

COMMUTING MAPS ON RANK- k MATRICES*

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Abstract. In this short note, a new proof and a slight improvement of the Franca Theorem are given. More precisely, it is proved: Let $n \geq 3$ be a natural number, and let $M_n(\mathbb{K})$ be the ring of all $n \times n$ matrices over an arbitrary field $\mathbb{K}$ with center Z . Fix a natural number $2 \leq s \leq n$. If $G : M_n(\mathbb{K}) \rightarrow M_n(\mathbb{K})$ is an additive map such that $G(x)x = xG(x)$ for every rank- s matrix $x \in M_n(\mathbb{K})$, then there exist an element $\lambda \in Z$ and an additive map $\mu : M_n(\mathbb{K}) \rightarrow Z$ such that $G(x) = \lambda x + \mu(x)$ for each $x \in M_n(\mathbb{K})$.

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1. Introduction. Let R be an associative ring with a nonempty subset S . A map $G : R \rightarrow R$ is a commuting map on S if $G(x)x = xG(x)$ for all $x \in S$. The well known Brešar Theorem implies that a commuting additive map f on a prime ring R must be of the form $f(x) = \lambda x + \mu(x)$ for some $\lambda \in C$ and some additive map $\mu : R \rightarrow C$, where C is the extended centroid of R (see [1, Theorem A]). Recently, Franca considered the commuting additive map on some subsets (which are not necessarily closed under addition) of the ring $M_n(\mathbb{K})$ of all $n \times n$ matrices over a field $\mathbb{K}$, for example the subset of all singular matrices (see [5, Theorem 1]), the subset of all invertible matrices (see [5, Theorem 3]), and the subset of all rank- k matrices (see [6, Theorem 3 and 4]) in $M_n(\mathbb{K})$. Franca's results imply that the theory of functional identities [3] can be explored on some subsets (which are not necessarily closed under addition), on which many important results had been proved in the active area of (linear) preserver problems (see the survey paper [4] for details). We state the Franca Theorem as follows:

THE FRANCA THEOREM ([6, Theorem 3]) *Let $n \geq 3$ be a natural number, $\mathbb{K}$ be a field with $\text{char} \mathbb{K} \neq 2, 3$. Fix $s \in \{2, \dots, n-1\}$. For an additive map G from $M_n(\mathbb{K})$ into $M_n(\mathbb{K})$ if $G(x)x = xG(x)$ holds for every rank- s matrix $x \in M_n(\mathbb{K})$, then there exist an element $\lambda \in Z$ and an additive map $\mu : M_n(\mathbb{K}) \rightarrow Z$ such that $G(x) = \lambda x + \mu(x)$ for each $x \in M_n(\mathbb{K})$.*

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In particular, for $s = n \geq 2$, Franca had obtained the same conclusion under the milder assumption of $|\mathbb{K}| \neq 2$ (see [5, Theorem 3]). In this short note, we will propose another proof of the Franca Theorem stated above also getting rid of the assumption of $\text{char}\mathbb{K} \neq 2, 3$.

In the following, we will always write $\mathbb{K}$ for a field, e_{ij} for the $n \times n$ matrix of the appropriate size with 1 in the position (i, j) and 0 in every other position, and Z the center of $M_n(\mathbb{K})$. For convenience, we take the sum over an empty set of matrices to be the zero matrix of the appropriate size, for example the symbol $\sum_{i=4}^s e_{ii}$ will be used to denote the zero matrix when $s = 3$.

2. Main result. The proof of Theorem 2.6 is based on the following several lemmas. For $s = 2$ we have Lemma 2.1

LEMMA 2.1. *For each $n \geq 3$, each rank-1 matrix in $M_n(\mathbb{K})$ can be expressed as a sum of three rank-2 matrices among which the sum of any two is rank-2.*

Proof. It is enough to prove that there exists a rank-1 matrix which is the sum of three rank-2 matrices among which the sum of any two is rank-2, since two matrices are equivalent if and only if they have the same rank.

The following is a rank-1 with the desired properties

$$e_{11} + e_{12} + e_{13} = [e_{11} + e_{22}] + [e_{12} + e_{23}] + [e_{13} - e_{22} - e_{23}]. \quad \square$$

For $s \geq 3$ and $|\mathbb{K}| > 2$ we have Lemma 2.2.

LEMMA 2.2. *For each s and n with $n \geq s \geq 3$, each rank- i ($i = 1, 2$) matrix in $M_n(\mathbb{K})$ with $|\mathbb{K}| > 2$ can be expressed as a sum of three rank- s matrices among which the sum of any two is rank- s .*

Proof. Since $|\mathbb{K}| > 2$ we can choose a nonzero element $\lambda \neq 1$ in $\mathbb{K}$.

