 0 & \sqrt{a^2+b^2} & d_3 & c \\ 0 & 0 & c & d_4 \end{bmatrix}.$$

By Theorem 2.2, we must have $s = 0$. Then

$$Q_2^T \begin{bmatrix} d_1 & f \\ f & d_2 \end{bmatrix} Q_2 = \begin{bmatrix} r & 0 \\ 0 & t \end{bmatrix}$$

which implies $r \neq t$, and

$$B = \begin{bmatrix} r & 0 & 0 & 0 \\ 0 & t & \sqrt{a^2+b^2} & 0 \\ 0 & \sqrt{a^2+b^2} & d_3 & c \\ 0 & 0 & c & d_4 \end{bmatrix}.$$

Moreover, Λ is the list of eigenvalues of B and U is the list of eigenvalues of $B(4)$.

Note that the r in the first row and column implies $r \in \Lambda$ and $r \in U$. Thus we may write $\Lambda = r, \alpha_1, \alpha_2, \alpha_3$, $U = r, \beta_1, \beta_2$, where $\Lambda' = \alpha_1, \alpha_2, \alpha_3$ is the list of eigenvalues of $B(1)$ and $U' = \beta_1, \beta_2$ is the list of eigenvalues of $\begin{bmatrix} t & \sqrt{a^2+b^2} \\ \sqrt{a^2+b^2} & d_3 \end{bmatrix}$. By Theorem 2.2, $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \alpha_3$.

In addition, if $r = \mu_2 = \lambda_3$, since $B(1) = \begin{bmatrix} t & \sqrt{a^2+b^2} & 0 \\ \sqrt{a^2+b^2} & d_3 & c \\ 0 & c & d_4 \end{bmatrix} \in \mathcal{S}(P_3)$, by Theorem A.1, we have

$$\mu_2 = r \neq t = \frac{\mu_1\mu_3(\lambda_1 + \lambda_2 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1\lambda_2\lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_2) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_2 - \mu_3)(\mu_3 - \lambda_4)}$$

as desired.

Similarly, if $r = \mu_2 = \lambda_2$, we have

$$\mu_2 = r \neq t = \frac{\mu_1\mu_3(\lambda_1 + \lambda_3 + \lambda_4 - \mu_1 - \mu_3) - \lambda_1\lambda_3\lambda_4}{(\lambda_1 - \mu_1)(\mu_1 - \lambda_3) + (\lambda_1 - \mu_1)(\mu_3 - \lambda_4) + (\lambda_3 - \mu_3)(\mu_3 - \lambda_4)}.$$

If $\lambda_2 = \mu_2 = \lambda_3$, the previous condition simplifies to $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$. $\square$

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A publication of the International Linear Algebra Society
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4.4. The case (paw, v), v degree 2.

We begin by ruling out one of the possible EI strings and its reversal.

OBSERVATION 4.4. Given $\Lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $U = \mu_1, \mu_2, \mu_3$ satisfying either

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4 \quad \text{or} \quad \lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4, \tag{4.3}$$

then (Λ, U) is not realizable for (paw, v), v degree 2.

Proof. The eigenvalues of $paw - v \cong P_3$ are μ_1, μ_2, μ_3, so it follows from Theorem 2.2 that they must be distinct. □

LEMMA 4.5. *Suppose* $\Lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ *and* $U = \mu_1, \mu_2, \mu_3$ *satisfy the interlacing inequalities but do not satisfy either condition in* (4.3). Let v be a degree 2 vertex of *paw* and w the pendant vertex of *paw*. Then (Λ, U) is realizable for (paw, v) if and only if (Λ, U) is realizable for (paw, w)

Proof. If $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, the conclusion is true by Theorem 1.1, so assume there is at least one equality among Λ and U.

($\Leftarrow$) Assume (Λ, U) is realizable for (paw, w), where w is the pendant vertex. Then there exists

$$A = \begin{bmatrix} d_1 & f & a & 0 \\ f & d_2 & b & c \\ a & b & d_3 & 0 \\ 0 & c & 0 & d_4 \end{bmatrix} \in \mathcal{S}(paw)$$

such that Λ is the list of eigenvalues of A and U is the list of eigenvalues of $A(4)$.

Let Q_2 and Q be as in the case (P_4, v), v degree 2. Then

$$B = Q^T A Q = \begin{bmatrix} r & s & 0 & \dfrac{-ac}{\sqrt{a^2+b^2}} \\ s & t & \sqrt{a^2+b^2} & \dfrac{bc}{\sqrt{a^2+b^2}} \\ 0 & \sqrt{a^2+b^2} & d_3 & 0 \\ \dfrac{-ac}{\sqrt{a^2+b^2}} & \dfrac{bc}{\sqrt{a^2+b^2}} & 0 & d_4 \end{bmatrix}.$$

Assume $s = 0$. Then $B \in \mathcal{S}(P_4)$ and (Λ, U) is realizable for (P_4, v), v degree 2. Since there is at least one equality among Λ and U, by Theorem 4.1, either $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$ or $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, which is condition (4.3), a contradiction.

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A publication of the International Linear Algebra Society
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Thus, $s \neq 0$. So $B \in \mathcal{S}(paw)$ and (Λ, U) is realizable for (paw, v), v degree 2.

($\Rightarrow$) Assume (Λ, U) is realizable for (paw, v), where v is a degree 2 vertex. Then there exists

$$A = \begin{bmatrix} d_1 & f & 0 & a \\ f & d_2 & c & b \\ 0 & c & d_3 & 0 \\ a & b & 0 & d_4 \end{bmatrix} \in \mathcal{S}(paw)$$

such that Λ is the list of eigenvalues of A and U is the list of eigenvalues of $A(4)$. Let Q_2 and Q be as in the case (P_4, v), v degree 2. Then

$$B = Q^T A Q = \begin{bmatrix} r & s & \dfrac{-ac}{\sqrt{a^2+b^2}} & 0 \\ s & t & \dfrac{bc}{\sqrt{a^2+b^2}} & \sqrt{a^2+b^2} \\ \dfrac{-ac}{\sqrt{a^2+b^2}} & \dfrac{bc}{\sqrt{a^2+b^2}} & d_3 & 0 \\ 0 & \sqrt{a^2+b^2} & 0 & d_4 \end{bmatrix}.$$

Assume $s = 0$. Then $B \in \mathcal{S}(P_4)$ and (Λ, U) is realizable for (P_4, v), v pendant. Since there is at least one equality among Λ and U this is impossible by Theorem 2.2.