The following is a rank-1 matrix which is the sum of three rank- s matrices among which the sum of any two is rank- s

$$\begin{aligned} e_{11} = & \left[e_{12} + e_{21} - \lambda e_{22} + \sum_{i=3}^s e_{ii} \right] + \left[(1 - \lambda)e_{11} - e_{12} - e_{21} - \lambda \sum_{i=3}^s e_{ii} \right] \\ & + \left[\lambda \sum_{i=1}^2 e_{ii} + (\lambda - 1) \sum_{i=3}^s e_{ii} \right]. \end{aligned}$$

The following is a rank-2 matrix which is the sum of three rank- s matrices among

which the sum of any two is rank- s

$$\begin{aligned} e_{11} + e_{22} = & \left[e_{11} + \lambda e_{12} + e_{23} + e_{31} + e_{32} + \sum_{i=4}^s e_{ii} \right] \\ & + \left[e_{13} + e_{21} + e_{22} + (\lambda - 1)e_{31} - e_{32} + e_{33} - \lambda \sum_{i=4}^s e_{ii} \right] \\ & + \left[-\lambda e_{12} - e_{13} - e_{21} - e_{23} - \lambda e_{31} - e_{33} + (\lambda - 1) \sum_{i=4}^s e_{ii} \right]. \quad \square \end{aligned}$$

For $s \geq 3$ and $|\mathbb{K}| = 2$ ($\mathbb{K} = \mathbb{Z}_2$), we have Lemmas 2.3 and 2.4.

LEMMA 2.3. For integers m and n with $\frac{n}{3} \geq m \geq 1$, each rank-2 matrix in $M_n(\mathbb{Z}_2)$ can be expressed as a sum of three rank- $3m$ matrices among which the sum of any two is rank- $3m$.

Proof. Set $A = 0$ for $m = 1$ and

$$A = \sum_{i=1}^{m-1} (e_{3i+1,3i+2} + e_{3i+1,3i+3} + e_{3i+2,3i+1} + e_{3i+3,3i+2})$$

for $m > 1$. Then the following is a rank-2 matrix as a sum of three rank- $3m$ matrices among which the sum of any two is rank- $3m$

$$\left[\sum_{i=1}^{3m} e_{ii} \right] + [e_{12} + e_{13} + e_{21} + e_{32} + A] + \left[e_{11} + e_{13} + e_{23} + e_{31} + e_{32} + A + \sum_{i=4}^{3m} e_{ii} \right]. \quad \square$$

LEMMA 2.4. For $n \geq s \geq 3$ with $3 \nmid s$, each rank- i ($i = 1, 2$) matrix in $M_n(\mathbb{Z}_2)$ can be expressed as a sum of three rank- s matrices among which the sum of any two is rank- s .

Proof. Case 1: Assume that $s = 3m + 1$ with $1 \leq m \leq \frac{n-1}{3}$. Set $A = 0$ for $m = 1$ and

$$A = \sum_{i=1}^{m-1} (e_{3i+2,3i+3} + e_{3i+2,3i+4} + e_{3i+3,3i+2} + e_{3i+4,3i+3})$$

for $m > 1$. Then the following is a rank-1 matrix as a sum of three rank- s matrices among which the sum of any two is rank- s

$$\begin{aligned} e_{13} = & \left[\sum_{i=1}^s e_{ii} \right] + [e_{12} + e_{21} + e_{22} + e_{34} + e_{43} + e_{44} + A] \\ & + \left[e_{11} + e_{12} + e_{13} + e_{21} + e_{33} + e_{34} + e_{43} + A + \sum_{i=5}^s e_{ii} \right]. \end{aligned}$$

The following is a rank-2 matrix as a sum of three rank- s matrices among which the sum of any two is rank- s

$$\begin{aligned} & \left[\sum_{i=1}^s e_{ii} \right] + [e_{12} + e_{21} + e_{22} + e_{34} + e_{43} + e_{44} + A] \\ & + \left[e_{11} + e_{12} + e_{13} + e_{21} + e_{24} + e_{33} + e_{34} + e_{43} + A + \sum_{i=5}^s e_{ii} \right]. \end{aligned}$$

Case 2: Assume that $s = 3m + 2$ with $1 \leq m \leq \frac{n-2}{3}$. Set $A = 0$ for $m = 1$ and

$$A = \sum_{i=1}^{m-1} (e_{3i+3,3i+4} + e_{3i+3,3i+5} + e_{3i+4,3i+3} + e_{3i+5,3i+4})$$

for $m > 1$. For the case $m = 1$ ($s = 5$), the symbol $\sum_{i=6}^s e_{ii}$ will be used to denote the $n \times n$ zero matrix. Then the following is a rank-1 matrix as a sum of three rank- s matrices among which the sum of any two is rank- s

$$\begin{aligned} e_{13} &= \left[\sum_{i=1}^s e_{ii} \right] + [e_{11} + e_{12} + e_{21} + e_{34} + e_{35} + e_{43} + e_{54} + A] \\ &+ \left[e_{12} + e_{13} + e_{21} + e_{22} + e_{33} + e_{34} + e_{35} + e_{43} + e_{44} + e_{54} + e_{55} + A + \sum_{i=6}^s e_{ii} \right]. \end{aligned}$$

The following is a rank-2 matrix as a sum of three rank- s matrices among which the sum of any two is rank- s

$$\begin{aligned} & \left[\sum_{i=1}^s e_{ii} \right] + [e_{11} + e_{12} + e_{21} + e_{34} + e_{35} + e_{43} + e_{54} + A] \\ & + \left[e_{12} + e_{13} + e_{21} + e_{22} + e_{24} + e_{33} + e_{34} + e_{35} + e_{43} + e_{44} + e_{54} + e_{55} + A + \sum_{i=6}^s e_{ii} \right]. \quad \square \end{aligned}$$

The following lemma is a key step of the proof for Theorem 2.6.

LEMMA 2.5. Let n, s be fixed integers with $n \geq s \geq 2$. Suppose that $G : M_n(\mathbb{K}) \rightarrow M_n(\mathbb{K})$ is an additive map such that $G(x)x = xG(x)$ for each rank- s matrix x . Then for a matrix D in $M_n(\mathbb{K})$ if D is a sum of three rank- s matrices among which the sum of any two is rank- s , then $G(D)D = DG(D)$.

Proof. For two rank- s matrices A and B such that $A + B$ is also rank- s , we have

$$[G(A), B] + [G(B), A] = 0$$

since

$$[G(A), A] = [G(B), B] = [G(A + B), A + B] = 0.$$

There exist rank- s matrices A_1, A_2, A_3 such that any one of $A_1 + A_2, A_1 + A_3, A_2 + A_3$ is rank- s and $D = \sum_{i=1}^3 A_i$. Then,

$$[G(D), D] = \sum_{i=1}^3 [G(A_i), A_i] + \sum_{1 \leq i < j \leq 3} ([G(A_i), A_j] + [G(A_j), A_i]) = 0. \quad \square$$

Now let us prove Theorem 2.6.

THEOREM 2.6. *Let $n \geq s \geq 2$ be fixed integers. For an additive map $G : M_n(\mathbb{K}) \rightarrow M_n(\mathbb{K})$ such that $G(x)x = xG(x)$ for every rank- s matrix $x \in M_n(\mathbb{K})$, there exist an element $\lambda \in Z$ and an additive map $\mu : M_n(\mathbb{K}) \rightarrow Z$ such that $G(x) = \lambda x + \mu(x)$ for each $x \in M_n(\mathbb{K})$.*

Proof. For the case $s = 2$, by Lemma 2.1 and 2.5, we have that $[G(D), D] = 0$ for each rank-1 matrix D in $M_n(\mathbb{K})$. Hence, we can summarize that $[G(D), D] = 0$ for each rank- i ($i = 1, 2$) matrix D in $M_n(\mathbb{K})$.

For the case $s \geq 3$ and $|\mathbb{K}| > 2$, by Lemma 2.2 and 2.5, we have that $[G(D), D] = 0$ for each rank- i ($i = 1, 2$) matrix D in $M_n(\mathbb{K})$.

For the case $s \geq 3$, $3 \nmid s$ and $|\mathbb{K}| = 2$, by Lemma 2.4 and 2.5, we have that $[G(D), D] = 0$ for each rank- i ($i = 1, 2$) matrix D in $M_n(\mathbb{K})$.

For the case $s = 3m$ ($1 \leq m \leq \frac{n}{3}$) and $|\mathbb{K}| = 2$, by Lemma 2.3 and 2.5, we have that $[G(D), D] = 0$ for each rank-2 matrix D in $M_n(\mathbb{K})$. And then, by Lemma 2.1 and 2.5, we have that $[G(D), D] = 0$ for each rank-1 matrix D in $M_n(\mathbb{K})$. Hence, in summary, we have that $[G(D), D] = 0$ for each rank- i ($i = 1, 2$) matrix D in $M_n(\mathbb{K})$.