Thus, $s \neq 0$. So $B \in \mathcal{S}(paw)$ and (Λ, U) is realizable for (paw, v), v pendant. ∎

Combining Lemma 4.5 and Theorem 4.3 we have the following theorem:

THEOREM 4.6. ((paw, v), v degree 2). *Given any $\Lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $U = \mu_1, \mu_2, \mu_3$ satisfying the interlacing inequalities, and v a degree 2 vertex of paw, then (Λ, U) is realizable for (paw, v) if and only if either $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$ or else Λ can be partitioned into $\alpha_1, \alpha_2, \alpha_3, \gamma$ and U can be partitioned into β_1, β_2, γ such that*

- $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \alpha_3$,
- *if* $\lambda_2 = \gamma = \mu_2 > \lambda_3$, *then* $\mu_1 > \lambda_2$ *and*
$$\mu_2 \neq \frac{\mu_1\mu_3(\lambda_1+\lambda_3+\lambda_4-\mu_1-\mu_3)-\lambda_1\lambda_3\lambda_4}{(\lambda_1-\mu_1)(\mu_1-\lambda_3)+(\lambda_1-\mu_1)(\mu_3-\lambda_4)+(\lambda_3-\mu_3)(\mu_3-\lambda_4)},$$
- *if* $\lambda_2 > \mu_2 = \gamma = \lambda_3$, *then* $\lambda_3 > \mu_3$ *and*
$$\mu_2 \neq \frac{\mu_1\mu_3(\lambda_1+\lambda_2+\lambda_4-\mu_1-\mu_3)-\lambda_1\lambda_2\lambda_4}{(\lambda_1-\mu_1)(\mu_1-\lambda_2)+(\lambda_1-\mu_1)(\mu_3-\lambda_4)+(\lambda_2-\mu_3)(\mu_3-\lambda_4)},$$
- *if* $\lambda_2 = \mu_2 = \lambda_3$, *then* $\lambda_1 + \lambda_4 \neq \mu_1 + \mu_3$.

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A publication of the International Linear Algebra Society
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Remark: On examination of the list of 23 EI strings near the beginning of Section 4, the partitioning is possible for exactly any (Λ, U) satisfying one of the strings 12, 20–23 or its reversal, where 20, 23 would require some additional restrictions on the specific value of the λ's and μ's.

4.5. The case (paw, v), v dominating.

THEOREM 4.7. ((paw, v), v dominating). *Given any $\Lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $U = \mu_1, \mu_2, \mu_3$ satisfying the interlacing inequalities, and v the dominating vertex of paw, then (Λ, U) is realizable for (paw, v) if and only if either $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$ or else Λ can be partitioned into $\alpha_1, \alpha_2, \alpha_3, \gamma$ and U can be partitioned into β_1, β_2, γ such that either*

- $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \alpha_3$, *(with no restriction on γ)*
- $\alpha_1 > \beta_1 = \alpha_2 = \beta_2 > \alpha_3$, $\gamma \neq \alpha_2$.

Remark: On examination of the list of 23 EI strings near the beginning of Section 4, the partitioning is possible for exactly any (Λ, U) satisfying one of the strings 2, 9, 10, 12, 17, 20–23 or its reversal.

Proof. The case $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$ follows from Theorem 1.1 so assume there is at least one equality among the Λ and U.

($\Leftarrow$) Case 1: $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \alpha_3$.

By Theorem 2.7, there exists $C_1 = \begin{bmatrix} \beta_1 & 0 & r \\ 0 & \beta_2 & t \\ r & t & \alpha_1 + \alpha_2 + \alpha_3 - \beta_1 - \beta_2 \end{bmatrix}$, $rt \neq 0$, such that $\alpha_1, \alpha_2, \alpha_3$ are the eigenvalues of C_1. Then

$$C_2 = \begin{bmatrix} \beta_2 & 0 & t \\ 0 & \beta_1 & r \\ t & r & \alpha_1 + \alpha_2 + \alpha_3 - \beta_1 - \beta_2 \end{bmatrix}$$

has the same eigenvalues.

Choose C to be either C_1 or C_2 so that $c_{11} \neq \gamma$, and let $B = \begin{bmatrix} \gamma & 0^T \\ 0 & C \end{bmatrix}$.

Then B has eigenvalues $\alpha_1, \alpha_2, \alpha_3, \gamma$ and $B(4)$ has eigenvalues β_1, β_2, γ.

Let $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ be an orthogonal matrix with $cs \neq 0$, let $Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \oplus I_2$, and let $A = Q^T BQ$. Then $A \in \mathcal{S}(paw)$ with 4 the dominating vertex, A has eigenvalues $\alpha_1, \alpha_2, \alpha_3, \gamma$ and $A(4)$ has eigenvalues β_1, β_2, γ.

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A publication of the International Linear Algebra Society
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Case 2: $\alpha_1 > \beta_1 = \alpha_2 = \beta_2 > \alpha_3$, $\gamma \neq \alpha_2$.

By Theorem 2.7, there exists $C = \begin{bmatrix} \beta_1 & 0 & r \\ 0 & \beta_2 & t \\ r & t & \alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2 \end{bmatrix}$, $rt \neq 0$, such that $\alpha_1, \alpha_2, \alpha_3$ are the eigenvalues of C.

Let $B = \begin{bmatrix} \gamma & 0^T \\ 0 & C \end{bmatrix}$, then B has eigenvalues $\alpha_1, \alpha_2, \alpha_3, \gamma$ and $B(4)$ has eigenvalues β_1, β_2, γ.

Let $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ be an orthogonal matrix with $cs \neq 0$. Let $Q = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \oplus I_2$, and let $A = Q^T BQ$.

Since $\gamma \neq \alpha_2 = \beta_1$, $A \in \mathcal{S}(paw)$ with 4 the dominating vertex, A has eigenvalues $\alpha_1, \alpha_2, \alpha_3, \gamma$ and $A(4)$ has eigenvalues β_1, β_2, γ.

($\Rightarrow$) Assume

$$A = \begin{bmatrix} d_1 & f & 0 & a \\ f & d_2 & 0 & b \\ 0 & 0 & d_3 & c \\ a & b & c & d_4 \end{bmatrix} \in \mathcal{S}(paw)$$

such that Λ is the list of eigenvalues of A and U is the list of eigenvalues of $A(4)$. We will show that (Λ, U) can be partitioned as indicated in the theorem.

Let Q_2 and Q be as in the case (P_4, v), v degree 2. Then

$$B = Q^T AQ = \begin{bmatrix} r & s & 0 & 0 \\ s & t & 0 & \sqrt{a^2+b^2} \\ 0 & 0 & d_3 & c \\ 0 & \sqrt{a^2+b^2} & c & d_4 \end{bmatrix}$$

and Λ is the list of eigenvalues of B and U is the list of eigenvalues of $B(4)$.

By Theorem 4.1, if $s \neq 0$, since there is at least one equality among Λ and U, it must be the case that either

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4 \qquad \text{or} \qquad \lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4.$$

In the former case letting $\gamma = \mu_2 = \lambda_3$, $\alpha_1 = \lambda_1$, $\alpha_2 = \lambda_2$, $\alpha_3 = \lambda_4$, $\beta_1 = \mu_1$, $\beta_2 = \mu_3$, we have the desired partition. The latter case is similar.

If $s = 0$, then $\Lambda = \alpha_1, \alpha_2, \alpha_3, r$ and $U = \beta_1, \beta_2, r$, where $\alpha_1, \alpha_2, \alpha_3$ are the eigenvalues of $B(1)$ and β_1, β_2 are the eigenvalues of $\begin{bmatrix} t & 0 \\ 0 & d_3 \end{bmatrix}$. Then by applying

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Theorem 2.7 to $B(1)$, we must have either $\alpha_1 > \beta_1 = \alpha_2 = \beta_2 > \alpha_3$ or $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \alpha_3$.

If $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \alpha_3$, by letting $\gamma = r$, we have the desired partition.

If $\alpha_1 > \beta_1 = \alpha_2 = \beta_2 > \alpha_3$, by the contrapositive of Observation 2.9, we have $r \neq \beta_1 = \alpha_2$. by letting $\gamma = r$, we have the desired partition. □

4.6. The case $(diamond, v)$**,** v **degree 2.**

LEMMA 4.8. *Let* $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$ *be real numbers with* $\alpha_1 > \alpha_2$ *and* $\beta_1 > \gamma > \beta_2$*, and let* v *be a degree 2 vertex of the diamond. Then there exists* $A \in \mathcal{S}(diamond)$ *such that* $\Lambda = \alpha_1, \alpha_2, \beta_1, \beta_2$ *are the eigenvalues of* A *and* $U = \{\alpha_1, \alpha_2, \gamma\}$ *are the eigenvalues of* $A(v)$.

Proof. Choose $\tau \in (\alpha_2, \alpha_1), \tau \neq \gamma$ and a, b, with $b \neq 0$, such that $\begin{bmatrix} a & b \\ b & \tau \end{bmatrix}$ has eigenvalues α_1, α_2. Also choose d, h, with $h \neq 0$, such that $\begin{bmatrix} \gamma & h \\ h & d \end{bmatrix}$ has eigenvalues β_1, β_2. Let

$$B = \begin{bmatrix} a & b & 0 & 0 \\ b & \tau & 0 & 0 \\ 0 & 0 & \gamma & h \\ 0 & 0 & h & d \end{bmatrix}.$$

Then B has eigenvalues $\alpha_1, \alpha_2, \beta_1, \beta_2$ and $B(4)$ has eigenvalues $\alpha_1, \alpha_2, \gamma$. Let $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ be an orthogonal matrix with $cs \neq 0$, and let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & s & 0 \\ 0 & -s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & cb & sb & 0 \\ cb & c^2\tau + s^2\gamma & cs(\tau - \gamma) & -sh \\ sb & cs(\tau - \gamma) & c^2\gamma + s^2\tau & ch \\ 0 & -sh & ch & d \end{bmatrix}.$$

Since $\tau \neq \gamma$ and $c, s, b, h \neq 0$, we have $A \in \mathcal{S}(diamond)$ with 4 a degree 2 vertex.

Then A has eigenvalues $\alpha_1, \alpha_2, \beta_1, \beta_2$ and

$$A(4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix} B(4) \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}$$

has eigenvalues $\alpha_1, \alpha_2, \gamma$. □

LEMMA 4.9. *Let* $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \gamma$ *be real numbers with* $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \alpha_3$*, and let* v *be a degree 2 vertex of the diamond. Then there exists* $A \in$

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A publication of the International Linear Algebra Society
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$\mathcal{S}$(diamond) such that $\Lambda = \alpha_1, \alpha_2, \alpha_3, \gamma$ are the eigenvalues of A and $U = \beta_1, \beta_2, \gamma$ are the eigenvalues of $A(v)$.

Proof. Case 1: $\gamma \notin (\beta_2, \beta_1)$.

By Theorem 2.2, choose a, b, d, r, t where $rt \neq 0$, such that $\begin{bmatrix} a & r & t \\ r & b & 0 \\ t & 0 & d \end{bmatrix}$ has eigenvalues $\alpha_1, \alpha_2, \alpha_3$ and $\begin{bmatrix} a & r \\ r & b \end{bmatrix}$ has eigenvalues β_1, β_2; necessarily $\beta_1 > a > \beta_2$. Let

$$B = \begin{bmatrix} \gamma & 0 & 0 & 0 \\ 0 & a & r & t \\ 0 & r & b & 0 \\ 0 & t & 0 & d \end{bmatrix}.$$

Then B has eigenvalues $\alpha_1, \alpha_2, \alpha_3, \gamma$ and $B(4)$ has eigenvalues β_1, β_2, γ. Since $a \in (\beta_2, \beta_1)$ and $\gamma \notin (\beta_2, \beta_1)$, $a \neq \gamma$. Let $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ be an orthogonal matrix with $cs \neq 0$, and let

$$A = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c^2\gamma + s^2 a & cs(\gamma - a) & -sr & -st \\ cs(\gamma - a) & s^2\gamma + c^2 a & cr & ct \\ -sr & cr & b & 0 \\ -st & ct & 0 & d \end{bmatrix}.$$

Since $\gamma \neq a$ and $c, s, r, t \neq 0$, we have $A \in \mathcal{S}(\mathit{diamond})$ with 4 a degree 2 vertex.

Then A has eigenvalues $\alpha_1, \alpha_2, \alpha_3, \gamma$ and

$$A(4) = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} B(4) \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has eigenvalues β_1, β_2, γ.

Case 2: $\gamma \in (\beta_2, \beta_1)$.

Then we have $\alpha_1 > \beta_1 > \alpha_2$, $\gamma > \beta_2 > \alpha_3$.

Subcase 1: $\gamma \neq \dfrac{\beta_1\beta_2(\alpha_1 + \alpha_2 + \alpha_3 - \beta_1 - \beta_2) - \alpha_1\alpha_2\alpha_3}{(\alpha_1 - \beta_1)(\beta_1 - \alpha_2) + (\alpha_1 - \beta_1)(\beta_2 - \alpha_3) + (\alpha_2 - \beta_2)(\beta_2 - \alpha_3)}$.

We apply Theorem 4.3 by identifying β_1, γ, β_2 with μ_1, μ_2, μ_3, respectively, α_1 with λ_1, α_3 with λ_4, and α_2 with either λ_2 or λ_3 depending on γ. It follows that

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A publication of the International Linear Algebra Society
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there exists

$$B = \begin{bmatrix} d_1 & a & b & 0 \\ a & d_2 & f & g \\ b & f & d_3 & 0 \\ 0 & g & 0 & d_4 \end{bmatrix} \in \mathcal{S}(paw)$$

such that the eigenvalues of B are $\Lambda = \alpha_1, \alpha_2, \alpha_3, \gamma$ and the eigenvalues of $B(4)$ are $U = \beta_1, \beta_2, \gamma$.

Then there exists an orthogonal matrix $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ such that

$$A = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{S}(diamond),$$

where the eigenvalues of A are Λ and the eigenvalues of $A(4)$ are U.

Subcase 2: $\gamma = \dfrac{\beta_1\beta_2(\alpha_1 + \alpha_2 + \alpha_3 - \beta_1 - \beta_2) - \alpha_1\alpha_2\alpha_3}{(\alpha_1 - \beta_1)(\beta_1 - \alpha_2) + (\alpha_1 - \beta_1)(\beta_2 - \alpha_3) + (\alpha_2 - \beta_2)(\beta_2 - \alpha_3)}$.

Making the same identification as in Subcase 1, and applying Theorem 4.2, there exists

$$B = \begin{bmatrix} d_1 & a & 0 & 0 \\ a & d_2 & b & f \\ 0 & b & d_3 & 0 \\ 0 & f & 0 & d_4 \end{bmatrix} \in \mathcal{S}(S_4)$$

such that the eigenvalues of B are $\Lambda = \alpha_1, \alpha_2, \alpha_3, \gamma$ and the eigenvalues of $B(4)$ are $U = \beta_1, \beta_2, \gamma$.