Obviously, the rank of $ae_{ij} + be_{kl}$ is at most 2, where $a, b \in \mathbb{K} \setminus \{0\}$. Thus,

$$[G(ae_{ij}), be_{kl}] + [G(be_{kl}), ae_{ij}] = 0$$

since

$$[G(ae_{ij}), ae_{ij}] = [G(be_{kl}), be_{kl}] = [G(ae_{ij} + be_{kl}), ae_{ij} + be_{kl}] = 0.$$

From now on we will follow Franca's idea and retell his corresponding proof of [6, Theorem 3]. Concretely, for any

$$D = \sum_{1 \leq i, j \leq n} a_{ij} e_{ij} \in M_n(\mathbb{K}),$$

where $a_{ij} \in \mathbb{K}$ and $1 \leq i, j \leq n$, we get

$$\begin{aligned} [G(D), D] &= \sum_{1 \leq i, j \leq n} [G(a_{ij}e_{ij}), a_{ij}e_{ij}] \\ &\quad + \sum_{(i,j) \neq (k,l)} ([G(a_{ij}e_{ij}), a_{kl}e_{kl}] + [G(a_{kl}e_{kl}), a_{ij}e_{ij}]) \\ &= 0. \end{aligned}$$

The desired result follows now from the well-known theorem on commuting maps due to Brešar (see the original paper [1], or the survey paper [2, Corollary 3.3], or the book [3, Corollary 5.28]). $\square$

For case $s = n = 2$, we have Lemma 2.7.

LEMMA 2.7. *If $|\mathbb{K}| > 2$, then every nonzero singular matrix in $M_2(\mathbb{K})$ can be expressed as a sum of three invertible matrices among which the sum of any two is invertible.*

Proof. The following is a nonzero singular matrix expressed as a sum of three invertible matrices among which the sum of any two is invertible

$$e_{11} = [e_{12} + e_{21} - \lambda e_{22}] + [(1 - \lambda)e_{11} - e_{12} - e_{21}] + \left[\lambda \sum_{i=1}^2 e_{ii} \right],$$

where $\lambda \neq 1$ is a nonzero element in $\mathbb{K}$. $\square$

The following example shows that the condition $|\mathbb{K}| > 2$ in Lemma 2.7 is necessary.

EXAMPLE 2.8. *In $M_2(\mathbb{Z}_2)$ if there exist invertible matrices $A_1, A_2, A_3 \in M_2(\mathbb{Z}_2)$ such that $A_i + A_j$ is invertible for all $1 \leq i < j \leq 3$, then $A_1 + A_2 + A_3 = 0$.*

Proof. Suppose that A_1, A_2 and A_3 are invertible, $A_i + A_j$ is invertible for all $1 \leq i \neq j \leq 3$, and $A_1 + A_2 + A_3 \neq 0$. We claim that $A_1, A_2, A_3, A_1 + A_2, A_1 + A_3, A_2 + A_3$ are six different invertible matrices. Firstly, for $1 \leq i < j \leq 3$, we have $A_i \neq A_j$, since $A_i + A_j \neq 0$. Secondly, for $i, j, k \in \{1, 2, 3\}$ such that $i \neq j, i \neq k$ and $j \neq k$, we obtain $A_i + A_j \neq A_i + A_k$, since $A_j \neq A_k$. Finally, for $i, j, k \in \{1, 2, 3\}$ such that $i \neq j, i \neq k$ and $j \neq k$, we obtain $A_i + A_j \neq A_k$, since $A_i + A_j + A_k \neq 0$. For $1 \leq i \neq j \leq 3$, it is obvious that $A_i + A_j \neq A_i$, since $A_j \neq 0$. However, the number of invertible matrices in $M_2(\mathbb{Z}_2)$ is six and their sum is the zero matrix. So, we will obtain a contradiction

$$A_1 + A_2 + A_3 = 3(A_1 + A_2 + A_3) = A_1 + A_2 + A_3 + (A_1 + A_2) + (A_1 + A_3) + (A_2 + A_3) = 0,$$

which completes the proof. $\square$

Keeping Lemma 2.5 and 2.7 in mind and using the same discussion as those in the proof of Theorem 2.6, we also obtain Theorem 2.9 as a special case of [5, Theorem 3].

THEOREM 2.9. *For $|\mathbb{K}| > 2$, if $G : M_2(\mathbb{K}) \rightarrow M_2(\mathbb{K})$ is an additive map such that $G(x)x = xG(x)$ for every invertible matrix $x \in M_2(\mathbb{K})$, then there exist an element $\lambda \in Z$ and an additive map $\mu : M_2(\mathbb{K}) \rightarrow Z$ such that $G(x) = \lambda x + \mu(x)$ for each $x \in M_2(\mathbb{K})$.*

For the condition $|\mathbb{K}| > 2$, Franca has indicated that it is necessary in Theorem 2.9 by [5, Example 1].

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