Then there exists orthogonal matrix $\begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ such that

$$A = \begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{S}(diamond),$$

where the eigenvalues of A are Λ and the eigenvalues of $A(4)$ are U. $\square$

THEOREM 4.10. $((diamond, v)$, v degree 2). *Given any $\Lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $U = \mu_1, \mu_2, \mu_3$ satisfying the interlacing inequalities, and v a degree 2 vertex of the diamond, then (Λ, U) is realizable for $(diamond, v)$ if and only if either*

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- $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$,
- *Λ can be partitioned into $\alpha_1, \alpha_2, \beta_1, \beta_2$ and U can be partitioned into $\alpha_1, \alpha_2, \gamma$ such that $\alpha_1 > \alpha_2$ and $\beta_1 > \gamma > \beta_2$,*
- *Λ can be partitioned into $\alpha_1, \alpha_2, \alpha_3, \gamma$ and U can be partitioned into β_1, β_2, γ such that $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \alpha_3$.*

Proof. The backward implication follows directly from the previous two Lemmas.

As for the forward implication, if (Λ, U) satisfies any of the EI strings 2, 3, 5, 6, 8–11, 13–16, 18, 19, they may be partitioned as in the second bulleted item, while if they satisfy any of 12, 17, 20–23, they may be partitioned as in the third bulleted item. Consequently we simply need to show the EI strings 1, 4, 7 cannot occur for $(diamond, v)$, v degree 2. By the argument before Theorem 4.2, EI string 1 cannot occur. So suppose string 4 or 7 occurs. Then there exists $A \in \mathcal{S}(diamond)$ such that $\operatorname{rank}(A - \lambda_3 I) = 2 = \operatorname{mr}(diamond)$ and $\operatorname{rank}(A(v) - \lambda_3 I) = 1$, where v is a degree 2 vertex. This contradicts Lemma 2.1. ◻

4.7. The case $(diamond, v)$, v dominating.

THEOREM 4.11. ($(diamond, v)$, v dominating). *Given any $\Lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $U = \mu_1, \mu_2, \mu_3$ satisfying the interlacing inequalities, and v a dominating vertex of the diamond, then (Λ, U) is realizable for $(diamond, v)$ if and only if $\mu_1 > \mu_2 > \mu_3$.*

Remark: On examination of the list of 23 EI strings near the beginning of Section 4, we see that $\mu_1 > \mu_2 > \mu_3$ if and only if (Λ, U) satisfies one of the strings 3, 5, 6, 8, 11–16, 18–23 or its reversal. It suffices to show (Λ, U) is realizable for exactly these strings.

Proof. ($\Rightarrow$) Suppose (Λ, U) is realizable for $(diamond, v)$ with v dominating. Then there exists $A \in \mathcal{S}(diamond)$ such that μ_1, μ_2, μ_3 are the eigenvalues of $A(v) \in \mathcal{S}(P_3)$. By Theorem 2.2, $\mu_1 > \mu_2 > \mu_3$.

($\Leftarrow$) Now suppose (Λ, U) is given satisfying the interlacing inequalities and $\mu_1 > \mu_2 > \mu_3$. We will construct an $A \in \mathcal{S}(diamond)$ such that (Λ, U) is realizable for $(diamond, v)$ with the dominating vertex v equal to 4.

Case 1. (Λ, U) satisfies one of the EI strings 3, 5, 6, 8, 11, 13–16, 18, or 19.

By Theorem 4.10, there exists

$$A = \begin{bmatrix} d_1 & f & a & 0 \\ f & d_2 & b & g \\ a & b & d_3 & h \\ 0 & g & h & d_4 \end{bmatrix} \in \mathcal{S}(diamond)$$

such that A has eigenvalues Λ and $A(4)$ has eigenvalues U.

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A publication of the International Linear Algebra Society
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Let Q_2 and Q be as in the case (P_4, v), v degree 2. Then

$$B = Q^T AQ = \begin{bmatrix} r & s & 0 & p \\ s & t & \sqrt{a^2+b^2} & q \\ 0 & \sqrt{a^2+b^2} & d_3 & h \\ p & q & h & d_4 \end{bmatrix}$$

where p, q are not 0. By Theorem 4.7 and our assumption on the EI string, (Λ, U) is not realizable for (paw, u) with u dominating. This requires $s \neq 0$, so that $B \in \mathcal{S}(diamond)$, with eigenvalues Λ and $B(4)$ has eigenvalues U. In other words (Λ, U) is realizable for $(diamond, v)$ with v dominating.

Case 2. (Λ, U) satisfies one of the EI strings 12, 20–23.

It follows from Theorems 4.6 and 4.2 that (Λ, U) is realizable for either (paw, v), v degree 2 or else (S_4, v), v pendant.

Subcase 1. (Λ, U) is realizable for (paw, v), v degree 2.

Then there exists

$$A = \begin{bmatrix} d_1 & a & 0 & 0 \\ a & d_2 & b & f \\ 0 & b & d_3 & g \\ 0 & f & g & d_4 \end{bmatrix} \in \mathcal{S}(paw)$$

such that A has eigenvalues Λ and $A(4)$ has eigenvalues U, where v is degree 2. Let Q_2 be an orthogonal 2 by 2 matrix diagonalizing $\begin{bmatrix} d_1 & a \\ a & d_2 \end{bmatrix}$ and let $Q = Q_2 \oplus I_2$. Then every entry of Q_2 is nonzero, so

$$B = Q^T AQ = \begin{bmatrix} r & 0 & w & x \\ 0 & t & y & z \\ w & y & d_3 & g \\ x & z & g & d_4 \end{bmatrix}$$

with $w, x, y, z \neq 0$. It follows that $B \in \mathcal{S}(diamond)$, B has eigenvalues Λ and $B(4)$ has eigenvalues U; i.e., (Λ, U) is realizable for (diamond, v), v dominating.

Subcase 2. (Λ, U) is realizable for (S_4, v), v pendant.

Then there exists

$$A = \begin{bmatrix} d_1 & a & 0 & 0 \\ a & d_2 & b & f \\ 0 & b & d_3 & 0 \\ 0 & f & 0 & d_4 \end{bmatrix} \in \mathcal{S}(S_4)$$

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A publication of the International Linear Algebra Society
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such that A has eigenvalues Λ and $A(v)$ has eigenvalues U, where v is pendant. Let Q_2 be an orthogonal 2 by 2 matrix diagonalizing $\begin{bmatrix} d_1 & a \\ a & d_2 \end{bmatrix}$ and let $Q = Q_2 \oplus I_2$. Then

$$B = Q^T A Q = \begin{bmatrix} r & 0 & w & x \\ 0 & t & y & z \\ w & y & d_3 & 0 \\ x & z & 0 & d_4 \end{bmatrix}$$

with $w, x, y, z \neq 0$ and B has eigenvalues Λ and $B(4)$ has eigenvalues U.

Let $P_2 = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$ be an orthogonal 2 by 2 matrix diagonalizing $\begin{bmatrix} t & y \\ y & d_3 \end{bmatrix}$. Then

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & s & 0 \\ 0 & -s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r & w' & w'' & x \\ w' & t' & 0 & z' \\ w'' & 0 & t'' & z'' \\ x & z' & z'' & d_4 \end{bmatrix}$$

with $w', w'', z', z'' \neq 0$ and C has eigenvalues Λ and $B(4)$ has eignvalues U.

It follows that $C \in \mathcal{S}(diamond)$ and (Λ, U) is realizable for (diamond, v), v dominating. $\square$

4.8. The case (C_4, v).

We come now to the final case of the connected 4-vertex graphs. Since C_4 is vertex transitive, there is only one case to consider.

Lemma 4.12. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$ be real numbers with $\alpha_1 > \gamma > \alpha_2, \beta_1 > \gamma > \beta_2$, and let v be a vertex of C_4. Then there exists $A \in \mathcal{S}(C_4)$ such that $\Lambda = \alpha_1, \alpha_2, \beta_1, \beta_2$ are the eigenvalues of A and $U = \beta_1, \beta_2, \gamma$ are the eigenvalues of $A(v)$.*

Proof. Choose a, b such that $\begin{bmatrix} a & b \\ b & \gamma \end{bmatrix}$ has eigenvalues β_1, β_2, and choose x, y such that $\begin{bmatrix} \gamma & x \\ x & y \end{bmatrix}$ has eigenvalues α_1, α_2.

Then

$$B = \begin{bmatrix} a & b & 0 & 0 \\ b & \gamma & 0 & 0 \\ 0 & 0 & \gamma & x \\ 0 & 0 & x & y \end{bmatrix}$$

has eigenvalues $\Lambda = \alpha_1, \alpha_2, \beta_1, \beta_2$ and $B(4)$ has eigenvalues $U = \beta_1, \beta_2, \gamma$.

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Let $R = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$ be any orthogonal 2 by 2 matrix with $cs \neq 0$. And let

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & s & 0 \\ 0 & -s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & cb & -sb & 0 \\ cb & \gamma & 0 & sx \\ -sb & 0 & \gamma & cx \\ 0 & sx & cx & y \end{bmatrix}$$

Then $A \in \mathcal{S}(C_4)$, A has eigenvalues $\Lambda = \alpha_1, \alpha_2, \beta_1, \beta_2$ and $A(4)$ has eigenvalues $U = \beta_1, \beta_2, \gamma$. ◻

THEOREM 4.13. (C_4, v). *Given any* $\Lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ *and* $U = \mu_1, \mu_2, \mu_3$ *satisfying the interlacing inequalities, then* (Λ, U) *is realizable for* (C_4, v) *if and only if either*

- $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, *or*
- (Λ, U) *satisfies any of the EI strings 3, 8, 11, 12, 15, 16, 19–23 or its reversal.*

Proof. The case $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$ follows from Theorem 1.1. So assume there is at least one equality among the Λ and U.

($\Rightarrow$) Suppose (Λ, U) is realizable for (C_4, v), and let $A \in \mathcal{S}(C_4)$ such that the eigenvalues of A are Λ and the eigenvalues of $A(v)$ are $U = \mu_1, \mu_2, \mu_3$. Since $C_4 - v = P_3$, $A(v) \in S(P_3)$ so that $\mu_1 > \mu_2 > \mu_3$ by Theorem 2.2. Therefore EI strings 1, 2, 4, 7, 9, 10, 17 are not realizable. It remains to show that strings 5, 6, 13, 14, 18 are not realizable. We write

$$A = \begin{bmatrix} a & 0 & w & x \\ 0 & b & y & z \\ w & y & c & 0 \\ x & z & 0 & d \end{bmatrix}$$

with $w, x, y, z \neq 0$ and without loss of generality, take v to be vertex 4.

Case 1. $b = a$

Then

$$A = \begin{bmatrix} a & 0 & w & x \\ 0 & a & y & z \\ w & y & c & 0 \\ x & z & 0 & d \end{bmatrix}.$$

Subcase 1. a is an eigenvalue of A.

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Then $A - aI$ is singular, so $\det(A - aI) = \begin{vmatrix} w & x \\ y & z \end{vmatrix}^2 = 0$ and $\begin{bmatrix} x \\ z \end{bmatrix}$ is a multiple of $\begin{bmatrix} w \\ y \end{bmatrix}$.

Let $Q_2 = \dfrac{1}{\sqrt{w^2+y^2}} \begin{bmatrix} y & w \\ -w & y \end{bmatrix}$ and let $Q = Q_2 \oplus I_2$, then

$$B = Q^T AQ = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & \sqrt{w^2+y^2} & \dfrac{wx+yz}{\sqrt{w^2+y^2}} \\ 0 & \sqrt{w^2+y^2} & c & 0 \\ 0 & \dfrac{wx+yz}{\sqrt{w^2+y^2}} & 0 & d \end{bmatrix}$$

has eigenvalues Λ and $B(4)$ has eigenvalues U.

Since $B(1) \in \mathcal{S}(P_3)$, it follows that Λ and U can be partitioned as in Theorem 4.7 with the (Λ, U) satisfying the inequalities in the first bullet. By the remark following this theorem, EI strings 5, 6, 13, 14, 18 are not realizable in this case.

Subcase 2. a is not an eigenvalue of A.

Then $wz - xy \neq 0$. Letting Q and B be as in Subcase 1, we have

$$B = \begin{bmatrix} a & 0 & 0 & p \\ 0 & a & \sqrt{w^2+y^2} & q \\ 0 & \sqrt{w^2+y^2} & c & 0 \\ p & q & 0 & d \end{bmatrix},$$

where $p = \dfrac{xy - wz}{\sqrt{w^2+y^2}} \neq 0$ and $q = \dfrac{wx+yz}{\sqrt{w^2+y^2}}$.

(i) $q = 0$

Then

$$B = \begin{bmatrix} a & 0 & 0 & p \\ 0 & a & \sqrt{w^2+y^2} & 0 \\ 0 & \sqrt{w^2+y^2} & c & 0 \\ p & 0 & 0 & d \end{bmatrix}$$

is a direct sum and (Λ, U) can be partitioned $\Lambda = \alpha_1, \alpha_2, \beta_1, \beta_2$ and $U = a, \alpha_1, \alpha_2$ with $\alpha_1 > a > \alpha_2$ and $\beta_1 > a > \beta_2$. Then (Λ, U) must satisfy an EI string of the form $f_1 f_2 >> f_5 f_6$ which excludes the strings 5, 6, 13, 14, and 18.

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(ii) $q \neq 0$

Then

$$B = \begin{bmatrix} a & 0 & 0 & p \\ 0 & a & \sqrt{w^2+y^2} & q \\ 0 & \sqrt{w^2+y^2} & c & 0 \\ p & q & 0 & d \end{bmatrix} \in \mathcal{S}(P_4)$$

with 4 a degree 2 vertex. By the remark following Theorem 4.1, (Λ, U) must satisfy EI string 17. So 5, 6, 13, 14, and 18 are again not possible.

Case 2. $b \neq a$

Again, letting Q_2, Q and B be as in Subcase 1, we have

$$B = Q^T A Q = \begin{bmatrix} r & s & 0 & p \\ s & t & \sqrt{w^2+y^2} & q \\ 0 & \sqrt{w^2+y^2} & c & 0 \\ p & q & 0 & d \end{bmatrix}.$$

Since $a \neq b$ and no entry of Q_2 is 0, $s \neq 0$. Moreover, since Q_2 is orthogonal, $\begin{bmatrix} p \\ q \end{bmatrix} = Q_2^T \begin{bmatrix} x \\ z \end{bmatrix} \neq 0$, so p, q are not both 0. Also, the Λ are the eigenvalues of B and the U are the eigenvalues of $B(4)$.

Subcase 1. $p = 0, q \neq 0$.

Then $B \in \mathcal{S}(S_4)$ with 4 a pendant vertex. By the remark following Theorem 4.2, this can only occur if (Λ, U) satisfies either the string 20, 23 or its reversal along with additional restrictions. In particular, 5, 6, 13, 14, and 18 are excluded.

Subcase 2. $p \neq 0, q = 0$

Then $B \in \mathcal{S}(P_4)$ with 4 a pendant vertex. By Theorem 2.2, all (Λ, U) are distinct, contradicting our assumption at the outset of the proof.

Subcase 3. $p \neq 0, q \neq 0$.

Then $B \in \mathcal{S}(paw)$ with 4 a degree 2 vertex. By the remark following Theorem 4.6, (Λ, U) satisfies 12, 20–23 or its reversal. So the strings 5, 6, 13, 14, and 18 cannot occur for C_4. This concludes the proof of the forward implication.

$(\Leftarrow)$

Case 1: (Λ, U) satisfies one of EI strings 3, 8, 11, 15, 16, 19

Immediately from Lemma 4.12, (Λ, U) is realizable for (C_4, v).

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A publication of the International Linear Algebra Society
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Case 2. (Λ, U) satisfies one of the EI strings 12, 20–23.

It follows from Theorems 4.6 and 4.2 that (Λ, U) is realizable for either (paw, v), v degree 2 or else (S_4, v), v pendant.

Subcase 1: (Λ, U) is realizable for $(S_4, v), v$ pendant.

Then there exists

$$A = \begin{bmatrix} d_1 & a & 0 & 0 \\ a & d_2 & b & f \\ 0 & b & d_3 & 0 \\ 0 & f & 0 & d_4 \end{bmatrix} \in \mathcal{S}(S_4)$$

such that A has eigenvalues Λ and $A(4)$ has eigenvalues U. Let Q_2 be an orthogonal 2 by 2 matrix diagonalizing $\begin{bmatrix} d_1 & a \\ a & d_2 \end{bmatrix}$ and let $Q = Q_2 \oplus I_2$. Then

$$B = Q^T AQ = \begin{bmatrix} r & 0 & w & x \\ 0 & t & y & z \\ w & y & d_3 & 0 \\ x & z & 0 & d_4 \end{bmatrix}$$

with $w, x, y, z \neq 0$, $B \in \mathcal{S}(C_4)$ and B has eigenvalues Λ and $B(4)$ has eigenvalues U. Thus, (Λ, U) is realizable for (C_4, v).

Subcase 2: (Λ, U) is realizable for $(paw, v), v$ degree 2.

Then there exists

$$A = \begin{bmatrix} d_1 & a & 0 & g \\ a & d_2 & b & f \\ 0 & b & d_3 & 0 \\ g & f & 0 & d_4 \end{bmatrix} \in \mathcal{S}(paw)$$

such that A has eigenvalues Λ and $A(4)$ has eigenvalues U. Let Q_2 be an orthogonal 2 by 2 matrix diagonalizing $\begin{bmatrix} d_1 & a \\ a & d_2 \end{bmatrix}$ and let $Q = Q_2 \oplus I_2$. Then

$$B = Q^T AQ = \begin{bmatrix} r & 0 & w & x \\ 0 & t & y & z \\ w & y & d_3 & 0 \\ x & z & 0 & d_4 \end{bmatrix}$$

with $w, y \neq 0$, at least one of x, z is nonzero and B has eigenvalues Λ and $B(4)$ has eigenvalues U.

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(i). $xz \neq 0$.

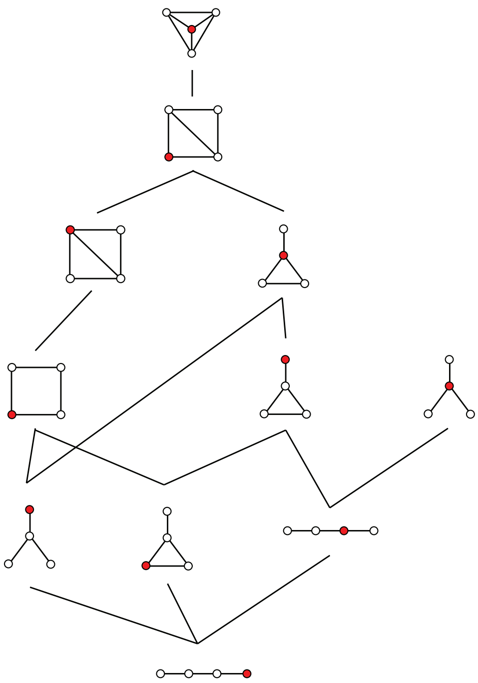
Then $B \in \mathcal{S}(C_4)$ and (Λ, U) is realizable for (C_4, v).

(ii). $x = 0, z \neq 0$ or $x \neq 0, z = 0$.

Then $B \in \mathcal{S}(P_4)$ and (Λ, U) is realizable for (P_4, v). By Theorem 2.2, (Λ, U) are all distinct. And by Theorem 1.1, (Λ, U) is realizable for (C_4, v). ▯

4.9. Summary.

The following chart and tables summarize Section 4. The red vertex represents the vertex which is deleted. A line connecting two graphs indicates that any (Λ, U) which is realizable for the lower graph is realizable for the higher graph. So the graphs with fewer restrictions on (Λ, U) appear higher in the table. As might be expected, when an edge is deleted from a graph, typically the set of realizable (Λ, U)-strings is a subset of the previously realizable strings, but there are notable exceptions. For example string 17 is not realizable for $(diamond, v)$, v dominating, but is realizable for (paw, v), v dominating. And string 25 is only realizable for (S_4, v), v dominating, a special case of Observation 2.9.



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A publication of the International Linear Algebra Society
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1.	$>$	$>$	$=$	$=$	$=$	$=$	13.	$>$	$>$	$>$	$=$	$>$	$=$
2.	$>$	$=$	$=$	$>$	$=$	$=$	14.	$>$	$>$	$=$	$>$	$>$	$=$
3.	$=$	$=$	$>$	$>$	$=$	$=$	15.	$>$	$=$	$>$	$>$	$>$	$=$
4.	$>$	$>$	$>$	$=$	$=$	$=$	16.	$=$	$>$	$>$	$>$	$>$	$=$
5.	$>$	$>$	$=$	$>$	$=$	$=$	17.	$>$	$>$	$>$	$=$	$=$	$>$
6.	$>$	$>$	$=$	$=$	$>$	$=$	18.	$>$	$>$	$=$	$>$	$=$	$>$
7.	$>$	$>$	$=$	$=$	$=$	$>$	19.	$>$	$=$	$>$	$>$	$=$	$>$
8.	$>$	$=$	$>$	$>$	$=$	$=$	20.	$>$	$>$	$=$	$=$	$>$	$>$
9.	$>$	$=$	$>$	$=$	$=$	$>$	21.	$>$	$>$	$>$	$>$	$>$	$=$
10.	$>$	$=$	$=$	$>$	$>$	$=$	22.	$>$	$>$	$>$	$>$	$=$	$>$
11.	$=$	$>$	$>$	$>$	$=$	$=$	23.	$>$	$>$	$>$	$=$	$>$	$>$
12.	$>$	$>$	$>$	$>$	$=$	$=$	24.	$>$	$>$	$>$	$>$	$>$	$>$
							25.	$>$	$=$	$=$	$=$	$=$	$>$

To the former list we have added the EI string with all strict inequalities and the palindromic EI string which is only realizable for (S_4, v), v a dominating vertex in order to list all the EI strings that are realized by some connected graph on 4 vertices.

Table 4.1
Summary of Results

Graph and specified vertex	Realizable EI strings
(K_4, v)	1–24
$(diamond, v)$, v degree 2	2, 3, 5, 6, 8-24
$(diamond, v)$, v dominating	3, 5, 6, 8, 11–16, 18–24
(C_4, v)	3, 8, 11, 12, 15, 16, 19–24
(paw, v), v dominating	2, 9, 10, 12, 17, 20–24
(paw, v), v pendant	12, 17, 20**, 21, 22, 23**, 24
(paw, v), v degree 2	12, 20**, 21, 22, 23**, 24
(S_4, v), v pendant	20*, 23*, 24
(S_4, v), v dominating	17, 24, 25
(P_4, v), v degree 2	17, 24
(P_4, v), v pendant	24

The (*),(**) indicates that there are additional conditions on (Λ, U). We note that the conditions in (*) and (**) are complementary restrictions.

5. The Λ problem for the connected 4-vertex graphs. We now turn to the Λ problem, which was mentioned at the beginning of the introduction, for connected 4-vertex graphs. Now that we have solved the (Λ, U)-problem for all such graphs, it is easy to adapt the solution to the Λ problem. There are now just 8 strings to consider.

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1. $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$
2. $\lambda_1 > \lambda_2 = \lambda_3 > \lambda_4$
3. $\lambda_1 > \lambda_2 > \lambda_3 = \lambda_4$
4. $\lambda_1 = \lambda_2 > \lambda_3 > \lambda_4$
5. $\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4$
6. $\lambda_1 > \lambda_2 = \lambda_3 = \lambda_4$
7. $\lambda_1 = \lambda_2 = \lambda_3 > \lambda_4$
8. $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$

The string 8 is not realizable for any connected graph, while, according to Theorem 3.1 [3], any of the strings 1 to 7 is realizable for K_4. The remaining connected graphs on 4 vertices all have minimum rank 2. But for any matrix with eigenvalues given by 6, 7, or 8, $\operatorname{rank}(A - \lambda_2 I) = 1$, so no string Λ satisfying 6, 7, or 8 is realizable for any of the remaining graphs. Moreover, Theorem 1.1 implies that any Λ with string 1 is realizable for any connected 4-vertex graph. So we need consider only strings 2, 3, 4, 5, and since string 4 is a reversal of 3, there are only 3 cases. We first state the result and then explain the conclusions.

THEOREM 5.1. *Let $\Lambda = \lambda_1, \lambda_2, \lambda_3, \lambda_4$ be a list of real numbers with*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4. \tag{5.1}$$

Then

- Λ *is realizable for P_4 if and only if $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$;*
- Λ *is realizable for S_4 if and only if $\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4$;*
- Λ *is realizable for the paw if and only if there is at most one equality in* (5.1);
- Λ *is realizable for C_4 or the diamond if and only if there are not two consecutive equalities in* (5.1);
- Λ *is realizable for K_4 if and only if $\lambda_1 > \lambda_4$.*

Proof. P_4: Theorem 2.2 requires the λ_i to be distinct.

S_4: Lemma 2.5 requires, $\lambda_1 > \lambda_2$ and $\lambda_3 > \lambda_4$. It follows from Theorem 2.7 that any Λ with $\lambda_1 > \lambda_2 = \lambda_3 > \lambda_4$ is realizable.

paw: That any string Λ with $\lambda_1 > \lambda_2 = \lambda_3 > \lambda_4$ is realizable follows from the last bulleted item in Theorem 4.3. And any Λ with $\lambda_1 > \lambda_2 > \lambda_3 = \lambda_4$ or its reversal is realizable because string 2 on page 31 is realizable for (paw, v) v dominating. Finally, the only string in the (Λ, U)-problem corresponding to the string Λ with $\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4$ is string 3 which is not realizable for the paw.

C_4, diamond: The strings Λ with $\lambda_1 > \lambda_2 = \lambda_3 > \lambda_4$, $\lambda_1 > \lambda_2 > \lambda_3 = \lambda_4$, and $\lambda_1 = \lambda_2 > \lambda_3 = \lambda_4$ are realizable for either C_4 or diamond because the strings 20, 12, and 3, respectively, on page 31 are realizable for C_4 and the diamond.

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A publication of the International Linear Algebra Society
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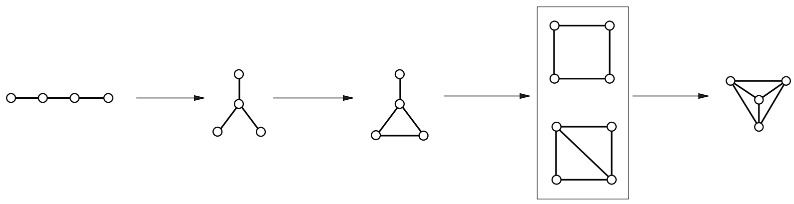
K_4: As already mentioned, any Λ with $\lambda_1 > \lambda_4$ is realizable by Theorem 3.1 in [3]. □

We summarize Theorem 5.1 with the diagram



Any Λ which is realizable for one of these graphs is realizable for any graph to the right. We observe that any Λ realizable for a connected graph G on 4 vertices is realizable for a connected graph obtained from G by deleting an edge. We do not know if this property holds for connected graphs on $n > 4$ vertices.

6. Conclusion.

The inverse eigenvalue problem for graphs has previously been solved only for special classes of graphs, for example trees, and even for these a complete solution is not yet available. For general graphs it has appeared intractible.

We have demonstrated that two of the well-known versions of the problem, which we have designated as the Λ problem and the (Λ, U)-problem are completely solvable for graphs on 4 or fewer vertices. Our solution clarifies for what graphs a solution might be obtainable in general. For example, Theorem 4.2 is sufficiently complex that we see little hope of solving the (Λ, U)-problem for (S_n, v), v pendant, even though a complete solution for (S_n, v), v dominating, is given in Theorem 2.7. Generalizations of Theorems 4.3 and 4.6 seem likewise remote. However, generalizations of Theorems 4.7 and 4.11 do seem worth pursuing.

One of the major obstacles in obtaining generalizations to graphs on n vertices by the methods we have employed is the interdependence of many of our arguments. For example our proof of Theorem 4.13 dealing with (C_4, v) uses previous results for (P_4, v), v degree 2, (paw, v), v dominating, (S_4, v), v pendant, (P_n, v), v pendant, and (paw, v), v degree 2. It appears that new methods must be found for specific graphs or graph classes on n vertices.

An avenue that should be pursued is establishing results of the form: If (Λ, U) is realizable for (G, v), then (Λ, U) is realizable for (H, w), where G and H are graphs on n vertices bearing a simple relationship. Likewise, the corresponding Λ version of this question ought to be investigated.

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Two specific open questions of this type are:

1. Let G be a graph on n vertices, let v be a vertex of G, let x, y be nonadjacent vertices of G, and let $e = xy$. Under what circumstances does it follow that if (Λ, U) is realizable for (G, v), then (Λ, U) is also realizable for $(G + e, v)$?
2. Let G be a graph on n vertices, Λ any set of n real numbers, and x, y be nonadjacent vertices of G. If Λ is realizable for G, is Λ realizable for $G + e$?

Finally, we note that Nylen's lemma (Lemma 2.1 above) was at times quite useful in demonstrating that a certain (Λ, U) is not realizable for a given graph. Are there additional results from the literature on minimum rank or the inverse inertia problem that can give additional restrictions to establish that a certain (Λ, U) or certain Λ is not realizable?

Appendix A. The graph P_3. Given $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \alpha_3$, we determine a, b, c, x, y, so that $\alpha_1, \alpha_2, \alpha_3$ are the eigenvalues of $A = \begin{bmatrix} a & x & 0 \\ x & b & y \\ 0 & y & c \end{bmatrix}$ and β_1, β_2 are the eigenvalues of $A(3) = \begin{bmatrix} a & x \\ x & b \end{bmatrix}$.

Necessarily,

$$a + b = \operatorname{tr} A(3) = \beta_1 + \beta_2, \tag{A.1}$$

$$a + b + c = \operatorname{tr} A = \alpha_1 + \alpha_2 + \alpha_3, \tag{A.2}$$

$$ab - x^2 = \det A(3) = \beta_1 \beta_2, \tag{A.3}$$

$$ab - x^2 + ac + bc - y^2 = E_2(A) = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3, \tag{A.4}$$

$$abc - ay^2 - cx^2 = \det A = \alpha_1\alpha_2\alpha_3. \tag{A.5}$$

From (A.1) and (A.2),

$$c = \alpha_1 + \alpha_2 + \alpha_3 - \beta_1 - \beta_2. \tag{A.6}$$

Substituting (A.1), (A.3), and (A.6) into (A.4) gives

$$\beta_1\beta_2 + (\beta_1 + \beta_2)(\alpha_1 + \alpha_2 + \alpha_3 - \beta_1 - \beta_2) - y^2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3.$$

So

$$y^2 = \beta_1\beta_2 + (\beta_1 + \beta_2)(\alpha_1 + \alpha_2 + \alpha_3) - (\beta_1 + \beta_2)^2 - \alpha_1\alpha_2 - \alpha_1\alpha_3 - \alpha_2\alpha_3$$

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which can be written as

$$y^2 = (\alpha_1-\beta_1)(\beta_1-\alpha_2)+(\alpha_1-\beta_1)(\beta_2-\alpha_3)+(\alpha_2-\beta_2)(\beta_2-\alpha_3) > 0. \tag{A.7}$$

Thus, there is $y \neq 0$ satisfying (A.7).

From (A.5), $(ab-x^2)c - ay^2 = \alpha_1\alpha_2\alpha_3$. Substituting from (A.3), (A.6), and (A.7),

$$a = \frac{\beta_1\beta_2(\alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2)-\alpha_1\alpha_2\alpha_3}{(\alpha_1-\beta_1)(\beta_1-\alpha_2)+(\alpha_1-\beta_1)(\beta_2-\alpha_3)+(\alpha_2-\beta_2)(\beta_2-\alpha_3)}. \tag{A.8}$$

Substituting (A.8) in $b = \beta_1+\beta_2-a$, expanding and algebraic manipulation yields

$$b = \frac{d+\beta_1\beta_2(\alpha_1+\alpha_2+\alpha_3-\beta_1-\beta_2)-\alpha_1\alpha_2\alpha_3}{(\alpha_1-\beta_1)(\beta_1-\alpha_2)+(\alpha_1-\beta_1)(\beta_2-\alpha_3)+(\alpha_2-\beta_2)(\beta_2-\alpha_3)},$$

where $d = (\alpha_1-\beta_1)(\beta_1-\alpha_2)(\beta_1-\alpha_3)+(\alpha_1-\beta_2)(\beta_2-\alpha_2)(\beta_2-\alpha_3)$.

From (A.3), $x^2 = ab-\beta_1\beta_2$.

Substituting from the last two equations and more algebraic manipulation gives

$$x^2 = \frac{(\alpha_1-\beta_1)(\beta_1-\alpha_2)(\beta_1-\alpha_3)(\alpha_1-\beta_2)(\alpha_2-\beta_2)(\beta_2-\alpha_3)}{[(\alpha_1-\beta_1)(\beta_1-\alpha_2)+(\alpha_1-\beta_1)(\beta_2-\alpha_3)+(\alpha_2-\beta_2)(\beta_2-\alpha_3)]^2} \tag{A.9}$$

Since the numerator and denominator are positive, there is a nonzero x satisfying (A.9).

We summarize with the following result.

THEOREM A.1. *Let $\alpha_1 > \beta_1 > \alpha_2 > \beta_2 > \alpha_3$. Set*

- $s = (\alpha_1-\beta_1)(\beta_1-\alpha_2)+(\alpha_1-\beta_1)(\beta_2-\alpha_3)+(\alpha_2-\beta_2)(\beta_2-\alpha_3)$,
- $d = (\alpha_1-\beta_1)(\beta_1-\alpha_2)(\beta_1-\alpha_3)+(\alpha_1-\beta_2)(\beta_2-\alpha_2)(\beta_2-\alpha_3)$,
- $p = (\alpha_1-\beta_1)(\beta_1-\alpha_2)(\beta_1-\alpha_3)(\alpha_1-\beta_2)(\alpha_2-\beta_2)(\beta_2-\alpha_3)$.

Then $A = \begin{bmatrix} a & x & 0 \\ x & b & y \\ 0 & y & c \end{bmatrix}$ has eigenvalues $\alpha_1,\alpha_2,\alpha_3$ and $\begin{bmatrix} a & x \\ x & b \end{bmatrix}$ has eigenvalues β_1,β_2 if and only if the entries of A satisfy the following equations:

- $a = \dfrac{\beta_1\beta_2 c-\alpha_1\alpha_2\alpha_3}{s}$,
- $b = \dfrac{d+\beta_1\beta_2 c-\alpha_1\alpha_2\alpha_3}{s}$,

Electronic Journal of Linear Algebra ISSN 1081-3810
A publication of the International Linear Algebra Society
Volume 27, pp. 742-778, October 2014

- $c = \alpha_1 + \alpha_2 + \alpha_3 - \beta_1 - \beta_2$,
- $x^2 = \dfrac{p}{s^2}$,
- $y^2 = s$.

The argument preceding the statement of the theorem verifies the necessity. Sufficiency can be checked using a computer algebraic system.

Acknowledgement. We thank the referee for several helpful comments that assisted us in making a number of improvements to the paper.

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